

Deformations of Algebras

Part III Essay

Abstract

The Deformation Theory of Associative Algebras was initiated by Murray Gerstenhaber in the 1960s, to parallel Analytical Deformation Theory. In his papers, Gerstenhaber described the rich connection of deformations of associative algebras and their Hochschild cohomology. After a brief look at Hochschild cohomology we will define deformations of algebras as done by Gerstenhaber and review a series of his original results. After introducing the Gerstenhaber bracket, we will look at the modern description of deformations via the Maurer-Cartan equation and DGLAs.

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Notation: Unless otherwise indicated, the following notations hold throughout this work:

- \mathbb{K} will denote a field, \otimes will denote the tensor product over field \mathbb{K} , $\otimes_{\mathbb{K}}$.
- A will denote an associative \mathbb{K} -Algebra, as defined in (1.1). Furthermore, a, b, c and a_1, a_2, \dots, a_n will always denote elements of the algebra A in question. We will refer to the multiplication map of algebra A by $\pi : A \otimes A \rightarrow A$ and for ease of notation, we will denote $\pi(a_1 \otimes a_2)$ by $a_1 a_2$.
- $A^{\otimes n}$ denotes the tensor product of n copies of A : $A \otimes A \otimes \dots \otimes A$, and if $n = 0$, $A^{\otimes 0} = \mathbb{K}$.
- We will refer to $A[[t]]$ as a \mathbb{K} -vectorspace by notation $A[[t]]$, and refer to it as an algebra by A_{π} . This notation is justified in section (2).
- From section (3) onwards, by a deformation, we mean an formal deformation.
- By a graded vector space structure on V , we refer to a decomposition $V = \bigoplus_{n \geq 0} V_n$ where V_n are vectorspaces, elements $v_n \in V_n$ are referred to as *homogeneous* elements with $|v_n| := n$.

1 Hochschild Cohomology

Since the deformation theory of associative algebras is tied in with Hochschild cohomology, we will first give a brief description of this cohomology theory. Our introduction to the topic is largely based on [22] and further detail can be found there.

Definition 1.1. For a field \mathbb{K} , a \mathbb{K} -vectorspace A is called an **associative \mathbb{K} -algebra** or simply a **\mathbb{K} -algebra**, if we have a \mathbb{K} -bilinear map $\pi : A \otimes A \rightarrow A$, such that

$$\pi(\pi(a \otimes b) \otimes c) = \pi(a \otimes \pi(b \otimes c))$$

for all $a, b, c \in A$.

A simple example of associative algebras is any field \mathbb{K} regarded as \mathbb{K} -vectorspace and its multiplication. Other important examples include the *group ring* $\mathbb{K}G$ for a group G , and the *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ of a Lie algebra \mathfrak{g} . Furthermore, basic notions such as algebra homomorphisms and modules are defined as usual to respect both the vectorspace structure and the multiplication. For a detailed account about associative algebras, one can refer to [17]. Additionally, recall that an A -bimodule M , has both left and right A -actions which respect each other, i.e. $(a_1 m) a_2 = a_1 (m a_2)$ for $a_1, a_2 \in A$ and $m \in M$.

To define the Hochschild cohomology of an algebra A , we must first restrict our attention to the sequence of A -bimodules

$$0 \longrightarrow \text{Hom}_{\mathbb{K}}(\mathbb{K}, A) \xrightarrow{d_1} \text{Hom}_{\mathbb{K}}(A, A) \xrightarrow{d_2} \text{Hom}_{\mathbb{K}}(A^{\otimes 2}, A) \xrightarrow{d_3} \dots \quad (1)$$

with maps $d_n : \text{Hom}_{\mathbb{K}}(A^{\otimes n-1}, A) \rightarrow \text{Hom}_{\mathbb{K}}(A^{\otimes n}, A)$ defined by

$$\begin{aligned} d_n(f)(a_1 \otimes a_2 \otimes \dots \otimes a_n) &= a_1 f(a_2 \otimes \dots \otimes a_n) + \sum_{i=1}^{n-1} (-1)^i f(a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n) \\ &\quad + (-1)^n f(a_1 \otimes \dots \otimes a_{n-1}) a_n \end{aligned} \quad (2)$$

Notice that every map in $\text{Hom}_{\mathbb{K}}(\mathbb{K}, A)$ is uniquely determined by where in A the unit of \mathbb{K} is taken to. Hence, $\text{Hom}_{\mathbb{K}}(\mathbb{K}, A) \cong A$ as an A -bimodule and $d_1 : A \rightarrow \text{Hom}_{\mathbb{K}}(A, A)$ is described by

$$(d_1(a_0))(a_1) = a_1 a_0 - a_0 a_1 \quad (3)$$

Furthermore, observe that $d_{n+1} d_n = 0$ for any $n \geq 1$:

$$\begin{aligned} d_{n+1} d_n(f)(a_1 \otimes a_2 \otimes \dots \otimes a_{n+1}) &= a_1 d_n(f)(a_2 \otimes \dots \otimes a_{n+1}) \\ &\quad + \sum_{i=1}^n (-1)^i d_n(f)(a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1}) + (-1)^{n+1} d_n(f)(a_1 \otimes \dots \otimes a_n) a_{n+1} \\ &= (a_1 a_2 f(a_3 \otimes \dots \otimes a_{n+1})) (1 - 1) + (a_1 f(a_2 \otimes \dots \otimes a_n) a_{n+1}) ((-1)^n + (-1)^{n+1}) \\ &\quad + \left(\sum_{i=2}^n (-1)^{i-1} a_1 f(a_2 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1}) \right) (1 - 1) \\ &\quad + \left(\sum_{i=1}^{n-1} (-1)^i f(a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n) a_{n+1} \right) ((-1)^n + (-1)^{n+1}) \\ &\quad + \sum_{i>j} (-1)^{i+j} f(a_2 \otimes \dots \otimes a_j a_{j+1} \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1}) \end{aligned}$$

$$\begin{aligned}
& + \sum_{j>i} (-1)^{i+j-1} f(a_2 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{n+1}) \\
& + (f(a_1 \otimes \cdots \otimes a_{n-1}) a_n a_{n+1}) ((-1)^{2n} + (-1)^{2n+1}) = 0
\end{aligned}$$

Hence, $Im(d_n) \subseteq Ker(d_{n+1})$ and for those familiar with Homological language, sequence (1) is a complex.

Definition 1.2. For \mathbb{K} -algebra A , we define its n th **Hochschild Cohomology** group, $HH^n(A)$, as the homology of complex (1) by

$$HH^n(A) = H^n(Hom_{\mathbb{K}}(A^{\otimes \cdot}, A)) = Ker(d_{n+1}) / Im(d_n)$$

for all $n \geq 0$ and d_0 is taken to be zero map. Moreover, elements in $Ker(d_{n+1})$ are called **Hochschild n -Cocycle** and elements in $Im(d_n)$ are called **Hochschild n -Coboundaries**. Furthermore, elements in $Hom_{\mathbb{K}}(A^{\otimes n}, A)$ are referred to as **Hochschild n -Cochains**.

Remark 1.3. Although we have presented the complex (1) and the maps d_i out of thin air, they arise rather naturally from the bar complex of A -bimodule A , and the correspondence of A -bimodules and left A^e -modules, where $A^e \cong A \otimes A^{op}$ is the enveloping algebra of A . A detailed account of the definition can be found in Chapter 1 of [22].

Even in low degrees Hochschild cohomology carries rich detail of the algebra:

At degree 0: By identity (3)

$$HH^0(A) = Ker(d_1)/Im(d_0) \cong Ker(d_1) = \{a_0 \in A \mid a_1 a_0 - a_0 a_1 = 0\}$$

Hence, $HH^0(A) = Z(A)$, the center of A .

At degree 1: Hochschild 1-cocycles correspond to \mathbb{K} -derivations of A . If $f \in Ker(d_2)$ then

$$\begin{aligned}
0 &= d_2(f)(a_1 \otimes a_2) = a_1 f(a_2) + (-1)f(a_1 a_2) + f(a_1) a_2 \\
&\Leftrightarrow f(a_1 a_2) = a_1 f(a_2) + f(a_1) a_2
\end{aligned} \tag{4}$$

for any $a_1, a_2 \in A$. Notice, (4) is called the *Leibniz rule* and any \mathbb{K} -linear map in $Hom_{\mathbb{K}}(A, A)$ which satisfies it is said to be a \mathbb{K} -derivation of A .

At degree 2: A map $f \in Hom_{\mathbb{K}}(A^{\otimes 2}, A)$ is a Hochschild 2-cocycle if $d_3(f) = 0$:

$$\Rightarrow 0 = d_3(f)(a_1 \otimes a_2 \otimes a_3) = a_1 f(a_2 \otimes a_3) - f(a_1 a_2 \otimes a_3) + f(a_1 \otimes a_2 a_3) - f(a_1 \otimes a_2) a_3$$

which is equivalent to f satisfying

$$a_1 f(a_2 \otimes a_3) + f(a_1 \otimes a_2 a_3) = f(a_1 a_2 \otimes a_3) + f(a_1 \otimes a_2) a_3 \tag{5}$$

for any $a_1, a_2, a_3 \in A$. Identity (5) will be essential to deformation theory as we will see in the next section.

On the calculation of Hochschild cohomology: Because \mathbb{K} is a field, and any \mathbb{K} -vectorspace is projective as a \mathbb{K} -module, Theorem 9.1.5 from [21] implies that the Hochschild cohomology of algebra A can be computed via the *Ext* functor. In particular,

$$HH^n(A) \cong Ext_{A^e}^n(A, A)$$

where $A^e = A \otimes A^{op}$ and there exists a correspondence between A -bimodule and left A^e -modules. Thereby, the Hochschild cohomology as defined in (1.2) is equal to the cohomology of any projective resolution, of A -bimodules, of A and can be calculated accordingly.

Example 1.4. Let $A = \mathbb{K}[\epsilon]/(\epsilon^2)$, where $\text{Char}(\mathbb{K}) \neq 2$. We can calculate its Hochschild cohomology via the following sequence of A -bimodules:

$$\dots \longrightarrow A \otimes A \xrightarrow{\tau_-} A \otimes A \xrightarrow{\tau_+} A \otimes A \xrightarrow{\tau_-} A \otimes A \xrightarrow{\pi} A \longrightarrow 0 \quad (6)$$

Where if $l_\epsilon : A \rightarrow A$ is left multiplication by ϵ and $r_\epsilon : A \rightarrow A$, right multiplication by ϵ , then

$$\tau_+ = l_\epsilon \otimes \text{Id}_A + \text{Id}_A \otimes r_\epsilon \quad \text{and} \quad \tau_- = l_\epsilon \otimes \text{Id}_A - \text{Id}_A \otimes r_\epsilon$$

Thereby, $\tau_- \tau_+ = \tau_+ \tau_- = 0$ since $\epsilon^2 = 0$. Observe that $A \otimes A$ is a free A -bimodules and projective: This is clear from the A -bimodule and A^e -module correspondence mentioned earlier and the fact that $A^e = A \otimes A$ since A is commutative. Moreover, since $\text{Char}(\mathbb{K}) \neq 2$:

$$\begin{aligned} \text{Ker}(\pi) &= \text{Im}(\tau_-) = \text{Ker}(\tau_+) = \text{Span}_{\mathbb{K}}\{1 \otimes \epsilon - \epsilon \otimes 1, \epsilon \otimes \epsilon\} \\ \text{Im}(\tau_+) &= \text{Ker}(\tau_-) = \text{Span}_{\mathbb{K}}\{1 \otimes \epsilon + \epsilon \otimes 1, \epsilon \otimes \epsilon\} \end{aligned}$$

Hence, sequence (6) is exact and thereby is a projective resolution. We calculate its cohomology by applying $\text{Hom}_{A\text{-bi}}(-, A)$ to the truncated complex

$$\begin{array}{ccccccc} \dots & \longrightarrow & A \otimes A & \xrightarrow{\tau_-} & A \otimes A & \xrightarrow{\tau_+} & A \otimes A & \xrightarrow{\tau_-} & A \otimes A & \longrightarrow & 0 \\ & & & & & & & \Downarrow & & & \\ 0 & \longrightarrow & \text{Hom}_{A\text{-bi}}(A \otimes A, A) & \xrightarrow{\tau_-^*} & \text{Hom}_{A\text{-bi}}(A \otimes A, A) & \xrightarrow{\tau_+^*} & \text{Hom}_{A\text{-bi}}(A \otimes A, A) & \longrightarrow & \dots & & \end{array} \quad (7)$$

Further, observe that

$$\text{Hom}_{A\text{-bi}}(A \otimes A, A) = \{g_a | a \in A, \text{ where } g_a(1 \otimes 1) = a\} \cong A$$

since an A -bimodule map in $\text{Hom}_{A\text{-bi}}(A \otimes A, A)$ is determined by where $1 \otimes 1$ is taken to. By an easy computation

$$\tau_-^*(g_a) = g_a(\tau_-) = 0 \quad \tau_+^*(g_a) = g_a(\tau_+) = 2\epsilon g_a$$

so that $\tau_-^* = 0$ and we can re-write (7) as

$$0 \longrightarrow A \xrightarrow{0} A \xrightarrow{2\epsilon} A \xrightarrow{0} A \xrightarrow{2\epsilon} A \longrightarrow \dots \quad (8)$$

Hence, $HH_0(A) \cong A$, $HH_{2m-1}(A) = \text{Ker}(2\epsilon) \cong \epsilon A$ and $HH_{2m}(A) \cong A/\epsilon A$ for $m \geq 1$. A similar result can be derived for $A = \mathbb{K}[\epsilon]/(\epsilon^n)$, for any n , and can be found in Example 1.1.16 of [22].

To calculate the Hochschild cohomology of more complicated examples, we need a more extensive Homological toolbox and notation, hence the above example will suffice and most importantly, will tie in nicely with the deformation Theory as introduced in the next chapters. One can refer to [22] and [18] for the calculation of the Hochschild cohomology of a large range of algebras.

2 Deformations of Associative Algebras

Given an associative algebra A over a field \mathbb{K} , we can define the vectorspace of formal power series $A[[t]] = A \otimes \mathbb{K}[[t]]$. The aim of deformation theory is to define a multiplication on $A[[t]]$, in order to enrich it with an associative algebra structure.

Observe that since elements $\alpha \in A[[t]]$ look like power series (i.e. $\alpha = \sum_{i \geq 0} a_i t^i$ for some $a_i \in A$), a $\mathbb{K}[[t]]$ -bilinear map $\nu : A[[t]] \otimes A[[t]] \rightarrow A[[t]]$ is uniquely determined by how it acts on elements of form $a \otimes b$ with $a, b \in A$. Moreover, for $a \otimes b$ with $a, b \in A$, $\nu_i(a \otimes b)$ are uniquely determined in the expansion

$$\nu(a \otimes b) = \nu_0(a \otimes b) + \nu_1(a \otimes b)t + \nu_2(a \otimes b)t^2 + \dots$$

Further, since ν is $\mathbb{K}[[t]]$ -bilinear, $\nu_i : A \otimes A \rightarrow A$ as determined by ν are also \mathbb{K} -bilinear.

In order for the new multiplication to be a deformation of the old, we require ν_0 to be the multiplication of A . In other words, when $t = 0$, $A[[t]]$ with the multiplication ν is just the original algebra A .

Definition 2.1. (I) For \mathbb{K} -algebra A , we say $\mu = \sum_{i \geq 0} \mu_i t^i$ with $\mu_i \in \text{Hom}_{\mathbb{K}}(A, A)$ is a **one-parameter deformation** of A if μ_0 is the multiplication of A . We refer to $A[[t]]$ with multiplication μ by A_μ .

(II) We say A_μ is an **formal deformation** if

$$\mu(\mu(a \otimes b) \otimes c) = \mu(a \otimes \mu(b \otimes c)) \quad (9)$$

for $a, b, c \in A$.

If $\mu = \pi$, we get the ring of formal power series $A[[t]]$ with the multiplication of A extended to $A[[t]]$, t -linearly. We will refer to this algebra as A_π , to avoid confusion between using $A[[t]]$ as an algebra and as a vectorspace.

Example 2.2. Let $A = \mathbb{K}[\epsilon]/(\epsilon^2)$. For any $\alpha \in \mathbb{K}$, we define $\rho_\alpha \in \text{Hom}_{\mathbb{K}}(A^{\otimes 2}, A)$ on the basis of A , $\{1, \epsilon\}$, by

$$\rho_\alpha(1 \otimes 1) = \rho_\alpha(1 \otimes \epsilon) = \rho_\alpha(\epsilon \otimes 1) = 0, \quad \rho_\alpha(\epsilon \otimes \epsilon) = \alpha \quad (10)$$

Thus for any sequence $\Delta = \{\alpha_i\}_{i=1}^\infty$ of elements $\alpha_i \in \mathbb{K}$, let $\mu_\Delta := \pi + \sum_{i \geq 1} \alpha_i t^i$. For any sequence Δ , μ_Δ is a one-parameter deformation; however, its associativity is less clear. Take the sequence to be non-zero only on the first term i.e. for $\alpha \in \mathbb{K}$ consider $\mu_\alpha := \pi + \rho_\alpha t$, then it is easy to check that A_{μ_α} is a formal deformation: Firstly, observe that we have relations

$$\mu_\alpha(1 \otimes 1) = 1, \quad \mu_\alpha(\epsilon \otimes 1) = \mu_\alpha(1 \otimes \epsilon) = \epsilon, \quad \mu_\alpha(\epsilon \otimes \epsilon) = \alpha t \quad (11)$$

We only need to check associativity on the basis $\{1, \epsilon\}$, and the relations above imply that $\mu_\alpha = \pi$ unless both entries are ϵ . Therefore, we only need to check the following cases

- $\mu_\alpha(\mu_\alpha(1 \otimes \epsilon) \otimes \epsilon) = \mu_\alpha(\mu_\alpha(\epsilon \otimes 1) \otimes \epsilon) = \mu_\alpha(\mu_\alpha(\epsilon \otimes \epsilon) \otimes 1) = \alpha t$
 $= \mu_\alpha(1 \otimes \mu_\alpha(\epsilon \otimes \epsilon)) = \mu_\alpha(\epsilon \otimes \mu_\alpha(1 \otimes \epsilon)) = \mu_\alpha(\epsilon \otimes \mu_\alpha(\epsilon \otimes 1))$
- $\mu_\alpha(\mu_\alpha(\epsilon \otimes \epsilon) \otimes \epsilon) = \alpha \epsilon t = \mu_\alpha(\epsilon \otimes \mu_\alpha(\epsilon \otimes \epsilon))$

Hence A_{μ_α} are formal deformations and from the relations in (11) it should be clear that we have an isomorphism of \mathbb{K} -algebras $A_{\mu_\alpha} \cong \mathbb{K}[\epsilon][[t]]/(\epsilon^2 - \alpha t)$. Observe that “at $t = 0$ ”, the deformation gives our original algebra: $(A_{\mu_\alpha})_{t=0} \cong \mathbb{K}[\epsilon]/(\epsilon^2) = A$.

By looking at the coefficient of t^n in the associativity condition (9) of μ , we see that A_μ is associative if and only if

$$\sum_{i=0}^n \mu_i(\mu_{n-i}(a \otimes b) \otimes c) = \sum_{i=0}^n \mu_i(a \otimes \mu_{n-i}(b \otimes c)) \quad (12)$$

holds for all $n \geq 0$. Since μ_0 is the multiplication of associative algebra A , (12) clearly holds for $n = 0$. Further, we can rewrite (12) as

$$\begin{aligned} \sum_{i=1}^{n-1} \mu_i(\mu_{n-i}(a \otimes b) \otimes c) - \sum_{i=1}^{n-1} \mu_i(a \otimes \mu_{n-i}(b \otimes c)) \\ = a\mu_n(b \otimes c) + \mu_n(a \otimes bc) - \mu_n(a \otimes b)c - \mu_n(ab \otimes c) \end{aligned} \quad (13)$$

Also, the RHS of (13) is equal to the Hochschild 3-coboundary of μ_n , $d_3\mu_n(a, b, c)$, where the map $d_3 : Hom_{\mathbb{K}}(A^{\otimes 2}, A) \rightarrow Hom_{\mathbb{K}}(A^{\otimes 3}, A)$ is as defined in (2) in the last section.

$$\sum_{i=1}^{n-1} \mu_i(\mu_{n-i}(a \otimes b) \otimes c) - \sum_{i=1}^{n-1} \mu_i(a \otimes \mu_{n-i}(b \otimes c)) = d_3\mu_n(a, b, c) \quad (\star_n)$$

Definition 2.3. In the expansion $\mu = \pi + \sum_{i \geq 1} \mu_i t^i$ of a one-parameter deformation μ , the first non-zero coefficient $\mu_n \in Hom_{\mathbb{K}}(A^{\otimes 2}, A)$ is called the **infinitesimal** of μ .

Observe that if μ_n is the infinitesimal of an associative deformation μ , all the terms on the LHS of (\star_n) will be zero, and thereby $d_3\mu_n = 0$. Hence:

Proposition 2.4. If μ is a formal deformation of algebra A , then the infinitesimal of μ is a Hochschild 2-cocycle.

Definition 2.5. For algebra A , we say Hochschild 2-cocycle $f \in Ker(d_3)$ is **integrable** if there exists a formal deformation of A , μ , such that f is the infinitesimal of μ .

Example 2.6. As we saw in Example (2.2), A_{μ_α} with $\mu_\alpha = \pi + \rho_\alpha t$ is a formal deformation for any $\alpha \in \mathbb{K}$. We can confirm Proposition (2.4): The expansion

$$d_3\rho_\alpha(a \otimes b \otimes c) = a\rho_\alpha(b \otimes c) - \rho_\alpha(ab \otimes c) + \rho_\alpha(a \otimes bc) - \rho_\alpha(a \otimes b)c$$

is zero for any combination of $a, b, c \in \{1, \epsilon\}$: since if at least one of a, b, c is 1, then two of the terms are zero and the other two cancel out. In the case that $a = b = c = \epsilon$, then the two middle terms of the expansion are zero and the other two terms cancel out. Therefore, $\rho_\alpha \in Ker(d_3)$ and since A_{μ_α} are formal deformations, then ρ_α are integrable.

Remark 2.7. If Hochschild 2-cocycle μ_n is integrable, and occurs as the infinitesimal of deformation $\mu = \pi + \sum_{i \geq 1} \mu_i t^i$, then in the can appear as the coefficient of t^m for any $m \geq 1$ and form a deformation μ^m with $\mu_i^m = 0$ for $1 \leq i \leq m-1$ and $\mu_{m+i}^m = \mu_{n+i}$ for $i \geq 0$. Therefore, in the literature when dealing with integrability problems, the infinitesimal is often assumed to occur at $n = 1$.

The natural question to ask is; which Hochschild 2-cocycles are integrable? Gerstenhaber attacked this problem inductively: we assume that for $\mu_1, \mu_2, \dots, \mu_{n-1} \in Hom_{\mathbb{K}}(A^{\otimes 2}, A)$, equations $(\star_i : 1 \leq i \leq n-1)$ hold. Then, we would like to find a $\mu_n \in Hom_{\mathbb{K}}(A^{\otimes 2}, A)$ such that (\star_n) holds. Of course this is only possible if and only if the LHS of (\star_n) is a Hochschild 3-coboundary and lies in $Im(d_3)$. Hence, from this point of view, the LHS of equation (\star_n) can be thought of as the ‘obstruction’ to our deformation problem.

Definition 2.8. If $\mu : A[[t]] \rightarrow A[[t]]$ is defined by decomposition $\mu = \pi + \sum_{i \geq 0}^{n-1} \mu_i t^i$, where $\mu_i \in \text{Hom}_{\mathbb{K}}(A^{\otimes 2}, A)$ and equations $(\star_i : 1 \leq i \leq n-1)$ hold, we call (A, μ) a **truncated deformation** and the LHS of (\star_n) , the **$(n-1)$ -th obstruction**.

In section (4), we will show that the $(n-1)$ -th obstruction of a truncated deformation is a Hochschild 3-cocycle, but we will first need to introduce the Gerstenhaber Bracket.

Example 2.9. For ρ_α as defined in Example (2.2), since $\text{Im}(\rho_\alpha)$ lies in \mathbb{K} for any $\alpha \in \mathbb{K}$, and $\rho_\alpha(-, -)$ is zero when one of the entries is from \mathbb{K} , then for any $\alpha, \beta \in \mathbb{K}$

$$\rho_\alpha(\rho_\beta(a \otimes b) \otimes c) = 0 = \rho_\alpha(a \otimes \rho_\beta(b \otimes c))$$

holds for $a, b, c \in \{1, \epsilon\}$ and thereby for any $a, b, c \in A = \mathbb{K}[\epsilon]/(\epsilon^2)$. Hence for any finite sequence $\Delta = \{\alpha_i\}_{i=1}^{n-1}$, the $(n-1)$ th obstruction of $\pi + \sum_{i \geq 1}^{n-1} \rho_{\alpha_i} t^i$ vanishes:

$$\sum_{i=1}^{n-1} \rho_{\alpha_i}(\rho_{\alpha_{n-i}}(a \otimes b) \otimes c) - \sum_{i=1}^{n-1} \rho_{\alpha_i}(a \otimes \rho_{\alpha_{n-i}}(b \otimes c)) = 0 - 0 = 0$$

Moreover, by Example (2.6), we know $d_3 \rho_\alpha = 0$ for any $\alpha \in \mathbb{K}$. Hence, we can extend the truncated deformation via any ρ_α i.e. for any $\alpha \in \mathbb{K}$, equation

$$\sum_{i=1}^{n-1} \rho_{\alpha_i}(\rho_{\alpha_{n-i}}(a \otimes b) \otimes c) - \sum_{i=1}^{n-1} \rho_{\alpha_i}(a \otimes \rho_{\alpha_{n-i}}(b \otimes c)) = d_3 \rho_\alpha$$

holds. So for any sequence $\Delta = \{\alpha_i\}_{i=1}^{\infty}$ of elements $\alpha_i \in \mathbb{K}$, $\mu_\Delta = \pi + \sum_{i \geq 1} \alpha_i t^i$ defines a formal deformation of $A = \mathbb{K}[\epsilon]/(\epsilon^2)$.

Remark 2.10. Much of the vocabulary of the theory such as integrability, obstructions and infinitesimals were chosen by Gerstenhaber to parallel the Analytical Theory of Deformations. Notice that if a one-parameter deformation μ is formal, it defines an associative algebra structure on vectorspace $A[[t]]$, where t is an indeterminate variable or ‘parameter’. Given this structure, we could then let t be any α element of A and define a new associative multiplication on A ; however, the issue which arises is the convergence of an expression $\sum_{i \geq 0} a_i \alpha^i$ in A , which is the difficulty of the analytical theory. This is why Gerstenhaber chooses to work over the ring of formal power series. In fact, in [6] Gerstenhaber works over $\mathbb{K}((t))$, while we have chosen to work over $\mathbb{K}[[t]]$. Additionally, it should be clear that given a truncated deformation $(A, \sum_{i \geq 0}^{n-1} \mu_i t^i)$, we can define an associative algebra structure on the vectorspace $A \otimes \mathbb{K}[t]/(t^n)$. Moreover, in this deformation, parameter t can take the value of any element of A with order n , since we are taking finite sums and do not need to worry about convergence. We discuss the generalisation of deformations to $A \otimes R$, where \mathbb{K} -algebra R satisfies certain conditions in section (5).

Example 2.11. The deformations A_{μ_α} from Example (2.2) can also be considered as deformations over $\mathbb{K}[t]/(t^2)$ and over $\mathbb{K}[t]/(t^2)$, $A_{\mu_\alpha} \cong \mathbb{K}[\epsilon, t]/(\epsilon^2 - \alpha t, t^2)$. As mentioned in the Remark above, we can replace parameter t by any element of order 2 in \mathbb{K} . If $\mathbb{K} = \mathbb{R}$, $\alpha = 1$ and $t = -1$, then

$$A_{\mu_1} \cong \mathbb{K}[\epsilon, t]/(\epsilon^2 - t, t^2) \quad \Rightarrow \quad (A_{\mu_1})_{t=-1} \cong \mathbb{R}[\epsilon]/(\epsilon^2 + 1) \cong \mathbb{C}$$

Notation: Since we are only concerned with formal deformations from this point on, we will simply write ‘a deformation A_μ ’, instead of a formal deformation.

3 Equivalence of Deformations

Since a deformation of \mathbb{K} -algebra A induces an associative algebra structure on vectorspace $A[[t]]$, the next natural question to ask is: When do two formal deformation induce isomorphic algebras? This question is resolved by defining the notion of equivalence between deformations.

Definition 3.1. We say one-parameter deformations A_μ and A_λ are **equivalent** if there exists a $\mathbb{K}[[t]]$ -linear map $\phi : A[[t]] \rightarrow A[[t]]$ such that

$$\phi = \phi_0 + \phi_1 t + \phi_2 t^2 + \cdots = \sum_{i \geq 0} \phi_i t^i$$

where $\phi_i \in \text{Hom}_{\mathbb{K}}(A, A)$ and $\phi_0 = \text{Id}_A$, and

$$\phi(\mu(a \otimes b)) = \lambda(\phi(a) \otimes \phi(b)) \quad (14)$$

or equivalently, diagram

$$\begin{array}{ccc} A[[t]] \otimes A[[t]] & \xrightarrow{\mu} & A[[t]] \\ \downarrow \phi \otimes \phi & & \downarrow \phi \\ A[[t]] \otimes A[[t]] & \xrightarrow{\lambda} & A[[t]] \end{array} \quad (15)$$

commutes.

Remark 3.2. It is known that an element $\alpha = \sum_{i \geq 0} r_i x^i$ is invertible in the ring of formal power series $R[[x]]$ if and only if r_0 is invertible. The inverse of α can be constructed inductively; $\beta = \sum_{i \geq 0} b_i x^i$, where $b_0 = r_0^{-1}$ and $b_n = -r_0^{-1} \left(\sum_{i=0}^{n-1} r_i b_i \right)$, so that $\alpha\beta = 1$.

Therefore, by the above Remark, any linear map $\phi \in \text{Hom}_{\mathbb{K}}(A, A)[[t]]$ of form $\phi = \text{Id}_A + \sum_{i \geq 1} \phi_i t^i$ is invertible and a \mathbb{K} -vectorspace isomorphism of $A[[t]]$. Moreover, its inverse has the form $\phi' = \text{Id}_A + \sum_{i \geq 1} \phi'_i t^i$ and equivalence, as defined above, is symmetric and indeed an equivalence relation, since reflexivity and transitivity clearly hold. Furthermore, if A_μ and A_λ are formal deformations and equivalent via a map ϕ , then by (14), ϕ is also \mathbb{K} -algebra isomorphism.

Definition 3.3. We say deformation A_μ is **trivial**, if it is equivalent to A_π , and we call algebra A **rigid** if all its formal deformations are trivial.

Given an equivalence of deformations $\phi : A_\mu \rightarrow A_\lambda$, consider the coefficient of t^n in (14):

$$\sum_{i+j=n} \phi_i(\mu_j(a \otimes b)) = \sum_{i+j+k=n} \lambda_i(\phi_j(a) \otimes \phi_k(b)) \quad (16)$$

where ϕ_0 is the identity map on A . Since μ_0 and λ_0 are the multiplication of A , then we can rewrite the above equation as

$$\sum_{\substack{i+j=n \\ i \neq 0}} \phi_i(\mu_j(a \otimes b)) - \sum_{\substack{i+j+k=n \\ j, k \neq n}} \lambda_i(\phi_j(a) \otimes \phi_k(b)) = a\phi_n(b) + \phi_n(a)b - \phi_n(ab) \quad (17)$$

Notice that the RHS of (17) is the Hochschild coboundary of $\phi_n \in \text{Hom}_{\mathbb{K}}(A, A)$:

$$\sum_{\substack{i+j=n \\ i \neq 0}} \phi_i(\mu_j(a \otimes b)) - \sum_{\substack{i+j+k=n \\ j, k \neq n}} \lambda_i(\phi_j(a) \otimes \phi_k(b)) = d_2 \phi_n(a \otimes b) \quad (**_n)$$

Hence, (**₁) holding implies the following result:

Proposition 3.4. *If deformations A_μ and A_λ are equivalent via a map ϕ as above, then $\mu_1 - \lambda_1 = d_2\phi_1$.*

Corollary 3.5. *If deformation A_μ is trivial, then μ_1 is a Hochschild 2-coboundary.*

Proof. Let $\phi : A_\mu \rightarrow A_\pi$ be an equivalence. By looking at the last Proposition, $\lambda_1 = 0$ and $\mu_1 = d_2\phi_1 \in \text{Im}(d_2)$. \square

Remark 3.6. *Recall from Remark (2.10) that a deformation over $\mathbb{K}[t]/(t^2)$ has form $\mu = \pi + \mu_1 t$. Further by Proposition (2.4) and (\star_1) holding for μ , we know that $\mu_1 \in \text{Ker}(d_3)$. On the other hand, since $t^2 = 0$, then Proposition (3.4) implies that two deformations μ and λ over $\mathbb{K}[t]/(t^2)$ are equivalent if and only if $\mu_1 - \lambda_1 \in \text{Im}(d_2)$. Hence, we have the bijection*

$$\{\text{Formal deformations of } A \text{ over } \mathbb{K}[t]/(t^2) \text{ upto equivalence}\} \longleftrightarrow HH^2(A) \quad (18)$$

Example 3.7. *Let $A = \mathbb{K}[\epsilon]/(\epsilon^2)$ and $\text{Char}(\mathbb{K}) \neq 2$. If $\varphi \in \text{Hom}_{\mathbb{K}}(A, A)$, then*

$$\begin{aligned} d_2\varphi(1 \otimes 1) &= 1\varphi(1) - \varphi(1) + \varphi(1)1 = \varphi(1) \\ d_2\varphi(1 \otimes \epsilon) &= 1\varphi(\epsilon) - \varphi(\epsilon) + \varphi(1)\epsilon = \varphi(1)\epsilon = d_2\varphi(\epsilon \otimes 1) \\ d_2\varphi(\epsilon \otimes \epsilon) &= \epsilon\varphi(\epsilon) - \varphi(0) + \varphi(\epsilon)\epsilon = 2\varphi(\epsilon)\epsilon \end{aligned}$$

Recall that deformations $\mu_\alpha = \pi + \rho_\alpha t$ for $\alpha \in \mathbb{K}$ are formal. Moreover, we know from Example (2.6) that ρ_α are 2-cocycles for any $\alpha \in \mathbb{K}$ and by the definition of ρ_α , we can deduce that if $\alpha \neq 0$, then

$$\rho_\alpha(\epsilon \otimes \epsilon) = \alpha \notin \epsilon A$$

However, by the above calculation for any $\varphi \in \text{Hom}_{\mathbb{K}}(A, A)$, $d_2\varphi(\epsilon \otimes \epsilon) \in \epsilon A$. Therefore, if $\alpha \neq 0$, then ρ_α is not a coboundary. By Corollary (3.5), this implies that deformations A_{μ_α} are nontrivial if $\alpha \neq 0$. Furthermore, since $\rho_\alpha - \rho_\beta = \rho_{\alpha-\beta}$ is not a coboundary if $\alpha \neq \beta$, then $\{\rho_\alpha \mid \alpha \in \mathbb{K}\}$ represent distinct elements in $HH^2(A)$. Additionally, by Proposition (3.4), $\{A_{\mu_\alpha} \mid \alpha \in \mathbb{K}\}$ must be a set of two by two non-equivalent deformations of A . Also, recall from Example (1.4) that $HH^2(A) \cong \mathbb{K}$. Hence,

$$HH^2(A) = \{\rho_\alpha + \text{Im}(d_2) \mid \alpha \in \mathbb{K}\}$$

Also as mentioned in Example (2.11), A_{μ_α} can be considered as deformations of A over $\mathbb{K}[t]/(t^2)$. In this case, Remark (3.6) implies that $\{A_{\mu_\alpha} \mid \alpha \in \mathbb{K}\}$ are all the deformations of A over $\mathbb{K}[t]/(t^2)$, upto equivalence.

Corollary (3.5) does not imply that the infinitesimal of a trivial deformation is a coboundary, since the infinitesimal might appear as the coefficient of t^n for some n bigger than 1. However, we will show in the next theorem that a non-trivial deformation, must be equivalent to a deformation whose infinitesimal is not a coboundary. But first we must observe that if A_μ is a deformation and $\phi_t \in \text{Hom}_{\mathbb{K}}(A, A)[[t]]$ has the form $\phi_t = Id_A + \sum_{i \geq 1} \phi_i t^i$, then $\phi_t \mu (\phi_t^{-1} \otimes \phi_t^{-1})$ is a deformation:

- Clearly since μ_0 is the multiplication of A and ϕ_t^{-1} has form $Id_A + \sum_{i \geq 1} \phi'_i t^i$ by Remark (3.2), then $\phi_t \mu (\phi_t^{-1} \otimes \phi_t^{-1})$ has form $\pi + \sum_{i \geq 1} \mu'_i t^i$ and thereby is a one-parameter deformation.
- Secondly, $\mu' = \phi_t \mu (\phi_t^{-1} \otimes \phi_t^{-1})$ is associative:

$$\begin{aligned} \mu'(\mu'(a \otimes b) \otimes c) &= \mu'(\phi_t \mu (\phi_t^{-1}(a) \otimes \phi_t^{-1}(b)) \otimes c) \\ &= \phi_t \mu (\phi_t^{-1} \phi_t \mu (\phi_t^{-1}(a) \otimes \phi_t^{-1}(b)) \otimes \phi_t^{-1}(c)) \\ &= \phi_t \mu (\mu (\phi_t^{-1}(a) \otimes \phi_t^{-1}(b)) \otimes \phi_t^{-1}(c)) \\ &= \phi_t \mu (\phi_t^{-1}(a) \otimes \mu (\phi_t^{-1}(b) \otimes \phi_t^{-1}(c))) \\ &= \phi_t \mu (\phi_t^{-1}(a) \otimes \phi_t^{-1} \phi_t \mu (\phi_t^{-1}(b) \otimes \phi_t^{-1}(c))) = \mu'(a \otimes \mu'(b \otimes c)) \end{aligned} \quad (19)$$

Theorem 3.8. *If deformation A_μ is a non-trivial deformation, then it is equivalent to a deformation A_λ such that the infinitesimal of λ , $\lambda_n \in \text{Ker}(d_3)$, is non-vanishing in $HH^2(A, A)$ i.e. λ_n is not a coboundary.*

Proof. By Proposition (2.4), the infinitesimal of any deformation is a 2-cocycle. Let A_μ be a non-trivial deformation. If its infinitesimal μ_n is a 2-coboundary, then $\mu_n \in \text{Im}(d_2)$ and $\mu_n = d_2 f_0$ for some $f_0 \in \text{Hom}_{\mathbb{K}}(A, A)$.

Let $\phi_0 : A[[t]] \rightarrow A[[t]]$ be defined by $\phi_0 = \text{Id}_A + f_0 t^n$. By Remark (3.2), ϕ_0 is invertible and by (19), $\mu' = \phi_0 \mu (\phi_0^{-1} \otimes \phi_0^{-1})$ is an equivalent deformation.

Claim. If μ'_m is the infinitesimal of μ' , then $m > n$.

The claim, if proved, directly implies the theorem: since either the infinitesimal of μ' is not a coboundary and the theorem holds, or we can repeat the process, to get an equivalent deformation. Assume that by repeating the process, we always get a coboundary as the infinitesimal. Then we get a sequence of equivalent deformations $\{\mu^{(i)}\}_{i \geq 0}$ where the infinitesimals of $\mu^{(i)}$, $\mu_{m_i}^{(i)}$, are all 2-coboundaries with $\mu_{m_i}^{(i)} = d_2 f_i$, where $\mu^{(i+1)} = \phi_i \mu^{(i)} (\phi_i^{-1} \otimes \phi_i^{-1})$ for $\phi_i = \text{Id}_A + f_i t^{m_i}$ and $\mu_0 = \mu$. Since the claim implies that $\{m_i\}_{i \geq 0}$ is a strictly increasing sequence, then

$$\Phi := \text{Id}_A + f_0 t^n + f_1 t^{m_1} + \dots = \text{Id}_A + \sum_{i \geq 0} f_i t^{m_i}$$

is well-defined in $\text{Hom}_{\mathbb{K}}(A, A)[[t]]$. Since Φ can be seen as the composition of all ϕ_i , the claim also implies that the infinitesimal of $\lambda := \Phi \mu (\Phi^{-1} \otimes \Phi^{-1})$ must be the coefficient of a t^M , where M is larger than all m_i . Because $\{m_i\}_{i \geq 0}$ is strictly increasing, then the coefficient of all t^i must be zero. So $\lambda = \pi$ and A_μ is a trivial deformation, which is a contradiction.

Proof of claim: Observe that by Remark (3.2) $\phi_0 = \text{Id}_A + f_0 t^n$ is invertible and

$$\phi_0^{-1} = \text{Id}_A - f_0 t^n + \phi_{n+1}^{-1} t^{n+1} + \dots$$

for some $\phi_{n+i}^{-1} \in \text{Hom}_{\mathbb{K}}(A, A)$, where $i \geq 1$. So

$$\begin{aligned} \mu'(a \otimes b) &= \phi_0 \mu (\phi_0^{-1}(a) \otimes \phi_0^{-1}(b)) = \phi_0 \mu ((a - f_0(a)t^n + \dots) \otimes (b - f_0(b)t^n + \dots)) \\ &= \phi_0 \mu (a \otimes b - (f_0(a) \otimes b)t^n - (a \otimes f_0(b))t^n + \dots) \\ &= \phi_0 (ab - f_0(a)bt^n - af_0(b)t^n + \mu_n(a \otimes b)t^n + \dots) \\ &= ab - f_0(a)bt^n - af_0(b)t^n + \mu_n(a \otimes b)t^n + f_0(ab)t^n + \dots \\ &= ab + (f_0(ab) - f_0(a)b - af_0(b) + d_2 f_0(a \otimes b))t^n + \dots \end{aligned}$$

where we've omitted the powers t^r of t with $r > n$. By the definition of d_2 , the coefficient of t^n in the above expression is zero. Therefore, the infinitesimal of μ' occurs at μ'_m with $m > n$. \square

Corollary 3.9. *If $HH^2(A) = 0$, then A is rigid.*

Let \mathfrak{g} be a finite dimensional semisimple Lie algebra, then $HH^2(\mathcal{U}(\mathfrak{g})) = 0$, where $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{g} and by Corollary (3.9), $\mathcal{U}(\mathfrak{g})$ is rigid. For a detailed proof of why $HH^2(\mathcal{U}(\mathfrak{g})) = 0$ refer to Exercise 2.8.1(c) in [20]. Another example of rigid algebras are **separable algebras**: when \mathbb{K} is a field, algebra A is said to be separable if it is a finite product of simple algebras whose centres are separable field extensions of \mathbb{K} . In Section 14 of [4], six equivalent definitions of separable algebras are presented, one being the absolute projectivity of A as an A -bimodule, which implies $HH^n(A) = 0$ for $n > 0$. Another equivalent definition requires the existence of an idempotent in A^e satisfying certain properties. In Example 2.6 of [18], the matrix ring $M_n(\mathbb{K})$ and group ring $\mathbb{K}G$, where the order of G is invertible in \mathbb{K} , are shown to be separable by finding this idempotent. The importance of separable algebras in deformation theory is due to the main result of section

14 of [4], which states that if S is a separable subalgebra of A , then for any deformation of A , we can find an equivalent deformation μ such that S is fixed, i.e for $s \in S$ and $a \in A$, $\mu(s, a) = sa$ and $\mu(a, s) = as$. As a consequence of this Theorem, if algebra A is *unital* and 1 is its unit, then $\mathbb{K} = \mathbb{K}1$ is a separable subalgebra of A , and for any deformation, we can find an equivalent deformation such that element 1 acts as a unit with respect to the new multiplication. The preservation of units by deformation was actually proved first in Theorem 17 of [7] and again implied by the relative preservation of general idempotents described in section 20 of [9]. Other results not covered here are Gerstenhaber and Schack's work in [9] and [4] on diagrams of algebras and their deformations, where the mentioned result on separable subalgebras translates to the deformation of monomorphism $S \hookrightarrow A$. Furthermore, another result described in Theorem 17 of [7] is the preservation of invertability of elements by deformations over $\mathbb{K}((t))$ and that deformations of *division rings* are themselves division rings. Further works of Gerstenhaber on division rings can be found in [7] and [8].

Remark 3.10. *Although $\mathcal{U}(\mathfrak{g})$ is rigid as an algebra, it is also enriched with a coalgebra structure which can have non-trivial deformations¹. Descriptions of the coalgebra structure of $\mathcal{U}(\mathfrak{g})$ can be found in [12]. Specifically, section XVI.5 describes Quantum Enveloping Algebras which are deformations of $\mathcal{U}(\mathfrak{g})$ as a bialgebra. By the above, these deformations will be trivial with respect to its algebra structure.*

4 Circle Product and Gerstenhaber Bracket

In [5], Gerstenhaber defines a Graded Lie structure on $C_*(A) := \bigoplus_{n \geq 0} \text{Hom}_{\mathbb{K}}(A^{\otimes n}, A)$, by what is now commonly called the *Gerstenhaber Bracket*. First we need to introduce the circle product as defined in the same paper².

Definition 4.1. *For $f \in \text{Hom}_{\mathbb{K}}(A^{\otimes m}, A)$ and $g \in \text{Hom}_{\mathbb{K}}(A^{\otimes n}, A)$, we define*

(I) *the circle product $f \circ g \in \text{Hom}_{\mathbb{K}}(A^{\otimes m+n-1}, A)$ by*

$$(f \circ g)(a_1 \otimes \cdots \otimes a_{m+n-1}) = \sum_{i=1}^m (-1)^{(n-1)(i-1)} f(a_1 \otimes \cdots \otimes g(a_i \otimes \cdots \otimes a_{i+n-1}) \otimes \cdots \otimes a_{m+n-1}) \quad (20)$$

(II) *and the cup product $f \smile g \in \text{Hom}_{\mathbb{K}}(A^{\otimes m+n}, A)$ by*

$$(f \smile g)(a_1 \otimes \cdots \otimes a_{m+n}) = f(a_1 \otimes \cdots \otimes a_m)g(a_{m+1} \otimes \cdots \otimes a_{m+n}) \quad (21)$$

Recall that in section (1), we defined maps $d_n : \text{Hom}_{\mathbb{K}}(A^{\otimes(n-1)}, A) \rightarrow \text{Hom}_{\mathbb{K}}(A^{\otimes n}, A)$ such that $d_{n+1}d_n = 0$. This construction corresponds to a map $d_* : C_*(A) \rightarrow C_*(A)$, where for a homogeneous element $a \in C_*(A) = \bigoplus_{n \geq 0} \text{Hom}_{\mathbb{K}}(A^{\otimes n}, A)$ with $|a| = n$, $d_*(a) = d_n(a)$ holds and d_* is extended linearly to all of $C_*(A)$.

Lemma 4.2. *For $f \in \text{Hom}_{\mathbb{K}}(A^{\otimes m}, A)$ and $g \in \text{Hom}_{\mathbb{K}}(A^{\otimes n}, A)$,*

$$d_*(f \circ g) = (-1)^{(n-1)}d_*f \circ g + f \circ d_*g + (-1)^{mn+n-1}f \smile g + (-1)^n g \smile f \quad (22)$$

holds.

¹Refer to Remark (4.5), where we briefly describe how analogous deformation theories can be defined for coalgebras and bialgebras.

²As Gerstenhaber himself remarks in [5], the paper was originally intended to be a part of his paper on deformation theory.

Proof. For $a_1 \otimes a_2 \otimes \cdots \otimes a_{m+n} \in A^{\otimes(m+n)}$, the LHS of (22) expands as bellow:

$$\begin{aligned} [d_*(f \circ g)](a_1 \otimes \cdots \otimes a_{m+n}) &= a_1(f \circ g)(a_2 \otimes \cdots \otimes a_{m+n}) \\ &\quad + \sum_{i=1}^{m+n-1} (-1)^i (f \circ g)(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{m+n}) \\ &\quad + (-1)^{m+n} (f \circ g)(a_1 \otimes \cdots \otimes a_{m+n-1}) a_{m+n} \end{aligned}$$

Since $(n-1)i = (n-1)(i-1) + (n-1)$, the RHS of (22) expands as bellow:

$$\begin{aligned} (-1)^{(n-1)} [d_* f \circ g](a_1 \otimes \cdots \otimes a_{m+n}) &= \\ &\sum_{i=1}^{m+1} (-1)^{(n-1)i} d_* f(a_1 \otimes \cdots \otimes g(a_i \otimes \cdots \otimes a_{i+n-1}) \otimes \cdots \otimes a_{m+n}) \\ &= \sum_{i=2}^{m+1} (-1)^{(n-1)i} a_1 f(a_2 \otimes \cdots \otimes g(a_i \otimes \cdots \otimes a_{i+n-1}) \otimes \cdots \otimes a_{m+n}) \\ &\quad + (-1)^{n-1} g(a_1 \otimes \cdots \otimes a_n) f(a_{n+1} \otimes \cdots \otimes a_{m+n}) \quad [= -(-1)^n g \smile f] \\ &\quad + \sum_{i=1}^m (-1)^{(n-1)i+m+1} f(a_1 \otimes \cdots \otimes g(a_i \otimes \cdots \otimes a_{i+n-1}) \otimes \cdots \otimes a_{m+n-1}) a_{m+n} \\ &\quad + (-1)^{(n-1)(m+1)+m+1} f(a_1 \otimes \cdots \otimes a_m) g(a_{m+1} \otimes \cdots \otimes a_{m+n}) \quad [= -(-1)^{mn+n-1} f \smile g] \\ &\quad + \sum_{i=2}^{m+1} (-1)^{(n-1)i+i-1} f(a_1 \otimes \cdots \otimes a_{i-1} g(a_i \otimes \cdots \otimes a_{i+n-1}) \otimes \cdots \otimes a_{m+n}) \\ &\quad + \sum_{i=1}^m (-1)^{(n-1)i+i} f(a_1 \otimes \cdots \otimes g(a_i \otimes \cdots \otimes a_{i+n-1}) a_{i+n} \otimes \cdots \otimes a_{m+n}) \\ &\quad + \sum_{i=1}^{m+1} \sum_{\substack{j < i-1 \\ \text{or } j > i+n-1}} (-1)^{(n-1)i+j} f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes g(a_i \otimes \cdots \otimes a_{i+n-1}) \otimes \cdots \otimes a_{m+n}) \\ [f \circ d_* g](a_1 \otimes \cdots \otimes a_{m+n}) &= \sum_{i=1}^m (-1)^{n(i-1)} f(a_1 \otimes \cdots \otimes d_* g(a_i \otimes \cdots \otimes a_{i+n}) \otimes \cdots \otimes a_{m+n}) \\ &= \sum_{i=1}^m (-1)^{n(i-1)} f(a_1 \otimes \cdots \otimes a_i g(a_{i+1} \otimes \cdots \otimes a_{i+n}) \otimes \cdots \otimes a_{m+n}) \\ &\quad + \sum_{i=1}^m (-1)^{n(i-1)+(n+1)} f(a_1 \otimes \cdots \otimes g(a_{i+1} \otimes \cdots \otimes a_{i+n-1}) a_{i+n} \otimes \cdots \otimes a_{m+n}) \\ &\quad + \sum_{i=1}^m \sum_{j=1}^n (-1)^{n(i-1)+j} f(a_1 \otimes \cdots \otimes g(a_i \otimes \cdots \otimes a_{i+j-1} a_{i+j} \otimes \cdots \otimes a_{i+n}) \otimes \cdots \otimes a_{m+n}) \end{aligned}$$

Hence, we see that terms of form $f(\cdots a_i g(\cdots))$ and $f(\cdots g(\cdots) a_{i+n} \cdots)$ in the RHS cancel out and by the definition of the circle product, after the addition of $(-1)^{mn+n-1} f \smile g + (-1)^n g \smile f$, the remaining terms on the RHS are equal to the expansion of the LHS. \square

Now we touch on the important relation between associativity and the circle product. Let $f, g \in \text{Hom}_{\mathbb{K}}(A^{\otimes 2}, A)$, then

$$f \circ g(a \otimes b \otimes c) = f(g(a \otimes b) \otimes c) - f(a \otimes g(b \otimes c)) \quad (23)$$

Hence, for $\mu_i \in \text{Hom}_{\mathbb{K}}(A^{\otimes 2}, A)$, where $0 \leq i \leq n-1$,

$$\sum_{i=1}^{n-1} \mu_i(\mu_{n-i}(a \otimes b) \otimes c) - \sum_{i=1}^{n-1} \mu_i(a \otimes \mu_{n-i}(b \otimes c)) = \sum_{i=1}^{n-1} \mu_i \circ \mu_{n-i}(a \otimes b \otimes c) \quad (*)$$

Recall that the LHS of the equation above is the $(n-1)$ -th obstruction for a truncated obstruction. This identity will help us show these obstructions are 3-cocycles:

Theorem 4.3. *If (A, μ) with $\mu = \sum_{i \geq 0}^{n-1} \mu_i t^i$ is a truncated deformation, as defined in (2.8), then the $(n-1)$ th obstruction*

$$\mathfrak{D}_{n-1}(a \otimes b \otimes c) := \sum_{i=1}^{n-1} \mu_i(\mu_{n-i}(a \otimes b) \otimes c) - \sum_{i=1}^{n-1} \mu_i(a \otimes \mu_{n-i}(b \otimes c))$$

is a Hochschild 3-cocycle, i.e. $d_4 \mathfrak{D}_{n-1} = 0$.

Proof. Since (A, μ) is a truncated deformation, then $(\star_k : 1 \leq k \leq n-1)$ hold and

$$\sum_{i=1}^{k-1} \mu_i(\mu_{k-i}(a \otimes b) \otimes c) - \sum_{i=1}^{k-1} \mu_i(a \otimes \mu_{k-i}(b \otimes c)) = d_3 \mu_k(a \otimes b \otimes c)$$

for $1 \leq k \leq n-1$. Then by Lemma (22)

$$\begin{aligned} d_* \left(\sum_{i=1}^{n-1} \mu_i \circ \mu_{n-i} \right) &= \sum_{i=1}^{n-1} \left((-1)^{2-1} d_* \mu_i \circ \mu_{n-i} + \mu_i \circ d_* \mu_{n-i} \right) \\ &\quad + \sum_{i=1}^{n-1} \left((-1)^{4+2-1} \mu_i \smile \mu_{n-i} + (-1)^2 \mu_{n-i} \smile \mu_i \right) \\ &= - \sum_{i=1}^{n-1} \sum_{j=1}^{i-1} \mu_j \circ \mu_{i-j} \circ \mu_{n-i} + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i-1} \mu_i \circ \mu_j \circ \mu_{n-i-j} \\ &\quad - \sum_{i=1}^{n-1} \mu_i \smile \mu_{n-i} + \sum_{i=1}^{n-1} \mu_{n-i} \smile \mu_i \\ &= - \sum_{\substack{i+j+k=n \\ i,j,k \geq 1}} \mu_i \circ \mu_j \circ \mu_k + \sum_{\substack{i+j+k=n \\ i,j,k \geq 1}} \mu_i \circ \mu_j \circ \mu_k + 0 = 0 \end{aligned}$$

By equation (*), $\mathfrak{D}_{n-1} = \sum_{i=1}^{k-1} \mu_i \circ \mu_{k-i}$. Thereby, $d_* \mathfrak{D}_{n-1} = 0$ and the $(n-1)$ -th obstruction is a Hochschild 3-cocycle. \square

Corollary 4.4. *For algebra A , if $HH^3(A) = 0$ then any Hochschild 2-cocycle is integrable.*

Proof. Since by the above Theorem, any obstruction to extending a truncated deformation with the 2-cocycle as its infinitesimal, is a 3-cocycle and if $HH^3(A) = 0$, the obstruction must be 3-coboundary, which is what equation (\star_n) required. \square

Remark 4.5. *In Gerstenhaber's papers on deformations of associative algebras, he comments that a similar theory can be applied to Lie algebras. In section 1 of [6] he demonstrates that the infinitesimal of a Lie algebra deformation must be a 2-cocycle in the respective cohomology theory. Indeed, in [10] and [11], T. Fox describes a deformation theory on \mathbb{T} -algebras for any Monad/Triple (\mathbb{T}, η, ν) on the category of \mathbb{K} -vectorspaces, using Triple cohomology. Furthermore, in section 8 of [11], even the circle product is described for a general triple. Hence, any result up to now holds for general triples on \mathbb{K} -vectorspaces with their respective cohomology theories.*

Example 4.6. We now have the sufficient tools to look at a more sophisticated example; $A = \mathbb{K}[x, y]$. Over field \mathbb{K} , A has basis $\mathcal{B} = \{x^\alpha y^\beta | \alpha, \beta \geq 0\}$. We can define the bilinear map $\psi : A \otimes A \rightarrow A$ on the basis via

$$\psi(a \otimes b) = \left(\frac{\partial a}{\partial x} \right) \left(\frac{\partial b}{\partial y} \right) \quad (24)$$

where $a, b \in \mathcal{B}$ and $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ are partial derivations in the usual sense. We can easily check that $d_3\psi = 0$ on the basis and thereby on all of A : If $a, b, c \in \mathcal{B}$

$$\begin{aligned} d_3\psi(a \otimes b \otimes c) &= a\psi(b \otimes c) - \psi(ab \otimes c) + \psi(a \otimes bc) - \psi(a \otimes b)c \\ &= a \frac{\partial b}{\partial x} \frac{\partial c}{\partial y} - \frac{\partial ab}{\partial x} \frac{\partial c}{\partial y} + \frac{\partial a}{\partial x} \frac{\partial bc}{\partial y} - \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} c \\ &= a \frac{\partial b}{\partial x} \frac{\partial c}{\partial y} - a \frac{\partial b}{\partial x} \frac{\partial c}{\partial y} - b \frac{\partial a}{\partial x} \frac{\partial c}{\partial y} + \frac{\partial a}{\partial x} \frac{\partial c}{\partial y} b + \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} c - \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} c = 0 \end{aligned}$$

Hence, $\psi \in \text{Ker}(d_3)$. Moreover, ψ is integrable and can occur as an infinitesimal. This is because $HH^3(A) = 0$: In fact for a polynomial algebra $\mathbb{K}[x_1, x_2, \dots, x_m]$,

$$HH^n(\mathbb{K}[x_1, x_2, \dots, x_m]) = \mathbb{K}[x_1, x_2, \dots, x_m] \otimes \bigwedge^n (\text{Span}_{\mathbb{K}}\{y_1, y_2, \dots, y_m\}) \quad (25)$$

where $\bigwedge^n V$ denotes the n th exterior power of $V = \text{Span}_{\mathbb{K}}\{y_1, y_2, \dots, y_m\}$. Therefore, when $m = 2$ variables are at work, the 3rd exterior power $\bigwedge^3 (\text{Span}_{\mathbb{K}}\{y_1, y_2\})$ has a basis of elements $y_{i_1} \wedge y_{i_2} \wedge y_{i_3}$, where $i_j \in \{1, 2\}$ and at least two coincide, giving $y_{i_1} \wedge y_{i_2} \wedge y_{i_3} = 0$. Consequently, $\bigwedge^3 (\text{Span}_{\mathbb{K}}\{y_1, y_2\}) = 0$ and $HH^3(A) = 0$. Result (25) is a specific case of the celebrated Hochschild-Kostant-Rosenberg Theorem [Theorem 9.4.7 [21]] and a detailed proof of the presented version can be found in Example 2.1.3 of [22].

Since $HH^3(A) = 0$, by Corollary (4.4), any 2-cocycle is integrable and thereby the 1st obstruction to extending $\mu = \pi + \psi t$ vanishes i.e. there exists a bilinear map $\mu_2 \in \text{Hom}_{\mathbb{K}}(A^{\otimes 2}, A)$ which satisfies

$$\begin{aligned} d_3\mu_2(a \otimes b \otimes c) &= \mathfrak{D}_1(a \otimes b \otimes c) = \psi \circ \psi(a \otimes b \otimes c) \\ &= \psi(\psi(a \otimes b) \otimes c) - \psi(a \otimes \psi(b \otimes c)) = \psi \left(\frac{\partial a}{\partial x} \frac{\partial b}{\partial y} \otimes c \right) - \psi \left(a \otimes \frac{\partial b}{\partial x} \frac{\partial c}{\partial y} \right) \\ &= \frac{\partial^2 a}{\partial x^2} \frac{\partial b}{\partial y} \frac{\partial c}{\partial y} + \frac{\partial a}{\partial x} \frac{\partial^2 b}{\partial x \partial y} \frac{\partial c}{\partial y} - \frac{\partial a}{\partial x} \frac{\partial^2 b}{\partial y \partial x} \frac{\partial c}{\partial y} - \frac{\partial a}{\partial x} \frac{\partial b}{\partial x} \frac{\partial^2 c}{\partial y^2} \\ &= \frac{\partial^2 a}{\partial x^2} \frac{\partial b}{\partial y} \frac{\partial c}{\partial y} - \frac{\partial a}{\partial x} \frac{\partial b}{\partial x} \frac{\partial^2 c}{\partial y^2} \end{aligned}$$

Observe that unlike Example (2.9) where A was $\mathbb{K}[\epsilon]/(\epsilon^2)$ and the obstruction to extending $\pi + \rho_\alpha$ was zero and trivially solvable, here the existence of μ_2 satisfying the above equation is non-trivial. On the other hand, similarly to the case of ρ_α , all non-vanishing 2-cocycles are of form ψ_κ where $\kappa \in A = \mathbb{K}[x, y]$ and

$$\psi_\kappa(a \otimes b) = \kappa \left(\frac{\partial a}{\partial x} \right) \left(\frac{\partial b}{\partial y} \right) \quad (26)$$

One can show that for polynomial algebras, all 2-coboundaries are those 2-cocycles which are symmetric. Starting with any symmetric bilinear map in $\text{Ker}(d_3)$, one can construct a map in $\text{Hom}_{\mathbb{K}}(A, A)$ so that its image under d_2 is the bilinear map. For the details of this method refer to Theorem 3.1 in [16]. Furthermore, observe that when $\kappa \neq 0$, ψ_κ is not symmetric:

$$\psi_\kappa(x \otimes y) = \kappa \neq 0 = \psi_\kappa(y \otimes x)$$

So $\psi_\kappa \notin \text{Im}(d_2)$ and additionally since $\psi_{\kappa_1} - \psi_{\kappa_2} = \psi_{\kappa_1 - \kappa_2}$, then $\psi_\kappa + \text{Im}(d_2)$ represent distinct elements of $HH^2(A)$. Further, since $\bigwedge^2(\text{Span}_{\mathbb{K}}\{y_1, y_2\}) = \text{Span}_{\mathbb{K}}\{y_1 \wedge y_2\} \cong \mathbb{K}$, then by (25), $HH^2(A) \cong \mathbb{K}[x, y]$. Hence,

$$HH^2(\mathbb{K}[x, y]) = \{\psi_\kappa + \text{Im}(d_2) | \kappa \in \mathbb{K}[x, y]\} \cong \mathbb{K}[x, y]$$

Therefore, for each $\kappa \in \mathbb{K}[x, y]$, we have a non-trivial formal deformation $\mu_\kappa = \pi + \sum_{i \geq 1} \mu_{(\kappa, i)} t^i$ such that $\mu_{\kappa, 1} = \psi_\kappa$. Furthermore, by Proposition (3.4) these deformations are pairwise non-equivalent since $\psi_{\kappa_1} - \psi_{\kappa_2} = \psi_{\kappa_1 - \kappa_2} \notin \text{Im}(d_2)$ if $\kappa_1 \neq \kappa_2$. It is essential to notice that since ψ_κ is not symmetric when $\kappa \neq 0$, then algebra A_{μ_κ} is a non-commutative. Deformations of this type feed into the subject of Non-commutative Geometry: it is well known that $\mathbb{K}[x, y]$ can be viewed as the algebra of polynomial functions on the affine 2-space \mathbb{K}^2 . With this idea in mind, one can assume that the deformed algebra must be the algebra of functions on a non-commutative geometry. More detail on this philosophy can be found in [20]. In order for the geometric aspects of the algebra to be preserved after deformation, more constraints are needed. In fact, the example of ψ as presented here arises naturally when looking at the Poisson structure on $\mathbb{K}[x, y]$. We will discuss this side of the theory briefly in the last section.

In [5], Gerstenhaber shows that a *pre-Lie algebra*, a vectorspace with a circle product satisfying certain properties, give rise to a Lie bracket and Lie structure on the vectorspace as follows:

Definition 4.7. For $f \in \text{Hom}_{\mathbb{K}}(A^{\otimes m}, A)$ and $g \in \text{Hom}_{\mathbb{K}}(A^{\otimes n}, A)$, we define the **Gerstenhaber bracket** $[f, g] \in \text{Hom}_{\mathbb{K}}(A^{\otimes m+n-1}, A)$ as

$$[f, g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f \quad (27)$$

Lemma 4.8. For $f \in \text{Hom}_{\mathbb{K}}(A^{\otimes m}, A)$, $g \in \text{Hom}_{\mathbb{K}}(A^{\otimes n}, A)$ and $h \in \text{Hom}_{\mathbb{K}}(A^{\otimes p}, A)$, the following hold:

- (I) $[f, g] = -(-1)^{(m-1)(n-1)} [g, f]$
- (II) $(-1)^{(m-1)(p-1)} [f, [g, h]] + (-1)^{(n-1)(m-1)} [g, [h, f]] + (-1)^{(p-1)(n-1)} [h, [f, g]] = 0$
- (III) $d_*[f, g] = (-1)^{(n-1)} [d_*f, g] + [f, d_*g]$

Proof. (I) clearly follows from the definition of the bracket. (II) follows from

$$\begin{aligned} [f, [g, h]] &= f \circ [g, h] - (-1)^{(m-1)(n+p-2)} [g, h] \circ f \\ &= f \circ g \circ h - (-1)^{(n-1)(p-1)} f \circ h \circ g - (-1)^{(m-1)(n+p-2)} g \circ h \circ f \\ &\quad + (-1)^{(n-1)(p-1)+(m-1)(n+p-2)} h \circ g \circ f \end{aligned}$$

So when multiplied by its coefficient in (II):

$$\begin{aligned} (-1)^{(m-1)(p-1)} [f, [g, h]] &= (-1)^{(m-1)(p-1)} f \circ g \circ h - (-1)^{(p-1)(m-1+n-1)} f \circ h \circ g \\ &\quad - (-1)^{(m-1)(n+p-2+p-1)} g \circ h \circ f \\ &\quad + (-1)^{(n-1)(p-1)+(m-1)(n+p-2+p-1)} h \circ g \circ f \\ \Rightarrow (-1)^{(m-1)(p-1)} [f, [g, h]] &= (-1)^{(m-1)(p-1)} f \circ g \circ h - (-1)^{(p-1)(m+n-2)} f \circ h \circ g \quad (\star) \\ &\quad - (-1)^{(m-1)(n-1)} g \circ h \circ f + (-1)^{(n-1)(p+m-2)} h \circ g \circ f \end{aligned}$$

Observe that the first and third terms in (\star) are cyclic permutations of f, g and h and the second and fourth terms in (\star) are the non-cyclic permutations. Hence, since the powers of -1 in the coefficients

respect the permutations, the terms on the LHS of (II) will all cancel out. (III) follows from identity (22) in Lemma (4.2):

$$\begin{aligned}
d_*[f, g] &= d_*\left(f \circ g - (-1)^{(m-1)(n-1)}g \circ f\right) \\
&= (-1)^{(n-1)}d_*f \circ g + f \circ d_*g + (-1)^{mn+n-1}f \smile g + (-1)^ng \smile f \\
&\quad - (-1)^{(m-1)(n-1)}\left((-1)^{(m-1)}d_*g \circ f + g \circ d_*f + (-1)^{mn+m-1}g \smile f + (-1)^mf \smile g\right) \\
&= (-1)^{(n-1)}\left(d_*f \circ g - (-1)^{(n-1)m}g \circ d_*f\right) + \left(f \circ d_*g - (-1)^{(m-1)n}d_*g \circ f\right) \\
&\quad + \left((-1)^{mn+n-1} - (-1)^{(mn-n-m+1)+m}\right)f \smile g \\
&\quad + \left((-1)^n - (-1)^{(mn-n-m+1)+mn+m-1}\right)g \smile f \\
&= (-1)^{(n-1)}[d_*f, g] + [f, d_*g] + 0 + 0 = (-1)^{(n-1)}[d_*f, g] + [f, d_*g]
\end{aligned}$$

□

Similar to d_* , since the Gerstenhaber bracket is defined on the homogeneous elements of $C_*(A) = \bigoplus_{n \geq 0} \text{Hom}_{\mathbb{K}}(A^{\otimes n}, A)$, it can be linearly extended to act on all of $C_*(A)$. Furthermore, the Gerstenhaber bracket is well-defined on Hochschild cohomology. First, observe that by Lemma (4.8) (III), for f and g as in the Lemma, we have

$$\begin{aligned}
d_*[d_*f, g] &= (-1)^{n-1}[0, g] + [d_*f, d_*g] = [d_*f, d_*g] \\
&= (-1)^n((-1)^n[d_*f, d_*g] + 0) = (-1)^nd_*[f, d_*g]
\end{aligned} \tag{\Theta}$$

Theorem 4.9. *The Gerstenhaber Bracket is well-defined on $HH^*(A) = \bigoplus_{n \geq 0} HH^n(A)$.*

Proof. (I) Let $f \in \text{Ker}(d_{m+1})$ and $g \in \text{Ker}(d_{n+1})$, then

$$d_*[f, g] = [d_*f, g] + [f, d_*g] = 0 + 0 = 0$$

So $[f, g] = 0 \in \text{Ker}(d_{n+m+1})$. Furthermore, if $d_*\alpha \in \text{Im}(d_m)$ and $d_*\beta \in \text{Im}(d_n)$, then by Lemma (4.8) (III) and (Θ),

$$\begin{aligned}
[f + d_*\alpha, g + d_*\beta] &= [f, g] + [d_*\alpha, g] + [f, d_*\beta] + [d_*\alpha, d_*\beta] \\
&= [f, g] + (-1)^{n-1}(d_*[\alpha, g] - [\alpha, d_*g]) + (d_*[f, \beta] - (-1)^{n-2}[d_*f, \beta]) \\
&\quad + d_*[d_*\alpha, \beta]
\end{aligned}$$

and since $f, g \in \text{Ker}(d_*)$, then

$$\begin{aligned}
[f + d_*\alpha, g + d_*\beta] &= [f, g] + (-1)^{n-1}d_*[\alpha, g] + d_*[f, \beta] + d_*[\alpha, d_*\beta] \\
&= [f, g] + d_*\left((-1)^{n-1}[\alpha, g] + [f, \beta] + [d_*\alpha, \beta]\right)
\end{aligned}$$

So $[-, -]$ is well-defined on $HH^*(A)$. □

Remark 4.10. *We have only used the cup product here to ease the notation in our proofs; however, the Cup product \smile has several important properties on its own. In fact, $HH^*(A)$ has a graded commutative ring structure with respect to the cup product³, which was one of the major results in Gerstenhaber's paper [5]. Due to this rich structure and the connection of the cup product and the bracket, which were explored by Gerstenhaber in [5], an algebra with a cup product and a bracket, satisfying similar conditions, is often called a Gerstenhaber algebra or G-algebra. For more detail, refer to Definition 1.4.6 of [22].*

³With a shift of 1 compared to the grading of $HH^*(A)$ as a DGLA as introduced in Theorems (4.12) and (4.14)

In fact both $C_*(A)$ and $HH^*(A)$ are enriched with the structure of graded Lie algebras via the Gerstenhaber bracket:

Definition 4.11. We say a graded \mathbb{K} -vector space $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ is a **graded Lie algebra** if there exists a bilinear $[-, -] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ which satisfies $[\mathfrak{g}_m, \mathfrak{g}_n] \subset \mathfrak{g}_{m+n}$ and

$$(I) \quad [a, b] = -(-1)^{|a||b|}[b, a] \text{ (Anti-commutativity)}$$

$$(II) \quad (-1)^{|a||c|}[a, [b, c]] + (-1)^{|b||a|}[b, [c, a]] + (-1)^{|c||b|}[c, [a, b]] = 0 \text{ (Graded Jacobi identity)}$$

for homogeneous elements $a, b, c \in \mathfrak{g}$. We say a \mathbb{K} -vector space \mathfrak{g} is **differential graded Lie algebra** or **DGLA**, if in addition to having a graded Lie algebra structure, we have a map $d : \mathfrak{g} \rightarrow \mathfrak{g}$ such that d is a differential i.e. $d^2 = 0$ and for homogeneous element a , da is homogeneous with $|da| = |a| + 1$. Further the differential and the bracket interact by the graded Leibniz rule:

$$d[a, b] = [da, b] + (-1)^{|a|}[a, db] \quad (28)$$

Notation. For a graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$, we denote the **shift** of V by \mathbf{i} , by $V^{+i} = \bigoplus_{n \in \mathbb{Z}} V_n^{+i}$, where $V_n^{+i} = V_{n+i}$. Also observe any $\mathbb{Z}_{\geq 0}$ -graded vector space $\bigoplus_{n \geq 0} V_n$ is \mathbb{Z} -graded with $V_n = 0$ for $n < 0$.

Theorem 4.12. Both $C_*(A)^{+1}$ and $HH^*(A)^{+1}$ form graded Lie algebras with respect to the Gerstenhaber bracket.

Notice that although identity (III) from Lemma (4.8) resembles the graded Leibniz rule, it does not agree with it completely. This can be fixed by taking the differential map $\delta_* : C_*(A) \rightarrow C_*(A)$ defined on homogeneous elements $f \in \text{Hom}_{\mathbb{K}}(A^{\otimes m}, A)$ by $\delta_*(f) = (-1)^{m-1}d_*(f)$, and extended to $C_*(A)$. Clearly, since $d_*^2 = 0$ and d_* is well-defined on the Hochschild complex $HH^*(A)$, then δ_* is a differential and is well-defined on $HH^*(A)$ as well.

Lemma 4.13. For $f \in \text{Hom}_{\mathbb{K}}(A^{\otimes m}, A)$, $g \in \text{Hom}_{\mathbb{K}}(A^{\otimes n}, A)$,

$$\delta_*[f, g] = [\delta_*f, g] + (-1)^{m-1}[f, \delta_*g] \quad (29)$$

holds.

Proof. This follows from (III) of Lemma (4.8):

$$\begin{aligned} \delta_*[f, g] &= (-1)^{(m+n-1)-1}d_*[f, g] = (-1)^{m+n} \left((-1)^{(n-1)}[d_*f, g] + [f, d_*g] \right) \\ &= (-1)^{(m-1)}[d_*f, g] + (-1)^{m-1+n-1}[f, d_*g] \\ &= [(-1)^{(m-1)}d_*f, g] + (-1)^{m-1}[f, (-1)^{n-1}d_*g] = [\delta_*f, g] + (-1)^{m-1}[f, \delta_*g] \end{aligned}$$

□

Theorem 4.14. Both $C_*(A)^{+1}$ and $HH^*(A)^{+1}$ have DGLA structures with respect to the Gerstenhaber bracket and the differential δ_* .

It is worth mentioning that maps δ_* and d_* , when restricted to $HH^*(A)^{+1}$, are just the zero map. The zero map when looked at as a differential is often referred to as the *zero differential*.

Remark 4.15. Many modern texts on deformation theory focused on DGLAs are often unclear about the choice of δ_* . Since in the literature, it is a known fact that complexes $C_*(A)^{+1}$ and $HH^*(A)^{+1}$ form DGLAs with the Gerstenhaber bracket, not much attention is given to this choice of the differential. In [20] and [14], when mentioning the "Hochschild DGLA", the differential is taken falsely

as d_* . However, [3] chooses to introduce the Hochschild differential map as δ_* instead of d_* . The text [15] was the only text the author found, that this change was explicitly stated. The importance of this choice, in addition to not agreeing with the common form of the Leibniz rule, is its effect on the Maurer-Cartan equation which we will introduce in the next section. If we take d_* as the differential, then the format of the Maurer-Cartan changes by a negative sign as indicated in equation (4.3.1) in [22]. We have chosen to introduce δ_* at this point to both respect the older theory and notation set by Gerstenhaber and Hochschild, and connect it with the modern theory.

A statement that is sometimes mentioned, is that "the Gerstenhaber bracket enriches $C_*(A)^{+1}$ and $HH^*(A)^{+1}$ with DGLA structures". This is because the differentials d_* and δ_* can be defined in terms of the bracket:

Lemma 4.16. For $f \in \text{Hom}_{\mathbb{K}}(A^{\otimes m}, A)$, $[f, \pi] = -d_*f$.

Proof. By the Definition of the bracket $[f, \pi] \in \text{Hom}_{\mathbb{K}}(A^{\otimes m+2-1}, A)$ and

$$\begin{aligned} [f, \pi](a_1 \otimes \cdots \otimes a_{m+1}) &= (f \circ \pi - (-1)^{(m-1)(2-1)} \pi \circ f)(a_1 \otimes \cdots \otimes a_{m+1}) \\ &= \sum_{i=1}^m (-1)^{(i-1)(2-1)} f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{m+1}) \\ &\quad - (-1)^{m-1} (f(a_1 \otimes \cdots \otimes a_m) a_{m+1} + (-1)^{(m-1)} a_1 f(a_2 \otimes \cdots \otimes a_{m+1})) \\ &= -d_*(f) \end{aligned}$$

□

Hence, we can write operation d_* as $(-1)[- , \pi]$. On the other hand, since $\delta_*(f) = (-1)^{m-1} d_*(f)$ for $f \in \text{Hom}_{\mathbb{K}}(A^{\otimes m}, A)$, then by Lemma (4.8) (I),

$$\delta_*(f) = (-1)^{(m-1)} (-[f, \pi]) = -(-1)^{(m-1)(2-1)} [f, \pi] = [\pi, f] \quad (30)$$

Which gives us a re-interpretation of δ_* as $[\pi, -]$. This identity is the last ingredient we need to describe deformations in terms of DGLAs.

5 DGLA Philosophy

By Theorem (4.14), $C_*(A)^{+1} = \bigoplus_{n \geq -1} \text{Hom}_{\mathbb{K}}(A^{\otimes n+1}, A)$ has a DGLA structure with the Gerstenhaber bracket and differential map δ_* . Operations $[-, -]$ and d_* can both be t -linearly extended to $C_*(A)^{+1}[[t]] = \bigoplus_{n \geq -1} \text{Hom}_{\mathbb{K}}(A^{\otimes n+1}, A)[[t]]$ and induce a DGLA structure on $C_*(A)^{+1}[[t]]$. Let μ be a one-parameter deformation of algebra A , thereby

$$\mu = \mu_0 + \mu_1 t + \mu_2 t^2 + \cdots = \sum_{i \geq 0} \mu_i t^i$$

where $\mu_i \in \text{Hom}_{\mathbb{K}}(A^{\otimes 2}, A)$. Hence $\mu \in \text{Hom}_{\mathbb{K}}(A^{\otimes 2}, A)[[t]]$.

Recall the relation between the circle product and associativity from (23). Since we have linearly extended the Gerstenhaber bracket (and the circle product), then

$$\mu(\mu(a \otimes b) \otimes c) - \mu(a \otimes \mu(b \otimes c)) = \mu \circ \mu$$

Suppose that $\text{Char}(\mathbb{K}) \neq 2$. Then

$$[\mu, \mu] = \mu \circ \mu - (-1)^{2-1} \mu \circ \mu = 2\mu \circ \mu$$

and thereby:

Proposition 5.1. *If $\text{Char}(\mathbb{K}) \neq 2$. Then $\mu \in \text{Hom}_{\mathbb{K}}(A^{\otimes n}, A)[[t]]$ defines a formal deformation if and only if $[\mu, \mu] = 0$.*

Recall that if μ is a deformation, then $\mu_0 = \pi$ where $\pi \in \text{Hom}_{\mathbb{K}}(A^{\otimes 2}, A)$ is the multiplication of algebra A . So $\mu = \pi + \mu'$ where $\mu' \in t\text{Hom}_{\mathbb{K}}(A^{\otimes 2}, A)[[t]]$.

$$[\mu, \mu] = [\pi + \mu', \pi + \mu'] = [\pi, \pi] + [\pi, \mu'] + [\mu', \pi] + [\mu', \mu']$$

Since π is associative, then $[\pi, \pi] = 0$. Further by Lemma (4.8) (I), $[\mu', \pi] = -(-1)^{(2-1)(2-1)}[\pi, \mu']$. Hence

$$[\mu, \mu] = 2[\pi, \mu'] + [\mu', \mu']$$

By Lemma (4.16) and the resulting identity (30), $[\pi, f] = \delta_*(f)$. Hence,

$$[\mu, \mu] = 2\delta_*(f) + [\mu', \mu']$$

Corollary 5.2. *If $\text{Char}(\mathbb{K}) \neq 2$. Then $\mu = \pi + \mu'$ with $\mu' \in t\text{Hom}_{\mathbb{K}}(A^{\otimes n}, A)[[t]]$ defines a formal deformation if and only if*

$$2\delta_*(f) + [\mu', \mu'] = 0 \quad (\mathcal{MC})$$

holds.

Equation (\mathcal{MC}) is called the **Maurer-Cartan equation**.

Since every deformation $\mu = \pi + \mu'$ is determined uniquely by μ' (not upto isomorphism), we have the bijection

$$\{\text{Formal deformations of } A, \mu\} \longleftrightarrow \{\mu' \in t\text{Hom}_{\mathbb{K}}(A^{\otimes n}, A)[[t]] \text{ satisfying } (\mathcal{MC})\} \quad (31)$$

By what we mentioned at the start of the section and Theorem (4.14), $C_*(A)^{+1}[[t]]$ has a DGLA structure and so does $tC_*(A)^{+1}[[t]]$

Definition 5.3. *For a DGLA $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$, we say $\alpha \in \mathfrak{g}_1$ is a **Maurer-Cartan element** if it satisfies the Maurer-Cartan Equation with the respective bracket and differential of the DGLA. The set of Maurer-Cartan elements of \mathfrak{g} is denoted by $\text{MCE}(\mathfrak{g})$*

In our case, $\mathfrak{g} = tC_*(A)^{+1}[[t]]$, and $\text{MCE}(\mathfrak{g}) \subset t\text{Hom}_{\mathbb{K}}(A^{\otimes 2}, A)[[t]]$. So we can rewrite the bijection (31) as

$$\{\text{Formal deformations of } A, \mu\} \longleftrightarrow \text{MCE}(tC_*(A)^{+1}[[t]]) \quad (32)$$

Notation. From now on we will denote DGLA $tC_*(A)^{+1}[[t]]$ by \mathfrak{C} .

Remark 5.4. *If a homogeneous element α of a DGLA $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ were to satisfy the Maurer-Cartan equation, then $\delta\alpha$ and $[\alpha, \alpha]$ must have the same degree:*

$$|\delta\alpha| = |\alpha| + 1 = 2|\alpha| = |[\alpha, \alpha]| \Rightarrow |\alpha| = 1$$

Since we are interested in deformations up to equivalence, we look to describe the equivalence of deformations in terms of DGLAs. Recall from (3.1) that an equivalence of deformations, ϕ_t , is of the form

$$\phi_t = Id_A + \phi_1 t + \phi_2 t^2 + \dots$$

where $\phi_i \in \text{Hom}_{\mathbb{K}}(A, A)$.

Assume $\text{Char}(\mathbb{K}) = 0$.

Since over characteristic 0, \log and \exp are well defined. Then let $\varphi_t = \log(\phi_t) \in \text{Hom}_{\mathbb{K}}(A, A)[[t]]$,

$$Id_A + \phi_1 t + \phi_2 t^2 + \dots = \phi_t = \exp(\varphi_t) = Id_A + \varphi_t + \frac{\varphi_t^2}{2} + \dots$$

Also recall the expansion of the logarithm function

$$\log(1 + a) = a - \frac{a^2}{2} + \frac{a^3}{3} + \dots$$

Which implies φ_t belongs to $tHom_{\mathbb{K}}(A, A)[[t]]$, since $a = \phi_t - Id_A \in tHom_{\mathbb{K}}(A, A)[[t]]$. So any equivalence of deformations ϕ_t can be written as $exp(\varphi_t)$ where $\varphi_t \in \mathfrak{C}_0 = tHom_{\mathbb{K}}(A, A)[[t]]$. Now, let $\varphi \in \mathfrak{C}_0 = tHom_{\mathbb{K}}(A, A)[[t]]$, and $\alpha' \in \text{MCE}(\mathfrak{C})$. Then by bijection (32), $\pi + \alpha'$ is a deformation and by (19)

$$\beta = exp(\varphi)(\pi + \alpha')(exp(-\varphi) \otimes exp(-\varphi))$$

is also a deformation, since $exp(-\varphi) = exp(\varphi)^{-1}$. By bijection (32), $\beta = \pi + \beta'$ where β' is a Maurer-Cartan element.

$$\begin{aligned} \pi + \beta' &= exp(\varphi)(\pi + \alpha')(exp(-\varphi) \otimes exp(-\varphi)) \\ \Rightarrow \quad exp(\varphi)(\pi + \alpha')(exp(-\varphi) \otimes exp(-\varphi)) - \pi &= \beta' \in \text{MCE}(\mathfrak{C}) \end{aligned}$$

Hence, elements of $exp(\mathfrak{C}_0) = \{exp(\varphi) | \varphi \in \mathfrak{C}_0\}$ act on $\text{MCE}(\mathfrak{C})$ via

$$exp(\varphi) \curvearrowright \alpha' = exp(\varphi)(\pi + \alpha')(exp(-\varphi) \otimes exp(-\varphi)) - \pi \in \text{MCE}(\mathfrak{C}) \quad (33)$$

Moreover, $exp(\mathfrak{C}_0)$ forms a group and this action respects that group structure: Observe that in any DGLA $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$, \mathfrak{g}_0 and $[-, -]$ form a Lie algebra, since the bracket takes elements of degree 0 to an element of degree 0, $[-, -] : \mathfrak{g}_0 \otimes \mathfrak{g}_0 \rightarrow \mathfrak{g}_0$.

Theorem 5.5. (Baker-Campbell-Hausdorff formula)[Theorem 2.1 [23]] *If \mathfrak{h} is a Lie algebra over a field \mathbb{K} with characteristic 0, and $\alpha, \beta \in \mathfrak{h}$, then $exp(\alpha)exp(\beta) = exp(\gamma)$ for γ , a formal infinite sum of elements in \mathfrak{h} ; in particular*

$$\gamma = \log(exp(\alpha)exp(\beta)) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{\substack{r_1+s_1>0 \\ \vdots \\ r_n+s_n>0}} \frac{[\alpha^{r_1}, \beta^{s_1}, \dots, \alpha^{r_n}, \beta^{s_n}]}{(\sum_{i=1}^n r_i + s_i) (\prod_{i=1}^n r_i s_i)} \quad (34)$$

where

$$[\alpha^{r_1}, \beta^{s_1}, \dots, \alpha^{r_n}, \beta^{s_n}] = \underbrace{[\alpha, [\alpha, [\dots, [\alpha, [\beta, [\beta, [\dots, [\beta, [a, [\dots]]]]]]]]]}_{r_1} \underbrace{]}_{s_1}$$

In the above Theorem, if γ is well-defined in \mathfrak{h} i.e. the infinite sum converges to an element of \mathfrak{h} , then $exp(\mathfrak{g}_0)$ forms a group with multiplication. In our case, it is important to notice that since $\alpha, \beta \in tHom_{\mathbb{K}}(A, A)[[t]]$, the power of t in $[\alpha^{r_1}, \beta^{s_1}, \dots, \beta^{s_n}]$ is at least $\sum_{i=1}^n r_i + s_i \geq n$. Hence, for any n , the coefficient of t^n is a finite sum, and $\log(exp(\alpha)exp(\beta))$ is well-defined and is an element of $exp(\mathfrak{C}_0) = tHom_{\mathbb{K}}(A, A)[[t]]$.

Now we show that the action of $exp(\mathfrak{C}_0)$ on $\text{MCE}(\mathfrak{C})$, as described in (33), respects the group structure of $exp(\mathfrak{C}_0)$:

$$\begin{aligned} exp(\beta) \curvearrowright (exp(\alpha) \curvearrowright \mu') &= exp(\beta) \curvearrowright (exp(\alpha)(\pi + \mu')(exp(-\alpha) \otimes exp(-\alpha)) - \pi) \\ &= exp(\beta)(\pi + exp(\alpha)(\pi + \mu')(exp(-\alpha) \otimes exp(-\alpha)) - \pi)(exp(-\beta) \otimes exp(-\beta)) - \pi \\ &= (exp(\beta)exp(\alpha))(\pi + \mu')(exp(-\alpha)exp(-\beta) \otimes exp(-\alpha)exp(-\beta)) - \pi \\ &= (exp(\beta)exp(\alpha)) \curvearrowright \mu' \end{aligned}$$

Hence, deformations are equivalent if and only if their corresponding Maurer-Cartan elements are in the same orbit of $exp(\mathfrak{C}_0)$. So we have the bijection

$$\left\{ \begin{array}{l} \text{Formal deformations of } A, \mu, \text{ over } K[[t]] \\ \text{upto equivalence} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Orbits of } \text{MCE}(\mathfrak{C}_0), \text{ under} \\ \text{the group action of } exp(\mathfrak{C}_0) \end{array} \right\} \quad (35)$$

The equivalence defined by the action of $\exp(\mathfrak{g}_0)$ on $\text{MCE}(\mathfrak{C}_0)$ as described in (33), is called **Gauge equivalence/action** and $\exp(\mathfrak{g}_0)$ is referred to as the **Gauge group**. In fact, by taking advantage of group-theoretic notation, bijection (35) is often presented as

$$\left\{ \begin{array}{l} \text{Formal deformations of } A, \mu, \text{ over } K[[t]] \\ \text{upto equivalence} \end{array} \right\} \longleftrightarrow \frac{\text{MCE}(\mathfrak{C}_0)}{\exp(\mathfrak{C}_0)} \quad (36)$$

The above bijection is the cornerstone of the so called ‘‘DGLA Philosophy’’ in deformation theory. Notice that the problem of classifying deformations of A over $\mathbb{K}[[t]]$ completely translates to a problem on the DGLA, $\mathfrak{C}_0 = tC_*(A)^{+1}[[t]]$. The DGLA philosophy as described by Mannetti in the first page of [14] states that ‘‘over characteristic 0, every deformation problem is governed by a DGLA via solutions of the Maurer-Cartan equation modulo gauge action’’.

Generalising our work:

For algebra A , let $\mathfrak{L} := C_*(A)^{+1}$ and observe that the DGLA we have been working with is

$$\mathfrak{C} = \mathfrak{L} \otimes t\mathbb{K}[[t]] = \bigoplus_{n \geq -1} \text{Hom}_{\mathbb{K}}(A^{\otimes n+1}, A) \otimes t\mathbb{K}[[t]]$$

- Firstly, notice that \mathbb{K} -algebra $\mathbb{K}[[t]]$ has a decomposition $\mathbb{K}[[t]] = \mathbb{K} \oplus (t)$ as a \mathbb{K} -vectorspace, where $(t) = t\mathbb{K}[[t]]$ is an ideal of $\mathbb{K}[[t]]$.

Definition 5.6. For a field \mathbb{K} , we say \mathbb{K} -algebra R is an **augmented algebra**⁴ if $R \cong \mathbb{K} \oplus R_+$, as a \mathbb{K} -vectorspace, for an ideal R_+ .

We used the fact that $\mathbb{K}[[t]] = \mathbb{K} \oplus (t)$ is augmented so that the definition of a one-parameter deformation over $\mathbb{K}[[t]]$ would make sense: We are using the existence of a decomposition

$$A \otimes \mathbb{K}[[t]] = (A \otimes \mathbb{K}) \oplus (A \otimes (t))$$

for the new multiplication on $A \otimes \mathbb{K}[[t]]$ to take form $\mu = \sum_{i \geq 0} \mu_i t^i$ with $\mu_0 = \pi$. For an augmented \mathbb{K} -algebra $R = \mathbb{K} \oplus R_+$, multiplication μ on $A \otimes R$ will decompose into $\mu = \mu_0 + \mu'$ with $\text{Im}(\mu_0) \subseteq A$ and $\text{Im}(\mu') \subseteq (A \otimes R_+)$. With these conditions, a deformation makes sense by requiring μ_0 to be the original multiplication of A .

- Secondly, observe that at the start of this section where we extend the Gerstenhaber bracket and circle product t -linearly to $tC_*(A)^{+1}[[t]]$, we are using the *commutativity* of $t\mathbb{K}[[t]]$. In fact, if we look at $tC_*(A)^{+1}[[t]]$ as $C_*(A)^{+1} \otimes t\mathbb{K}[[t]] = \bigoplus_{n \geq -1} \text{Hom}_{\mathbb{K}}(A^{\otimes n+1}, A) \otimes t\mathbb{K}[[t]]$, we are extending these operations via

$$(f \otimes a) \circ (g \otimes b) = (f \circ g \otimes ab)$$

where we would previously write

$$(ft^m) \circ (gt^n) = (f \circ g)t^{m+n}$$

We require the commutativity of $t\mathbb{K}[[t]]$ in order for the anti-commutativity condition from Definition (4.11) to hold in $C_*(A)^{+1} \otimes t\mathbb{K}[[t]]$:

$$\begin{aligned} (f \circ g \otimes ab) &= (f \otimes a) \circ (g \otimes b) = -(-1)^{|f||g|} (g \otimes b) \circ (f \otimes a) \\ &= -(-1)^{|f||g|} (g \circ f \otimes ba) = (f \circ g \otimes ba) \end{aligned}$$

- Throughout all our calculations, we have used the fact that any sum of the form $\sum_{i \geq 0} \alpha_i t^i$ with

⁴An equivalent definition can be found in [20], Definition 1.10.10.

$\alpha_i \in \mathbb{K}$, represents an element in $\mathbb{K}[[t]]$. Another way to put this phenomena is to say for any sequence $(\alpha_0, \alpha_1, \alpha_2, \dots)$ with α_i in \mathbb{K} , there exists an element $\alpha \in \mathbb{K}[[t]]$ such that

$$\alpha \equiv \sum_{i \geq 0}^{n-1} \alpha_i t^i \pmod{R_+^n}$$

where $R_+ = (t) = t\mathbb{K}[[t]]$.

Definition 5.7. For ideal $I \triangleleft R$ consider the inverse sequence

$$R/I \leftarrow R/I^2 \leftarrow R/I^3 \leftarrow \dots$$

The inverse limit of the sequence, $\hat{R} = \varprojlim R/I^n$ is called the **completion** of R with respect to the **I -adic topology**.

By the definition of the inverse limit:

$$\hat{R} = \{(r_0, r_1, r_2, \dots) : r_n \in R/I^n \text{ and } r_{n+1} \equiv r_n \pmod{I^n}\} \quad (37)$$

It is easy to see that there exists a natural map $\iota : R \rightarrow \hat{R}$. Hence, the assumption we have used for $\mathbb{K}[[t]]$ is in fact equivalent to ι being surjective, where \hat{R} is the completion of augmented ring $R = \mathbb{K} \oplus R_+$ with respect to ideal R_+ . In fact for $R = \mathbb{K}[[t]]$, ι is an isomorphism and in order to fully generalise our work, we also require ι to be injective:

- An important fact that we used in this section was that the Baker-Campbell-Hausdorff formula is well-defined on $\mathcal{L}_0 \otimes t\mathbb{K}[[t]]$. The property of $\mathbb{K}[[t]]$ that we used, is that ideal $t\mathbb{K}[[t]]$ is *pronilpotent*:

Definition 5.8. We say ideal R_+ of ring R is **pronilpotent**⁵ if $\bigcap_{n \geq 0} R_+^n = 0$.

In fact, the ideal R_+ being pronilpotent is equivalent to the map $\iota : R \rightarrow \hat{R}$ being injective.

Definition 5.9. We say ring R with ideal $I \triangleleft R$ is **complete** with respect to the **I -adic topology**, if the natural map $\iota : R \rightarrow \hat{R}$ is an isomorphism. For an augmented ring $R = \mathbb{K} \oplus R_+$, we simply say R is a **complete augmented ring** if it is complete with respect to the R_+ -adic topology.

To elaborate further, after Theorem (5.5) we argued that the number of elements in the expansion of $\log(\exp(\alpha)\exp(\beta))$ with the n th power of t is finite. For a pronilpotent ideal R_+ , since $\bigcap_{n \geq 0} R_+^n = 0$, the number of elements in the expansion intersecting with $R_+^r \setminus (\bigcap_{n \geq 0}^r R_+^n)$ are finite. And by (37), elements in $R \cong \hat{R}$ can be written as sequences (r_0, r_1, r_2, \dots) , where $r_n \in R/R_+^n$ and $r_{n+1} \equiv r_n \pmod{R_+^n}$, which via ι correspond to formal sums $r_0 + \sum_{n \geq 0} (r_{n+1} - r_n)$ in R , where

$$r_{n+1} - r_n \in R_+^r \setminus (\bigcap_{n \geq 0}^r R_+^n)$$

and $r_0 \in \mathbb{K}$. So the infinite sum in the BCH formula is well-defined if R_+ is pronilpotent and R is complete with respect to R_+ .

Furthermore, we have continuously utilized $\mathbb{K}[[t]]$ being complete with respect to the (t) -adic topology when extending addition to $A \otimes \mathbb{K}[[t]]$, t -linearly. In fact, this t -linear extension is often written using the notation $\hat{\otimes}$, sometimes called the *topological tensor product*, and constructed as follows

$$A \hat{\otimes} R = \varprojlim_{n \rightarrow \infty} A \otimes_R R/R_+^n$$

However, since t -linear extension over $\mathbb{K}[[t]]$ is commonly used and well understood by Mathematicians, the author has chosen to avoid this bit of detail and notation until now.

By the above explanations, we can generalise our deformation theory to any *commutative complete augmented* \mathbb{K} -algebra $R = \mathbb{K} \oplus R_+$:

⁵The definition of pronilpotent ideals as presented here is due to [19].

Definition 5.10. For algebra A with $\mathfrak{L} = C_*(A)^{+1}$ and commutative complete augmented \mathbb{K} -algebra $R = \mathbb{K} \oplus R_+$, a **formal deformation of A over R** , is a map $\mu : (A \hat{\otimes} R) \hat{\otimes} (A \hat{\otimes} R) \rightarrow A \hat{\otimes} R$ such that μ is associative and

$$\mu \equiv \pi \pmod{R_+}$$

Furthermore, since our theory on DGLAs can be generalised, we have a bijection

$$\left\{ \begin{array}{l} \text{Formal deformations of } A, \mu, \text{ over } R, \\ \text{upto equivalence} \end{array} \right\} \longleftrightarrow \frac{\text{MCE}(\mathfrak{L} \hat{\otimes} R_+)}{\text{exp}(\mathfrak{L}_0 \hat{\otimes} R_+)} \quad (38)$$

Observe that examples of algebras satisfying the required conditions include $R = \mathbb{K}[t]/(t^n)$ for $n \geq 2$, with $R_+ = t\mathbb{K}[t]/(t^n)$, where their deformation theory sometimes referred to as **(n - 1)-th order deformations**, as well as $R = \mathbb{K}[[t_1, t_2, \dots, t_n]]$ with $R_+ = (t_1, t_2, \dots, t_n)$ giving us **several-parameter deformations**.

Example 5.11. Recall from Remark (2.10), that when $R = \mathbb{K}[t]/(t^2)$, there exists a bijection between deformations of an algebra A over $\mathbb{K}[t]/(t^2)$ and $HH^2(A)$. We can confirm this via the DGLA method as well: Let $\mathfrak{L} = C_*(A)^{+1}$ and $\mu' \in \mathfrak{L}_1 \otimes t\mathbb{K}[t]/(t^2)$. Then $\mu' = \alpha t$ for $\alpha \in \text{Hom}_{\mathbb{K}}(A^{\otimes 2}, A)$ and

$$[\mu', \mu'] = 2\mu' \circ \mu' = 2(\alpha \circ \alpha)t^2 = 0$$

So $\mu' \in \mathfrak{L}_1 \otimes (t)$ is a Maurer-Cartan element if and only if $\delta_*(\mu') = 0 = -d_3\mu'$ and

$$\text{MCE}(\mathfrak{L} \otimes t\mathbb{K}[t]/(t^2)) = \text{Ker}(d_3)$$

Further the gauge action is induced by elements $\text{exp}(\gamma)$ where $\gamma \in \mathfrak{L}_0 \otimes (t) = t\text{Hom}_{\mathbb{K}}(A, A)$. Hence, since $t^2 = 0$, such a γ has form $\gamma = \varphi t$ where $\varphi \in \text{Hom}_{\mathbb{K}}(A, A)$ and $\text{exp}(\varphi t) = \text{Id}_A + \varphi t$. Thereby, the gauge action described in (33) becomes

$$\begin{aligned} \text{exp}(\varphi t) \curvearrowright \alpha t &= \text{exp}(\varphi t)(\pi + \alpha t)(\text{exp}(-\varphi t) \otimes \text{exp}(-\varphi t)) - \pi \\ &= (\text{Id}_A + \varphi t)(\pi + \alpha t)((\text{Id}_A - \varphi t) \otimes (\text{Id}_A - \varphi t)) - \pi \\ &= \pi((\text{Id}_A - \varphi t) \otimes (\text{Id}_A - \varphi t)) + \alpha(\text{Id}_A \otimes \text{Id}_A)t + \varphi\pi(\text{Id}_A \otimes \text{Id}_A)t - \pi \\ &= \pi - \pi - \pi(\varphi \otimes \text{Id}_A)t - \pi(\text{Id}_A \otimes \varphi)t + \alpha t + \varphi(\pi)t \\ &= (\alpha - d_2\varphi)t \end{aligned}$$

Thereby,

$$\frac{\text{MCE}(\mathfrak{L} \otimes t\mathbb{K}[t]/(t^2))}{\text{exp}(\mathfrak{L}_0 \otimes t\mathbb{K}[t]/(t^2))} \cong \text{Ker}(d_3) / \text{Im}(d_2) = HH^2(A)$$

Which confirms bijection (18):

$$\left\{ \begin{array}{l} \text{Formal deformations of } A, \text{ over } \mathbb{K}[t]/(t^2) \\ \text{upto equivalence} \end{array} \right\} \longleftrightarrow \frac{\text{MCE}(\mathfrak{L} \otimes t\mathbb{K}[t]/(t^2))}{\text{exp}(\mathfrak{L}_0 \otimes t\mathbb{K}[t]/(t^2))} \longleftrightarrow HH^2(A)$$

Remark 5.12. In many papers on deformation theory and DGLAs, including [14], $\text{Def}_{\mathfrak{L}}(R)$ is first defined in the case where $R = \mathbb{K} \oplus R_+$ and R_+ is a nilpotent ideal, or equivalently R is local Artinian with residue field \mathbb{K} . Observe that for a commutative complete augmented ring $R = \mathbb{K} \oplus R_+$, the rings R/R_+^n all have a nilpotent maximal ideal R_+ and R is equal to the inverse limit of these rings. Hence, once we define the theory for nilpotent R_+ , we can extend the theory to our case where R_+ is pronilpotent and R a commutative complete augmented ring.

We denote the set on the RHS of (38) by

$$\text{Def}_{\mathfrak{L}}(R) := \frac{\text{MCE}(\mathfrak{L} \otimes R_+)}{\text{exp}(\mathfrak{L}_0 \otimes R_+)}$$

For each DGLA L , $\text{Def}_L : \text{Art}_{\mathbb{K}} \rightarrow \text{Set}$, where $\text{Art}_{\mathbb{K}}$ is the category of local Artinian \mathbb{K} -algebras and Set the category of sets, is functorial and for each DGLA we obtain a functor Def_L called the **deformation functor**. More detail on this construction can be found in [14] and [3].

A natural question to ask is when two algebras have the same deformation theory, which translates to their cochain DGLAs admitting the same Deformation functor.

Definition 5.13. *We say a morphism of DGLAs $\Phi : L \rightarrow N$ is a **quasi-isomorphism** if the induced morphism on their cohomology DGLAs, $H(\Phi) : H(L) \rightarrow H(N)$ is an isomorphism.*

Theorem 5.14. [Corollary 3.2 [14]] *Let $\Phi : L \rightarrow N$ be a quasi-isomorphism of DGLAs. Then the induced morphism $\text{Def}_L \rightarrow \text{Def}_N$ is an isomorphism.*

Observe that in our case $H(C_*(A)^{+1}) = HH^*(A)^{+1}$ and thereby, over characteristic 0, the deformation theory of an algebra is completely unique to its Hochschild cohomology.

6 Prelude to Deformation Quantization

Much of the interest in Algebraic Deformation Theory is due to its connections with deformations in Geometry and its connections with Theoretical Physics. In fact parameter t is often denoted by \hbar , since the case of interest in Physics is when the parameter equals *Planck's constant*. The deformation philosophy in Mathematical Physics as described in the survey [2] says: "Intuitively, classical mechanics is the limit of quantum mechanics when $\hbar = h/2\pi$ goes to zero". As described in section III of the same paper, Classical mechanics is concerned with the theory of Poisson manifolds and the deformation of interest are deformations of the algebra of functions on the manifold. Hence, we devote the last section of this essay to Poisson algebras and Deformation Quantization:

Definition 6.1. *A commutative associative algebra A is said to be a **Poisson algebra** if A is equipped with a Lie bracket $\{-, -\}$ such that*

$$\{a, bc\} = \{a, b\}c + b\{a, c\} \quad (39)$$

*holds. The bracket $\{-, -\}$ is then called the **Poisson bracket** on A .*

Any commutative algebra can in fact be endowed with a Poisson structure via its Hochschild cohomology:

Theorem 6.2. *Let A be a commutative algebra and ρ a Hochschild 2-cocycle such that $[\rho, \rho] = 0$, then the \mathbb{K} -bilinear map $\{-, -\}$ defined by*

$$\{a, b\} = \rho(a \otimes b) - \rho(b \otimes a) \quad (40)$$

defines a Poisson bracket on A .

Proof. First, we show $\{-, -\}$ is a Lie bracket. Observe that since $[\rho, \rho] = 0$, then

$$\rho(\rho(a \otimes b) \otimes c) - \rho(a \otimes \rho(b \otimes c)) = \frac{1}{2}\rho \circ \rho(a, b, c) = \frac{1}{2}[\rho, \rho](a, b, c) = 0$$

and

$$\{a, \{b, c\}\} = \{a, \rho(b \otimes c) - \rho(c \otimes b)\}$$

$$= \rho(a \otimes \rho(b \otimes c)) - \rho(a \otimes \rho(c \otimes b)) - \rho(\rho(b \otimes c) \otimes a) + \rho(\rho(c \otimes b) \otimes a)$$

then, the Jacobi identity follows directly:

$$\begin{aligned} \{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = \\ \rho(a \otimes \rho(b \otimes c)) - \rho(a \otimes \rho(c \otimes b)) - \rho(\rho(b \otimes c) \otimes a) + \rho(\rho(c \otimes b) \otimes a) \\ + \rho(b \otimes \rho(c \otimes a)) - \rho(b \otimes \rho(a \otimes c)) - \rho(\rho(c \otimes a) \otimes b) + \rho(\rho(a \otimes c) \otimes b) \\ + \rho(c \otimes \rho(a \otimes b)) - \rho(c \otimes \rho(b \otimes a)) - \rho(\rho(a \otimes b) \otimes c) + \rho(\rho(b \otimes a) \otimes c) = 0 \end{aligned}$$

Furthermore, since $d_3\rho = 0$ and A is commutative, then $\{a, bc\} = \{a, b\}c + b\{a, c\}$ follows from

$$\begin{aligned} 0 = d_3\rho(a \otimes b \otimes c) &= a\rho(b \otimes c) - \rho(ab \otimes c) + \rho(a \otimes bc) - \rho(a \otimes b)c \\ \Rightarrow \{a, bc\} &= \rho(a \otimes bc) - \rho(bc \otimes a) = \rho(a \otimes cb) - \rho(cb \otimes a) \\ &= d_3\rho(a \otimes c \otimes b) + d_3\rho(c \otimes b \otimes a) + \rho(ac \otimes b) + \rho(a \otimes c)b - c\rho(b \otimes a) - \rho(c \otimes ba) \\ &= d_3\rho(a \otimes c \otimes b) + d_3\rho(c \otimes b \otimes a) - d_3\rho(c \otimes a \otimes b) + c\rho(a \otimes b) - \rho(c \otimes a)b \\ &\quad + \rho(a \otimes c)b - c\rho(b \otimes a) = 0 + 0 - 0 + \{a, b\}c + \{a, c\}b = \{a, b\}c + b\{a, c\} \end{aligned}$$

□

Observe that for a commutative algebra A , and $\varphi \in \text{Hom}_{\mathbb{K}}(A, A)$

$$d_2\varphi(a \otimes b) = a\varphi(b) - \varphi(ab) + \varphi(a)b = \varphi(b)a - \varphi(ba) + b\varphi(a) = d_2\varphi(b \otimes a)$$

Hence, for Hochschild 2-cocycle $\rho \in \text{Ker}(d_3)$, ρ and $\rho + d_2\varphi$ induce the same Poisson bracket via (40).

Definition 6.3. A deformation quantization of a Poisson algebra A with Poisson bracket $\{-, -\}$ is a formal deformation A_μ with $\mu = \pi + \sum_{i \geq 1} \mu_i t^i$ and $\mu_i \in \text{Hom}_{\mathbb{K}}(A^{\otimes n}, A)$ such that⁶

$$\mu(a \otimes b) - \mu(b \otimes a) \equiv \{a, b\}t \pmod{t^2} \quad (41)$$

Example 6.4. For $A = \mathbb{K}[x, y]$, recall the definition of map ψ from Example (4.6). As we demonstrated previously, ψ is a Hochschild 2-cocycle. In fact it is well-known that

$$\{a, b\} = \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial b}{\partial x} \frac{\partial a}{\partial y} = \psi(a \otimes b) - \psi(b \otimes a)$$

defines a Poisson bracket on A , as confirmed by Theorem (6.2). In fact in Example (4.6), we showed that the truncated deformation $\pi + \psi t$ can extend to a formal deformation A_{μ_1} , giving a deformation quantization of $\mathbb{K}[x, y]$ with respect to this Poisson structure. A similar deformation quantization exists for any polynomial algebra and the formal deformation is referred to as the Moyal product [Example 1.4.1 [13]]. Although, in this case, the Moyal product gives an example of deformation quantization if its definition required the Poisson bracket to occur as the coefficient of t . This is slightly different from our example, but both can be recovered from each other.

A Poisson manifold is a smooth manifold \mathcal{M} with a Poisson bracket on its algebra of smooth functions $C^\infty(\mathcal{M})$. The main question of Quantization theory is ‘‘Does an arbitrary Poisson manifold have a deformation quantization?’’, where a deformation quantization of a Poisson manifold \mathcal{M} is a deformation quantization of $C^\infty(\mathcal{M})$, $\mu = \pi + \sum_{i \geq 1} \mu_i t^i$ where μ_i are bidifferential operators. This question was first worked on for the case of Symplectic manifolds in [1], where the idea of

⁶In many texts, there is an additional coefficient of $\frac{1}{2}$ on the LHS of (41), to indicate it being the anti-symmetric part of bilinear map μ .

deformation quantization was first pushed forward. Formal deformations of the mentioned form with the additional property of having differential operators as coefficients are often referred to as **\star -products**. The solution in the case of Poisson manifolds is due to M. Kontsevich and Theorem (5.14) is key to his solution. In [13], Kontsevich describes a quasi-isomorphism of two DGLAs, the Hochschild complex restricted to differential operators, and the complex of *poly-vector fields* on \mathcal{M} . Maurer-Cartan elements of the first DGLA, as we've seen in section (5), are formal \star -products on $C^\infty(\mathcal{M})$ and the MC elements of the second DGLA are *formal Poisson deformations* of $C^\infty(\mathcal{M})$. A quasi-isomorphism of graded differential algebras was already known from the Hochschild-Konstant-Rosenberg Theorem, but it did not respect the brackets of both sides. This issue was solved by Kontsevich in [13], where he extends this map to a quasi-isomorphism of L_∞ -algebras. DGLAs are a specific case of L_∞ -algebras. Hence by Theorem (5.14), this results to a bijection between formal \star -products and formal Poisson deformations on $C^\infty(\mathcal{M})$, which by Kontsevich's map then implies the existence of a deformation quantization. The account of the proof described here is influenced by the sketches given in [19] and section 8 of [3] and the proof in full detail can be found in [13].

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