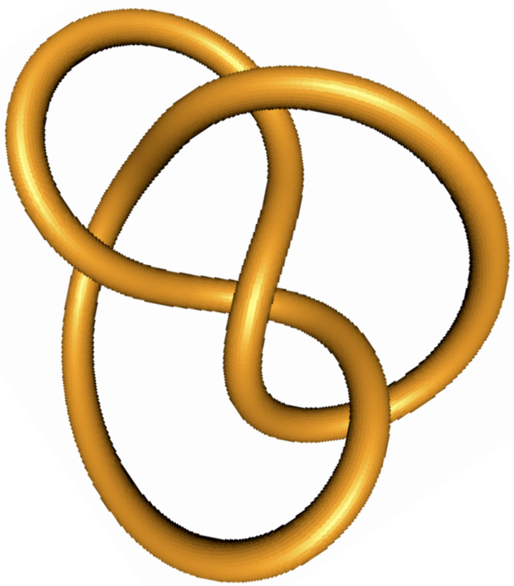
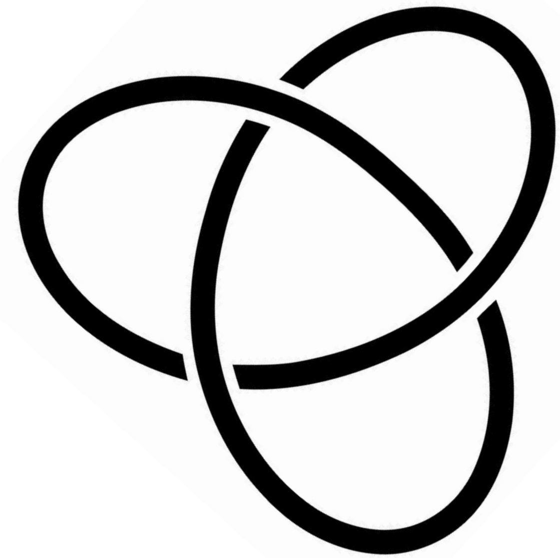
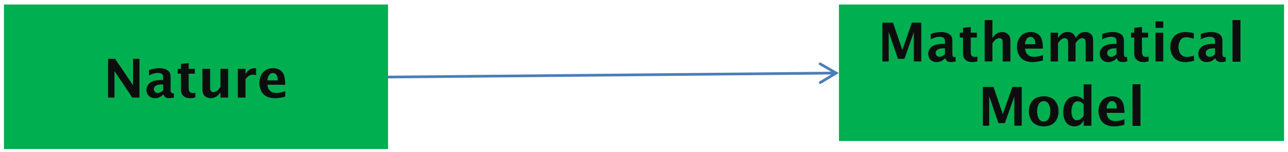


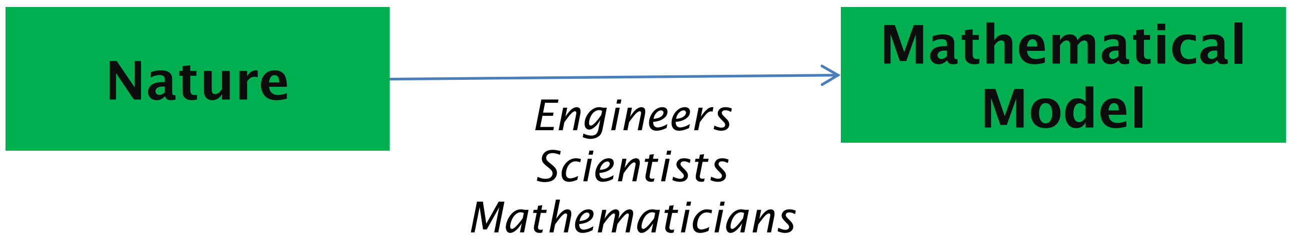


Knots

(*and how to tame them…*)

By Aryan Ghobadi


Nature
Mathematical Model


Nature
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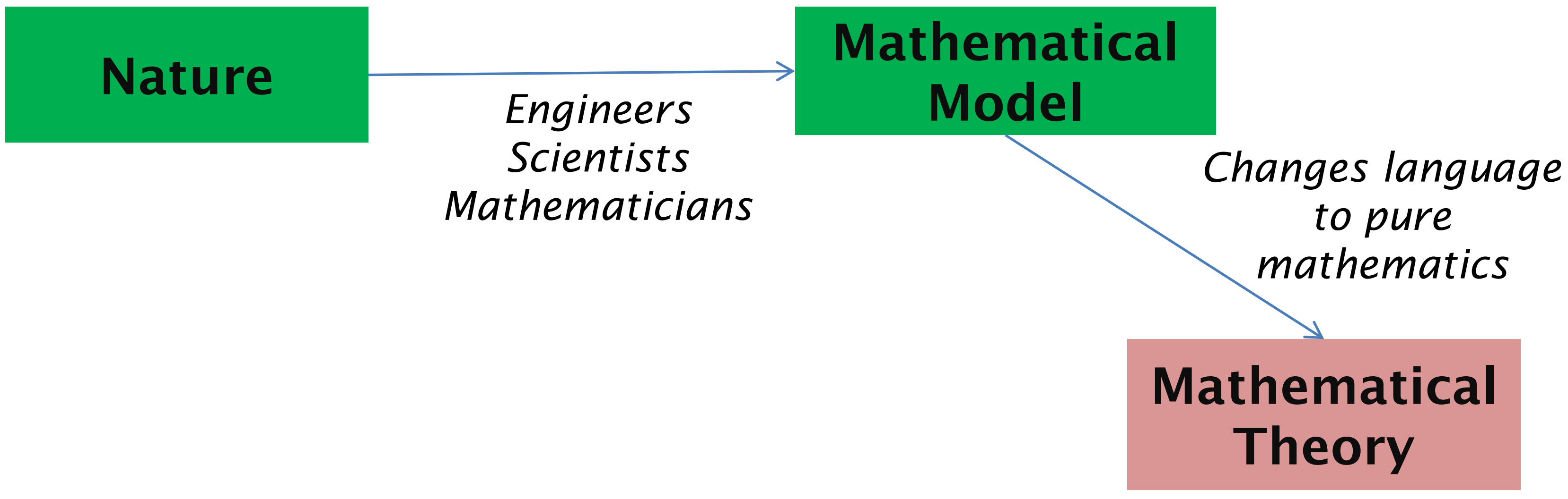
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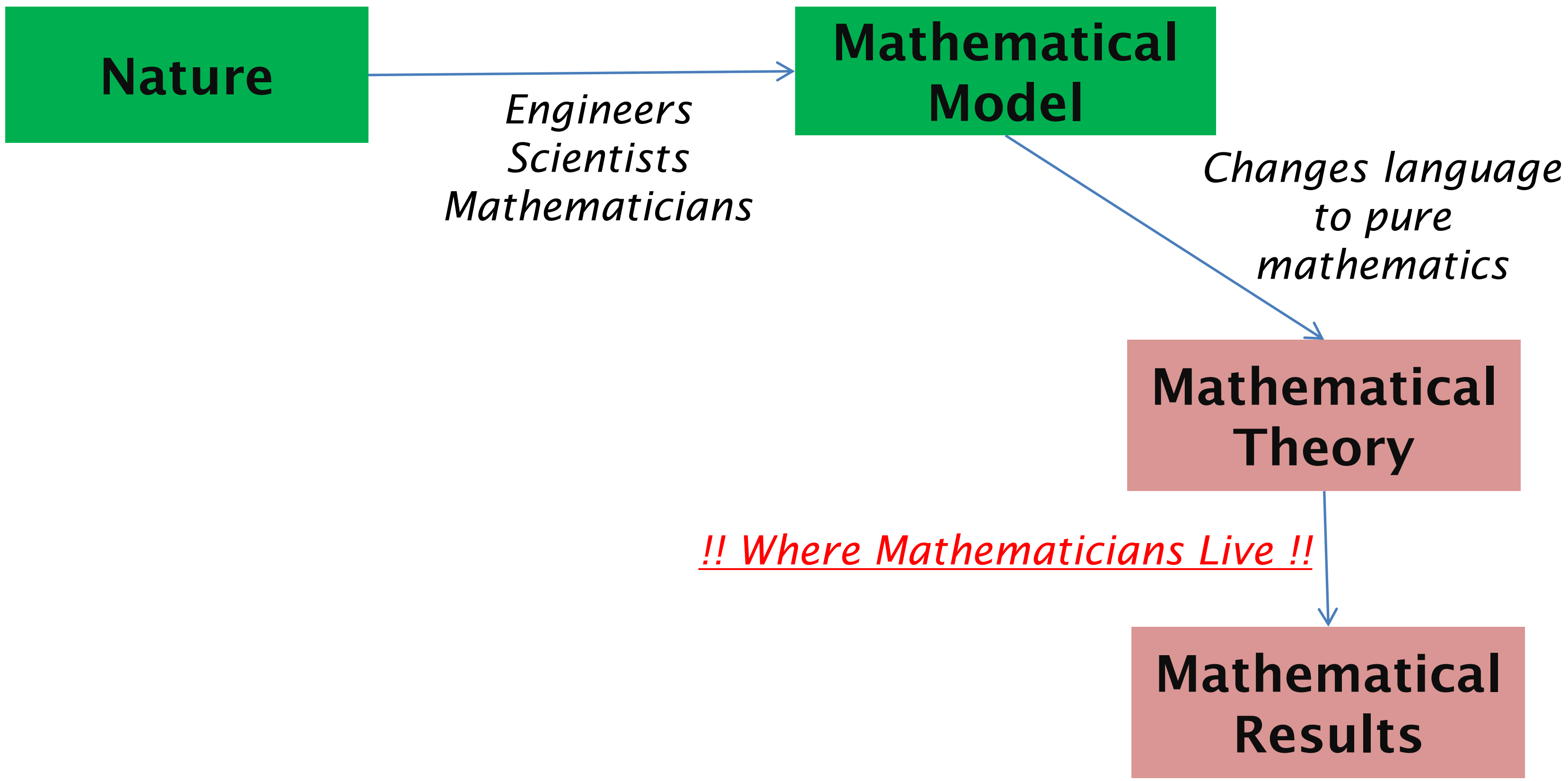
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!! Where Mathematicians Live !!
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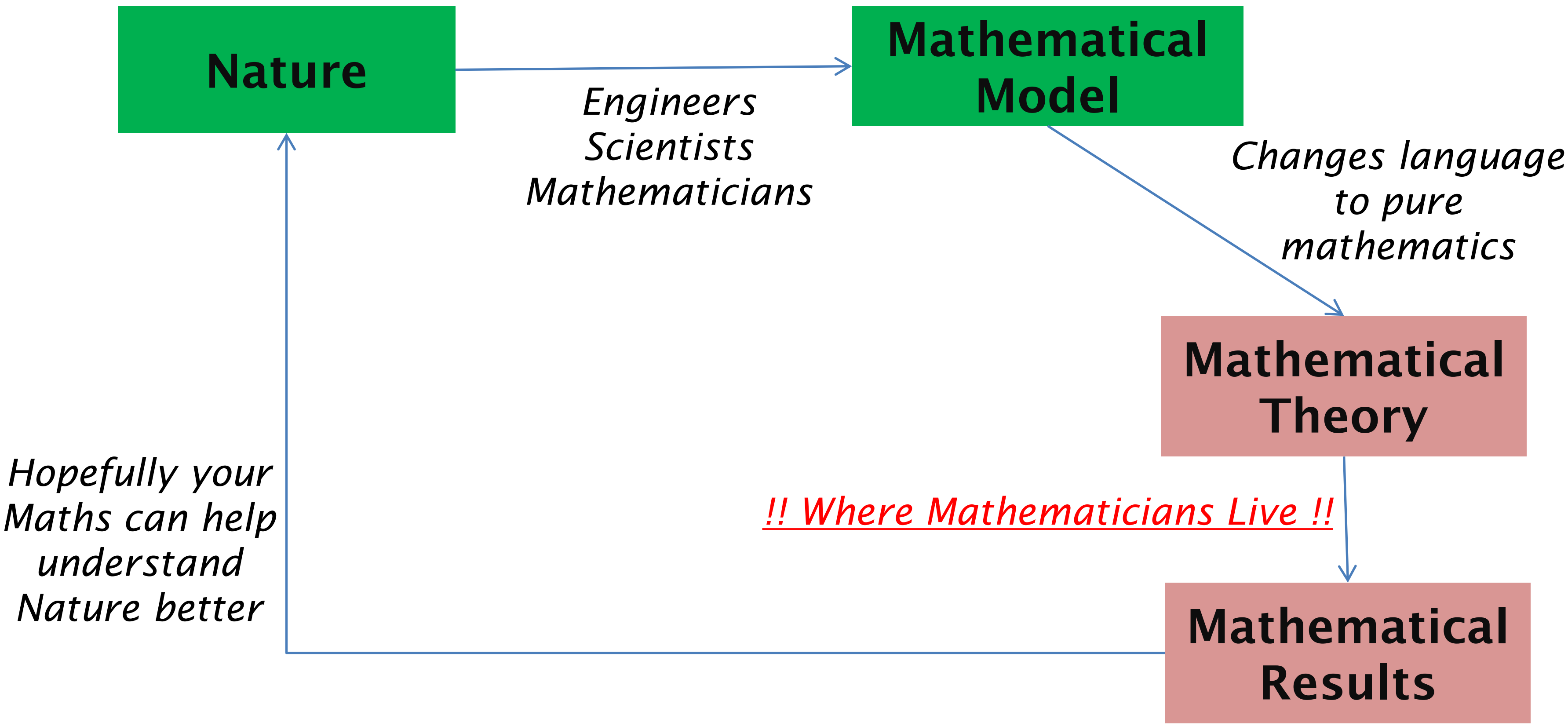
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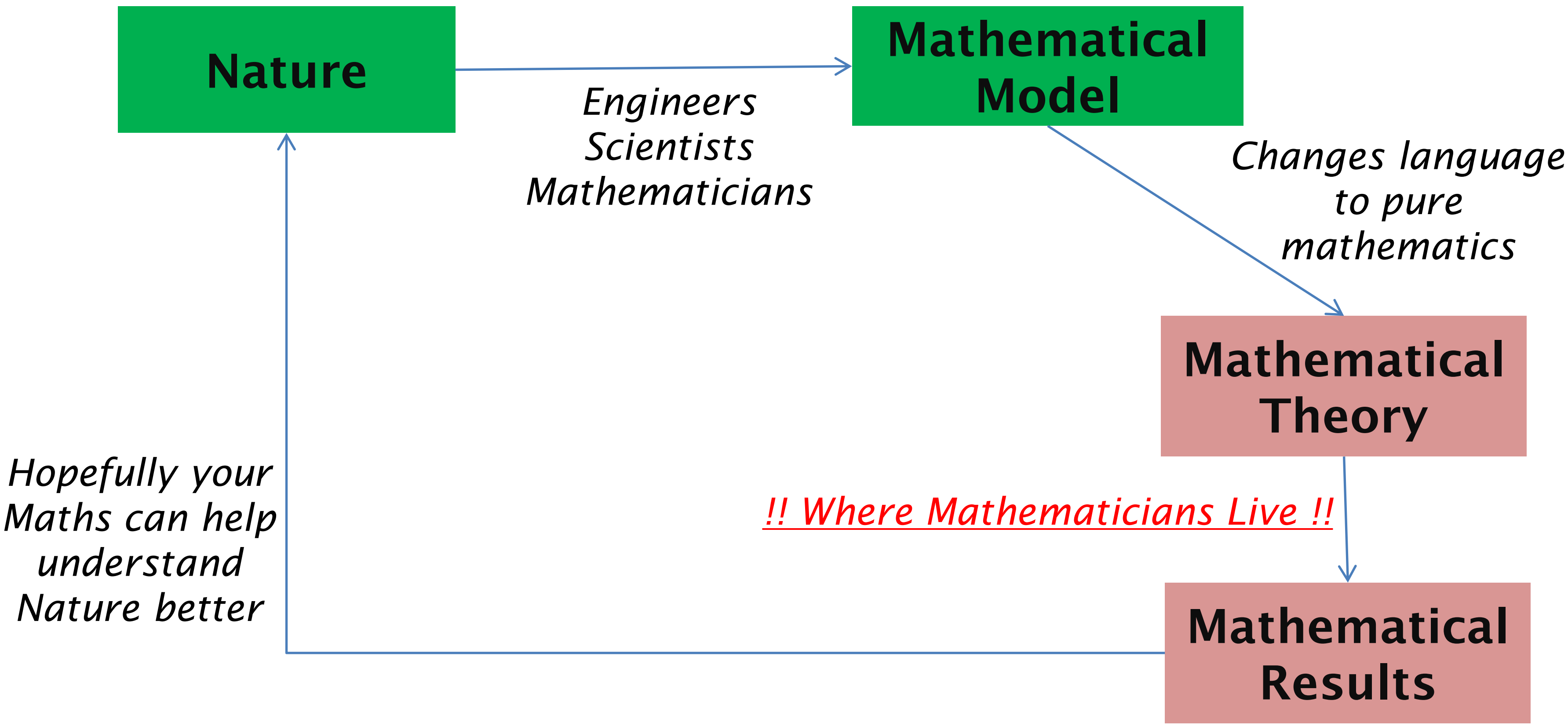
Mathematical Results
Hopefully your Maths can help understand Nature better



- *Mathematical results might not solve the big problem at first but add to overall Mathematical knowledge*
- *Good mathematics should connect with other good mathematics!*

What are Knots?

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- A closed “line” in 3 dimensional space, without intersection

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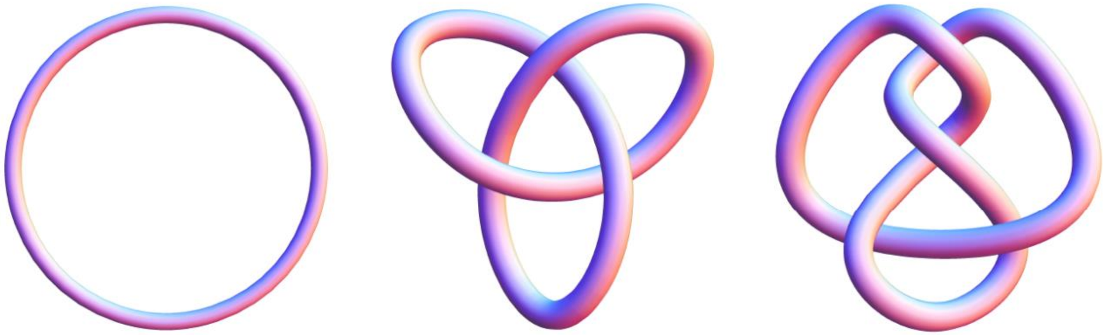

(a) Unknot (b) Trefoil (c) Figure eight

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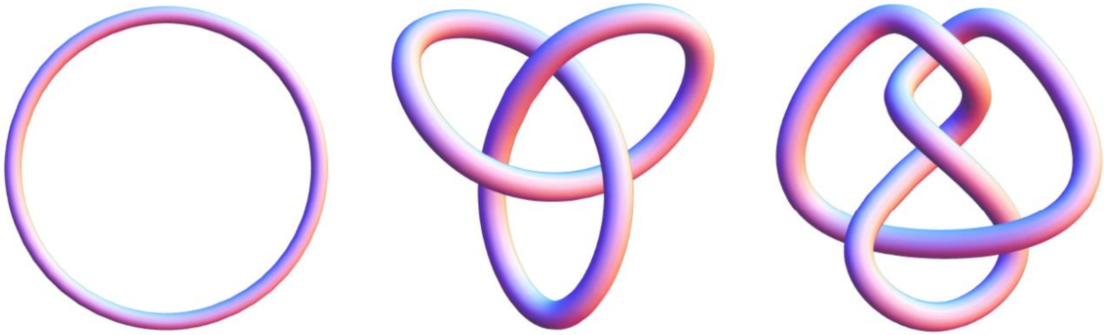

(a) Unknot (b) Trefoil (c) Figure eight

- **More formally:**

A "smooth" function $f: [0,1] \rightarrow \mathbb{R}^3$ such that $f(0) = f(1)$ and that's the only case where $f(x) = f(y)$, for $x \neq y$.

How do we look at Knots?

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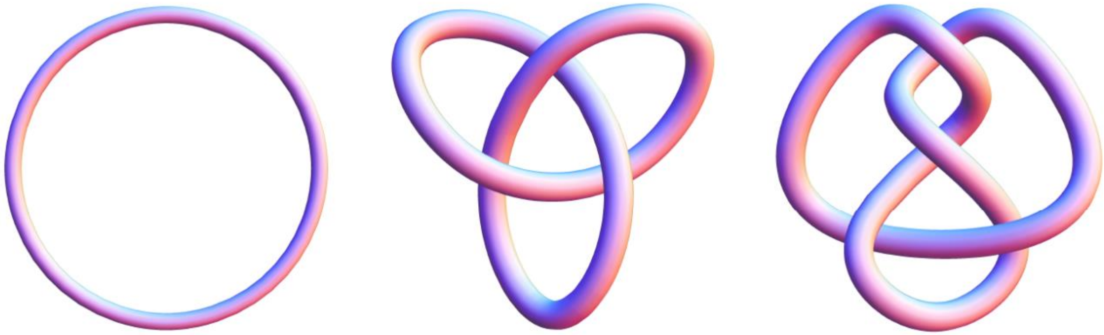
- As 2 dimensional diagrams



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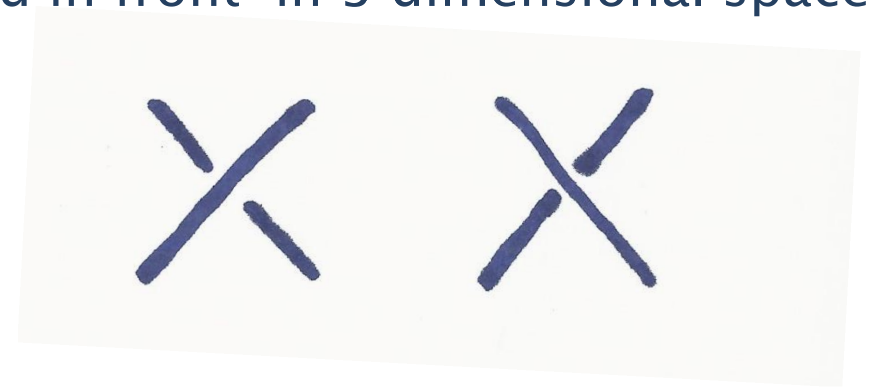
- Crossing behind and in front in 3 dimensional space are represented as



Topological View

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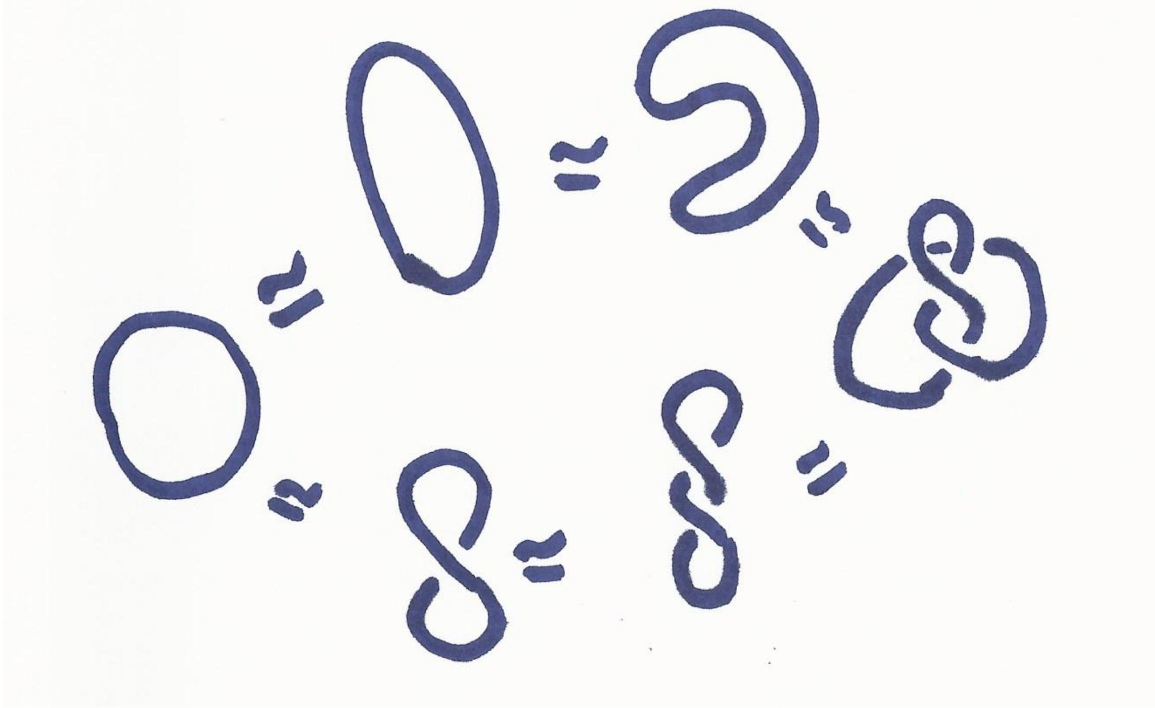

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- **More formally: (Ambient Isotopy)**

Given two knots $k, \bar{k}: [0,1] \to \mathbb{R}^3$, there exists a continuous map

$$F: \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$$

Such that $F(k(x), 0) = k(x)$ and $F(k(x), 1) = \bar{k}(x)$.

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- One Knot has INFINITELY many equivalent Diagrams.
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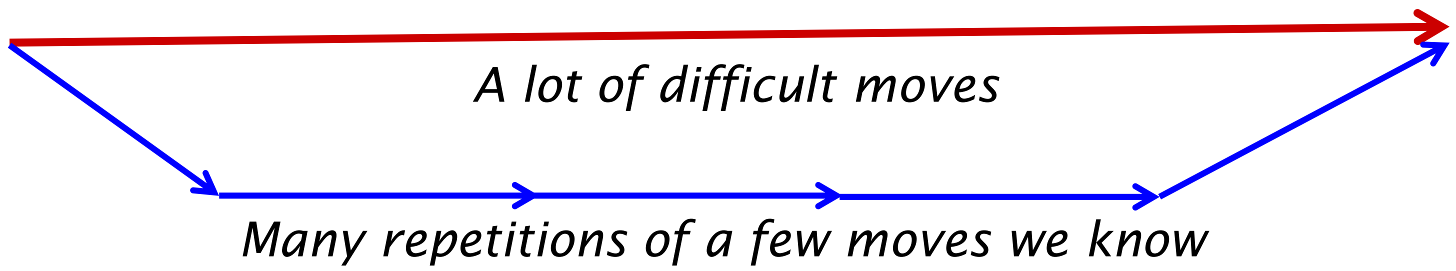

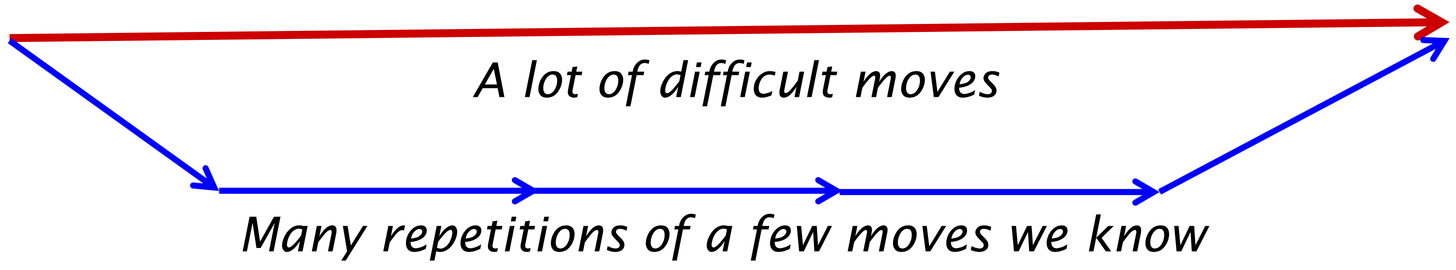
A lot of difficult moves
Many repetitions of a few moves we know

Reidmeister Moves

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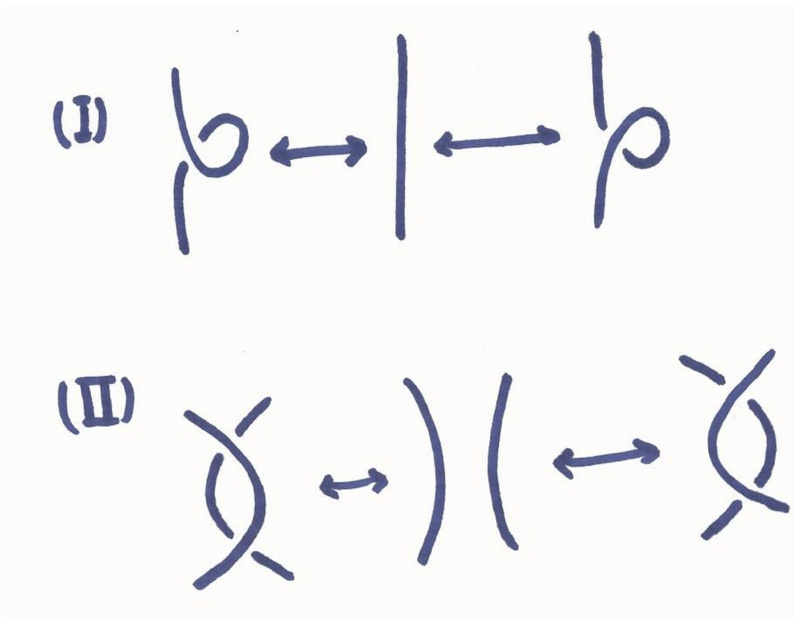
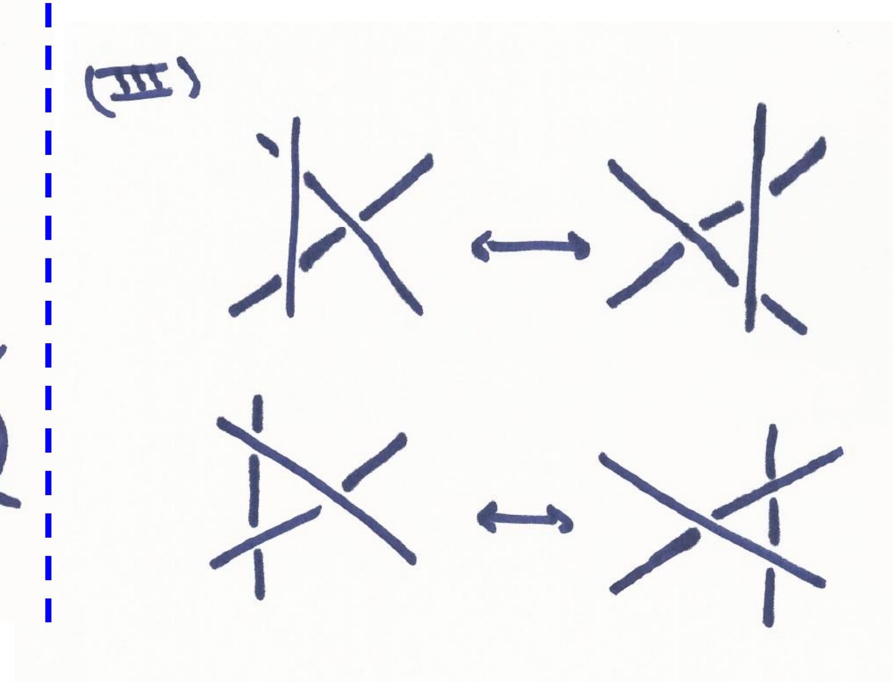
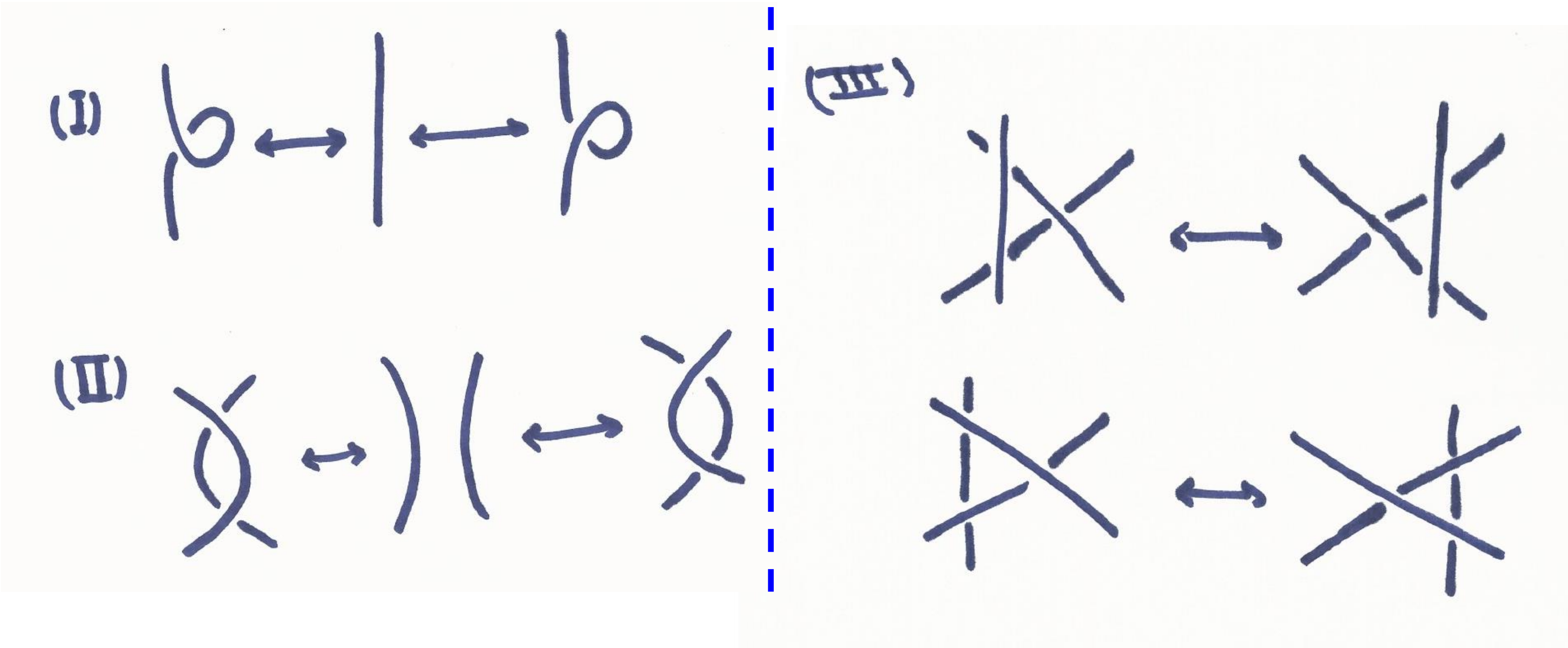




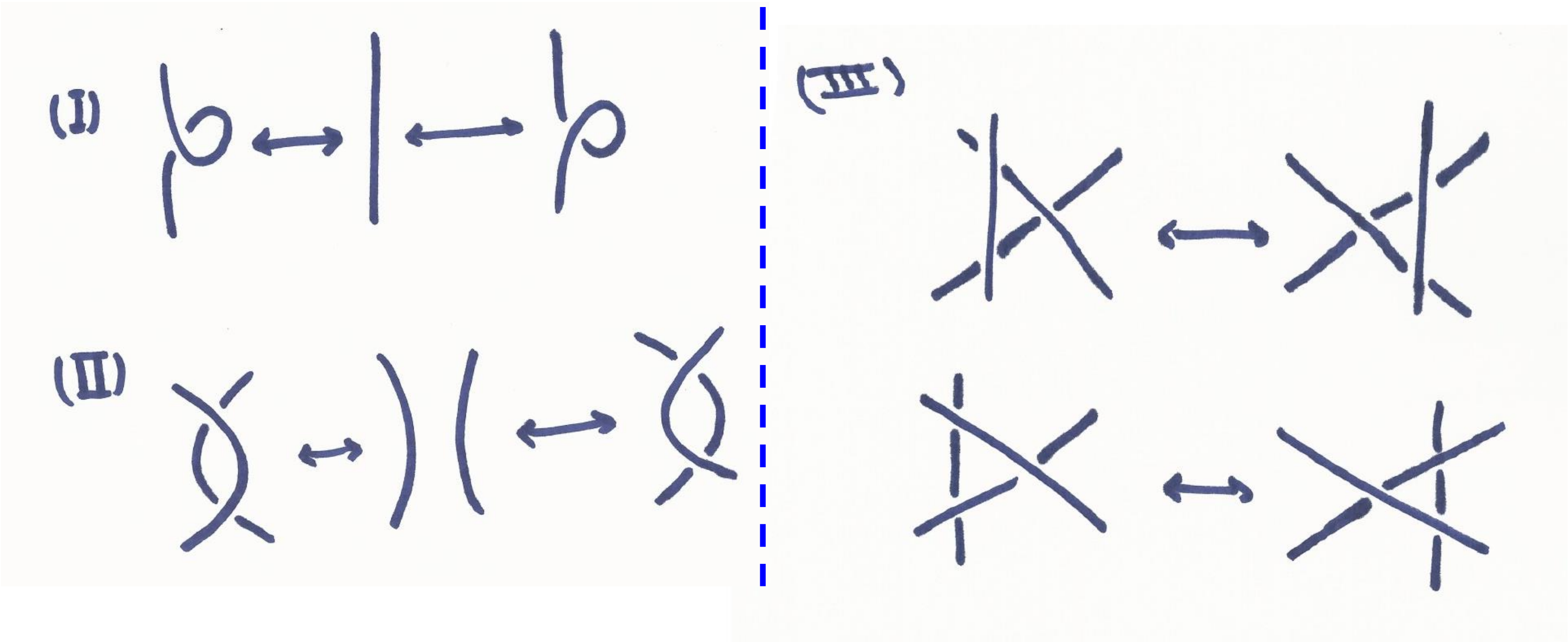
Reidmeister Moves



- **More formally: (Reidmeister's Theorem, 1927)**

Given two knots $k, \bar{k}: [0,1] \rightarrow \mathbb{R}^3$, they are equivalent if and only if one can be transformed to the other by finitely many Reidmeister moves.

Reidmeister Moves



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Do all possible (I) moves at least!

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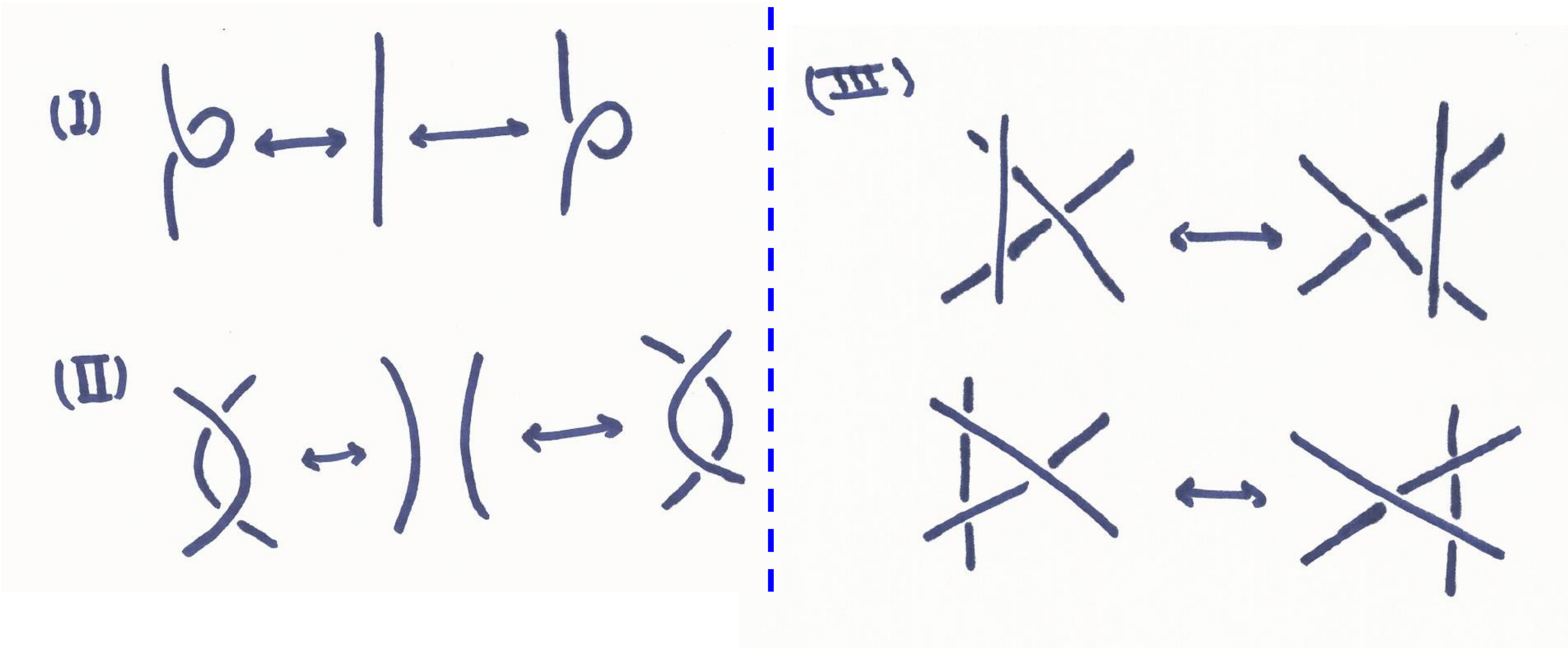

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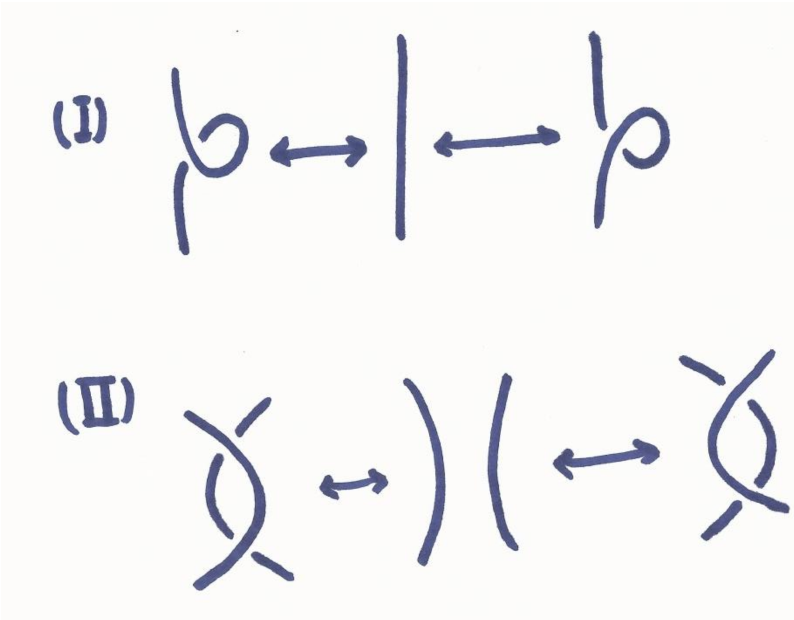

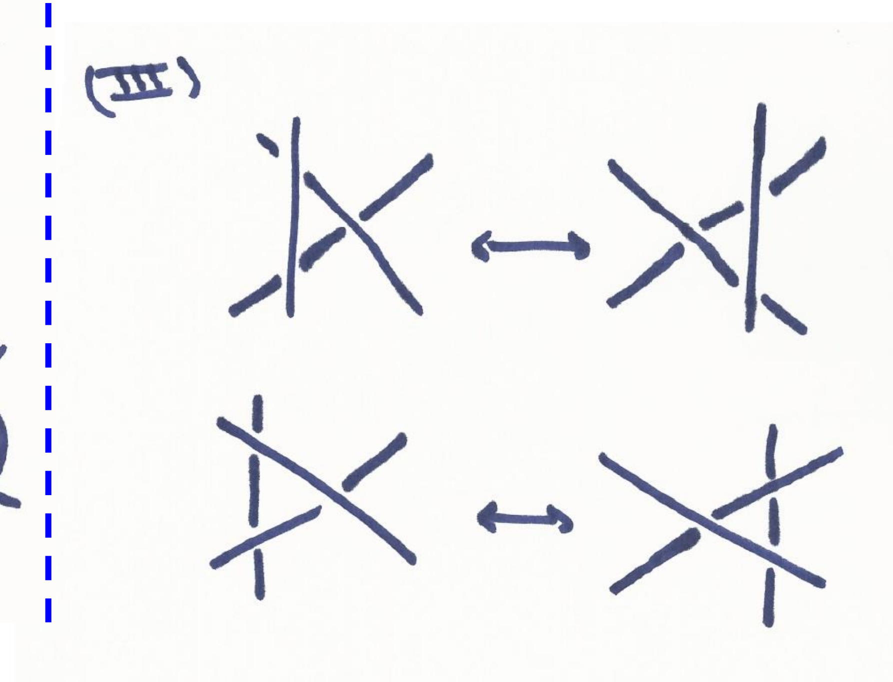

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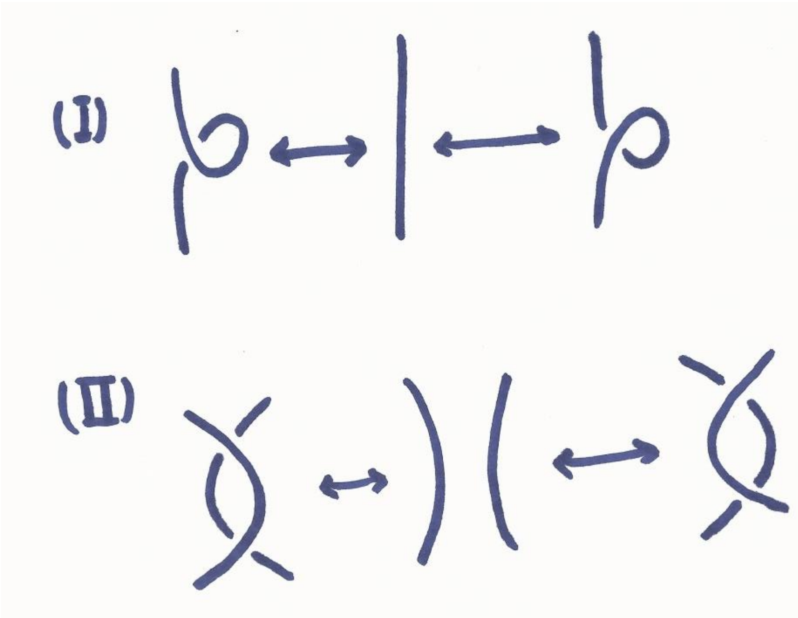

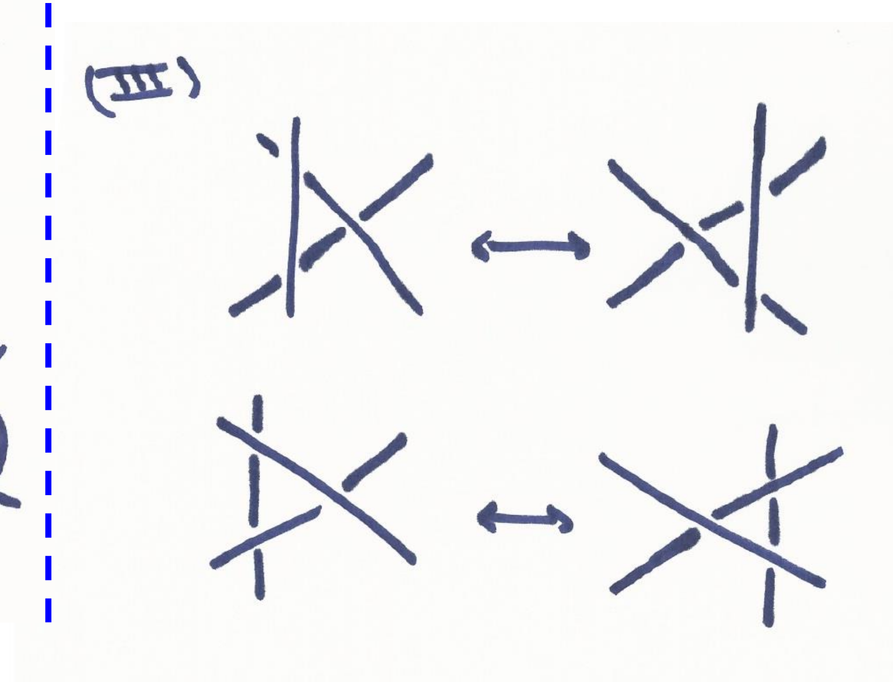

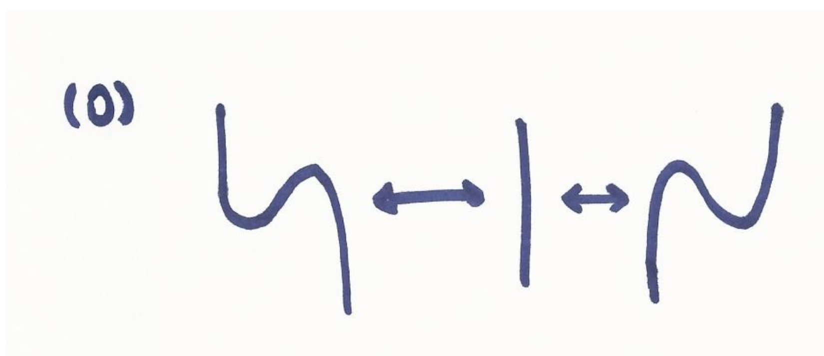

How to tell Knots apart?

Diagrams for the same knot → Reidmesiter → *Same Knot*

How do we tell if knot diagrams are definitely different

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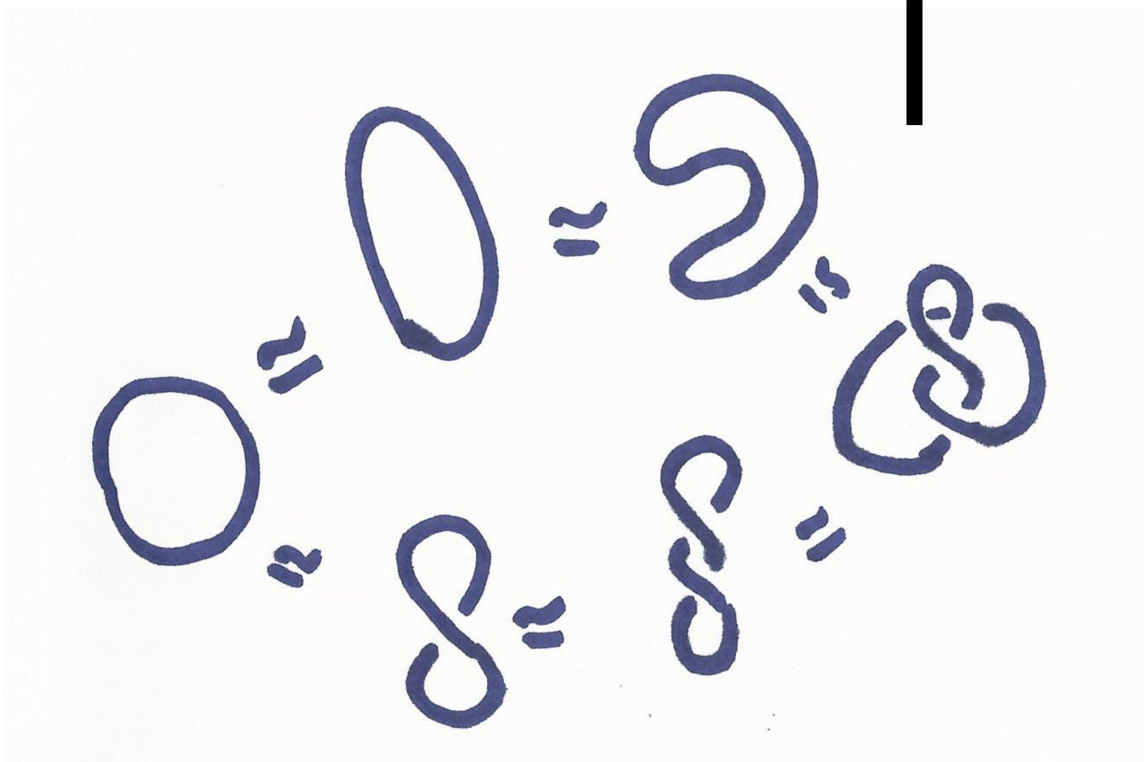

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Answer: (Invariance)

Assign a “number” to each Knot called it’s <u>invariant</u> so that

- Equivalent knots get the same number

Then, if two knots get different numbers, then they’re not equivalent.

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Then, if two knots get different numbers, then they're not equivalent.

- No guarantee that an assignment tells apart all Knots
- The more it does so, the better (a more *coarse* invariant)

Ideas for Invariants

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- Crossing Number (not an invariant)
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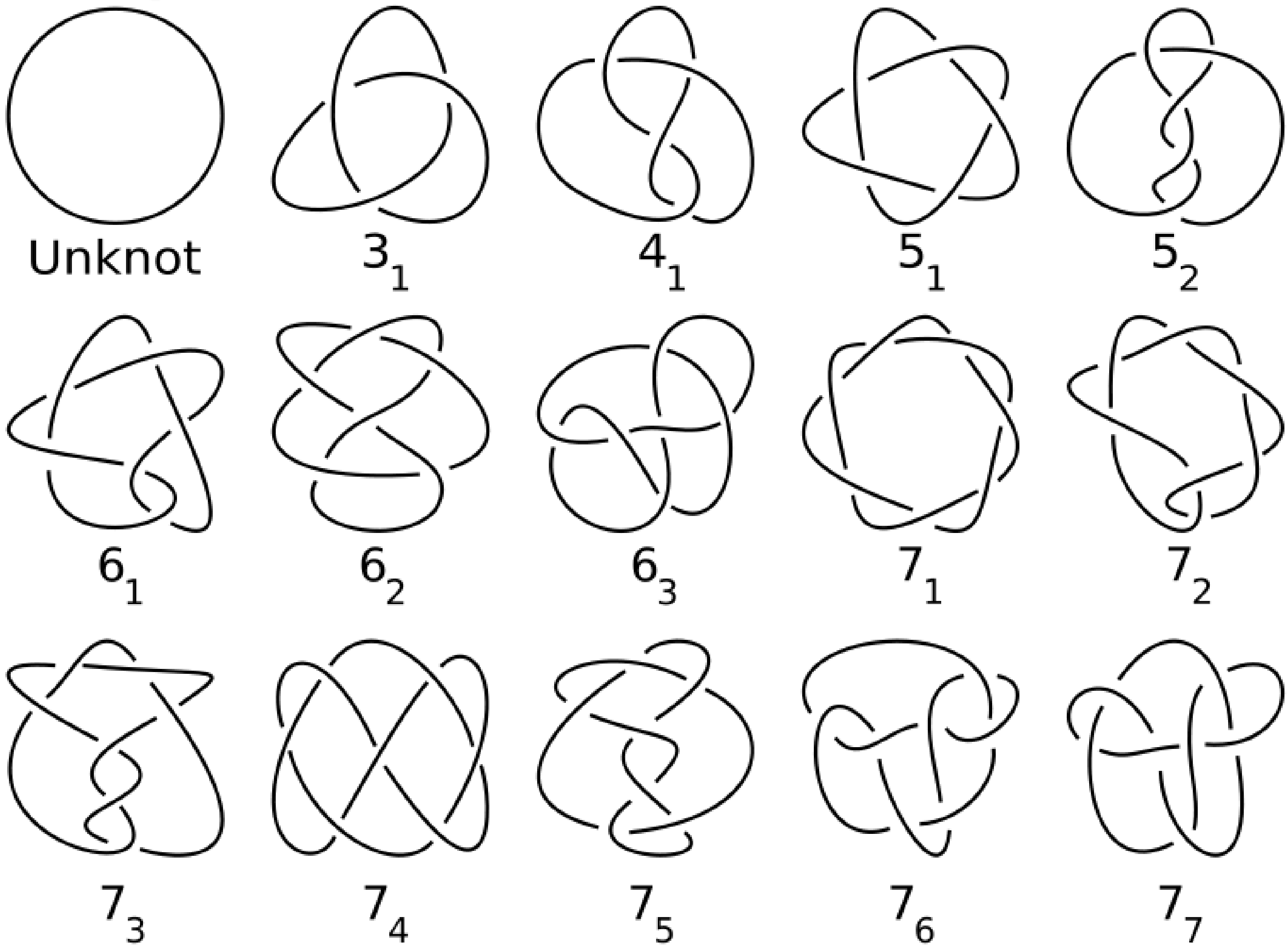

- For a better invariant we need to work with more complicated numbers than $\mathbb{N} = \{1,2,3,4,\dots\}$

- Assign Polynomials to Knots $\mathbb{Z}[t]$

$$p(t) \in \mathbb{Z}[t] \quad \Rightarrow \quad p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$$

For example $5t^3 + 8t^2 - 2$, $3t^7 + 4t^3$

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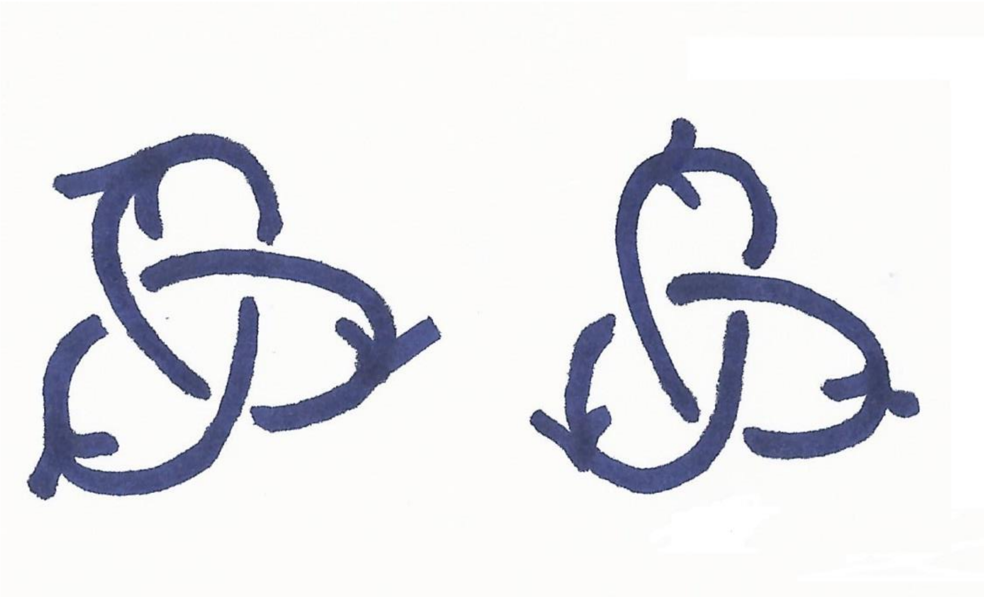

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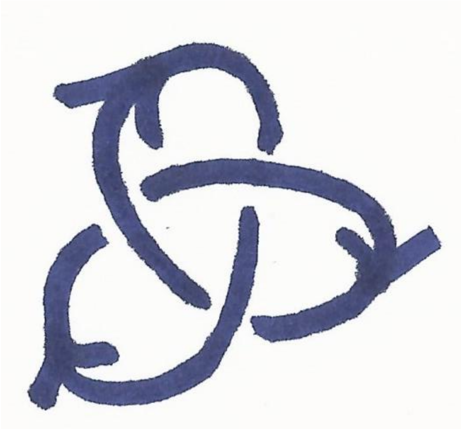

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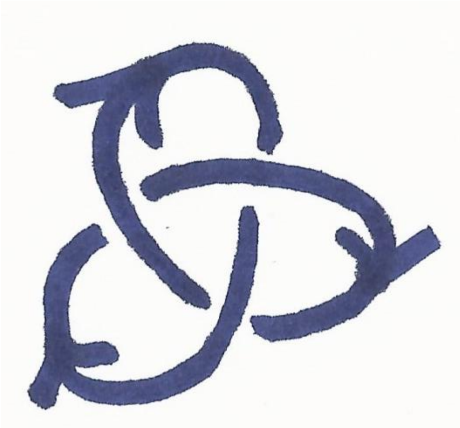

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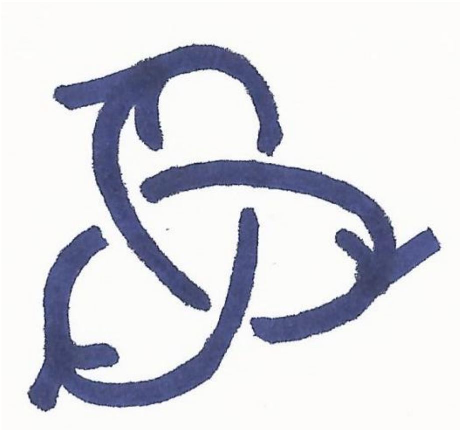

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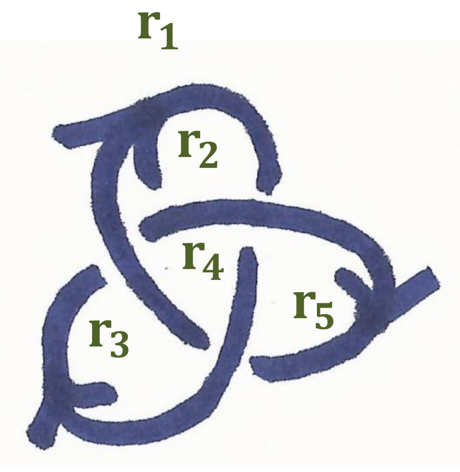

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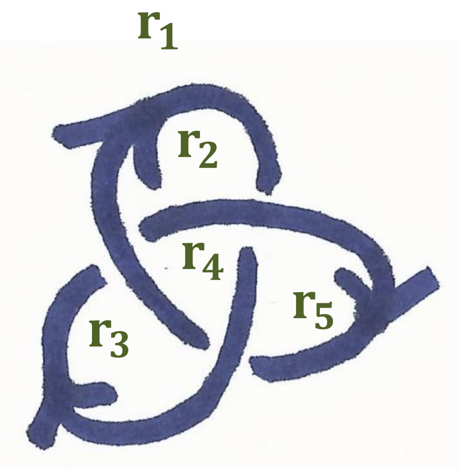

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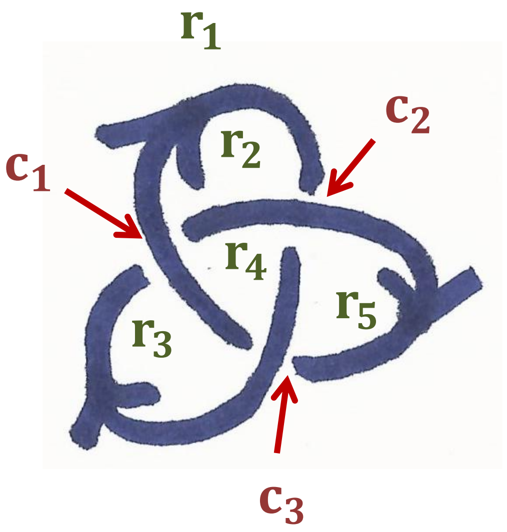

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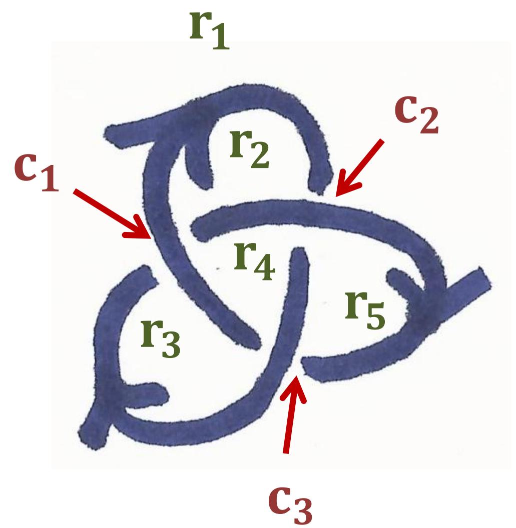

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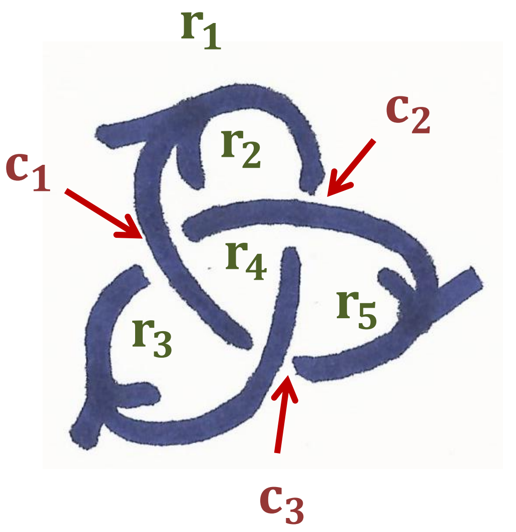
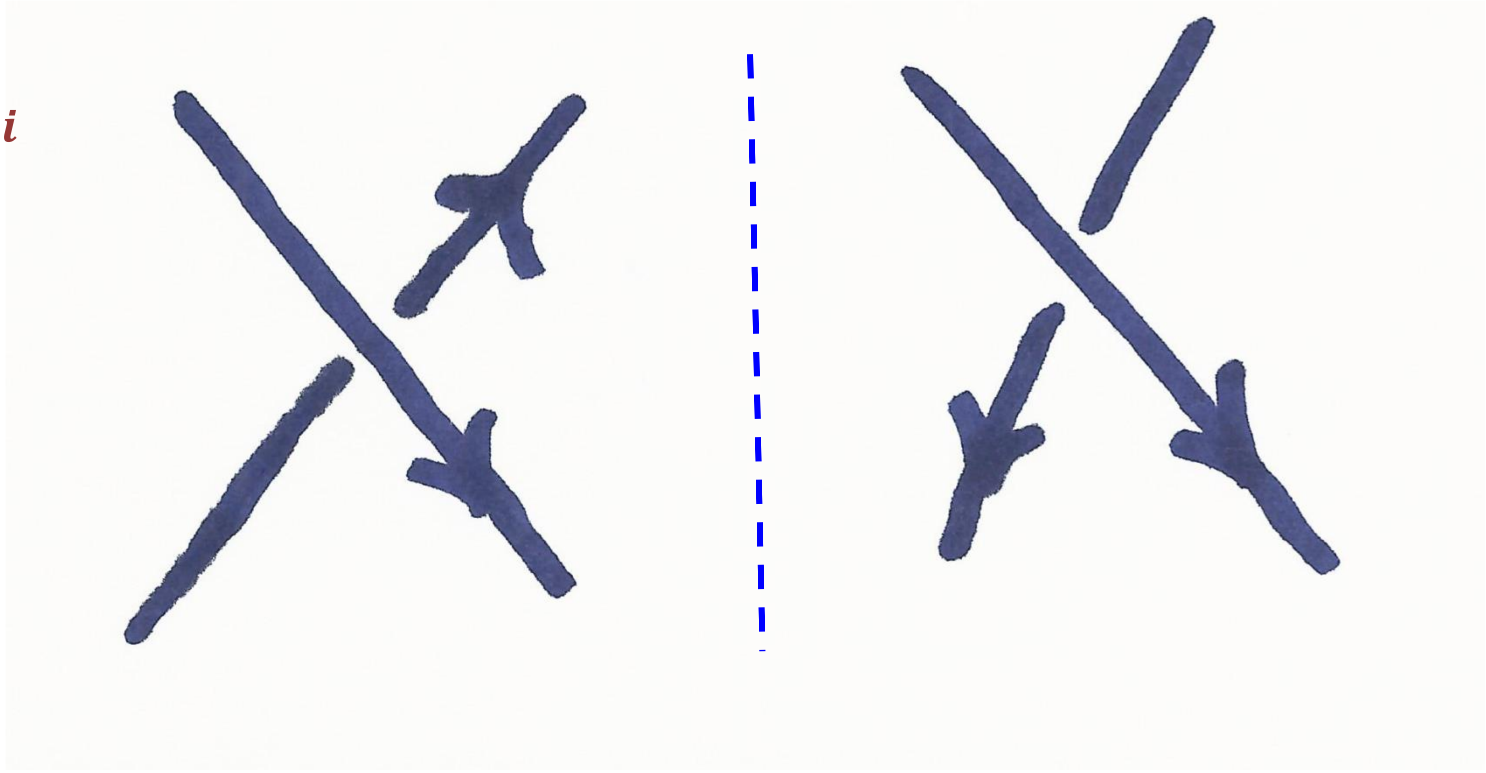


$$M = \begin{matrix} & r_1 & r_2 & r_3 & r_4 & r_5 \\ c_1 & & & & & \\ c_2 & & & & & \\ c_3 & & & & & \end{matrix}$$

Alexander Polynomial

c_i



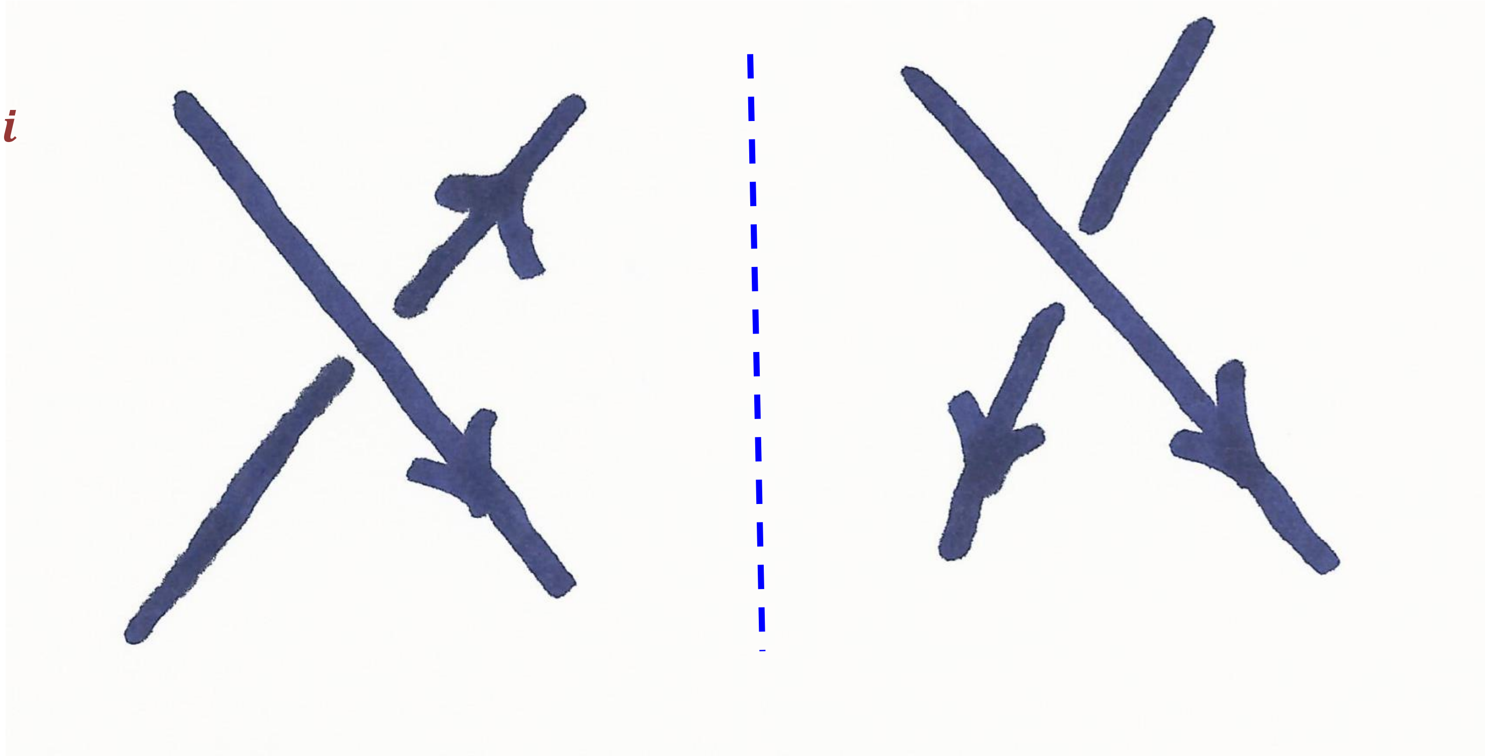
Alexander Polynomial

- Fill Matrix: For each crossing c_i, *(row i)*

- use the following pictures to fill the columns of its 4 neighboring regions

- and 0 in all other entries

c_i



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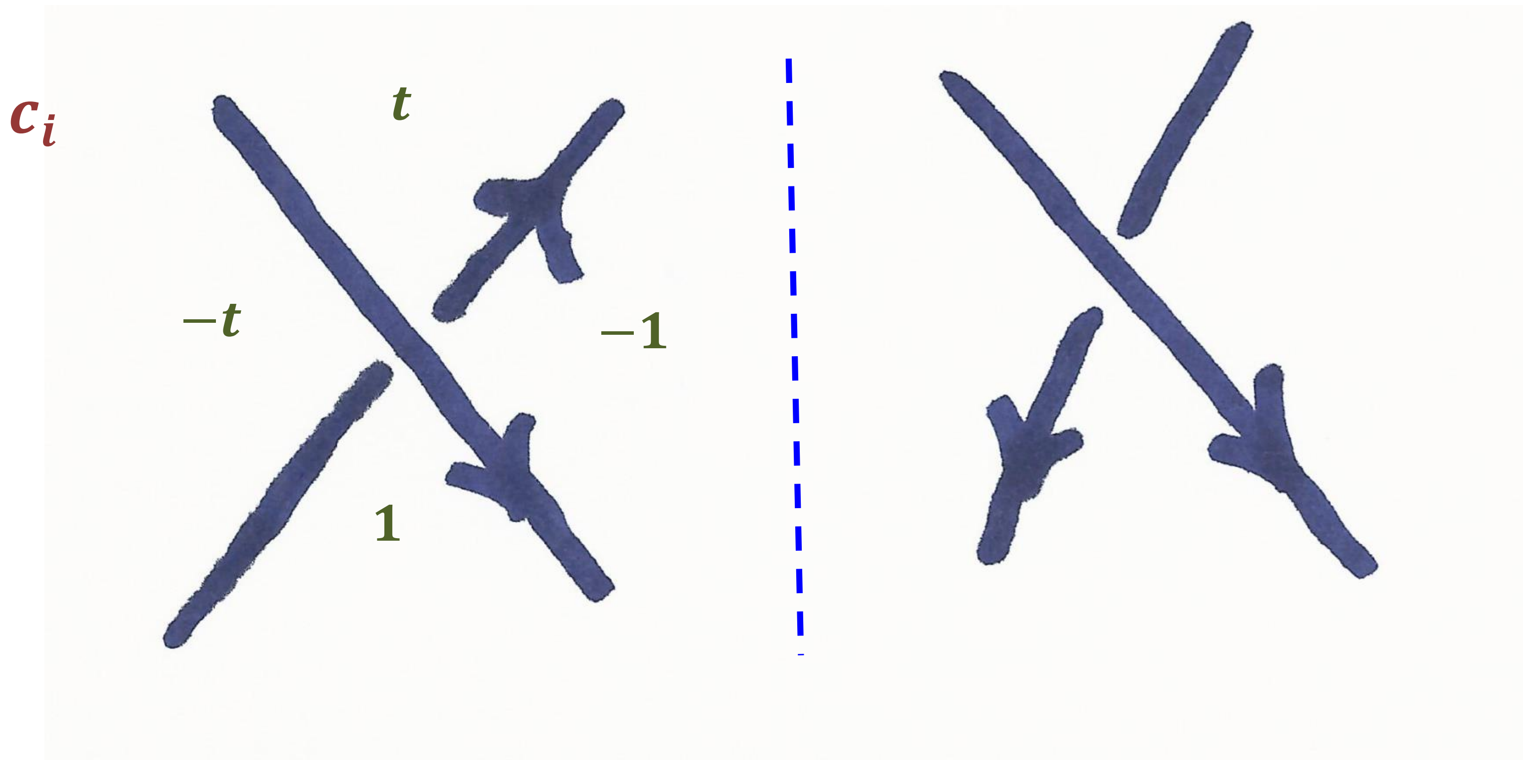

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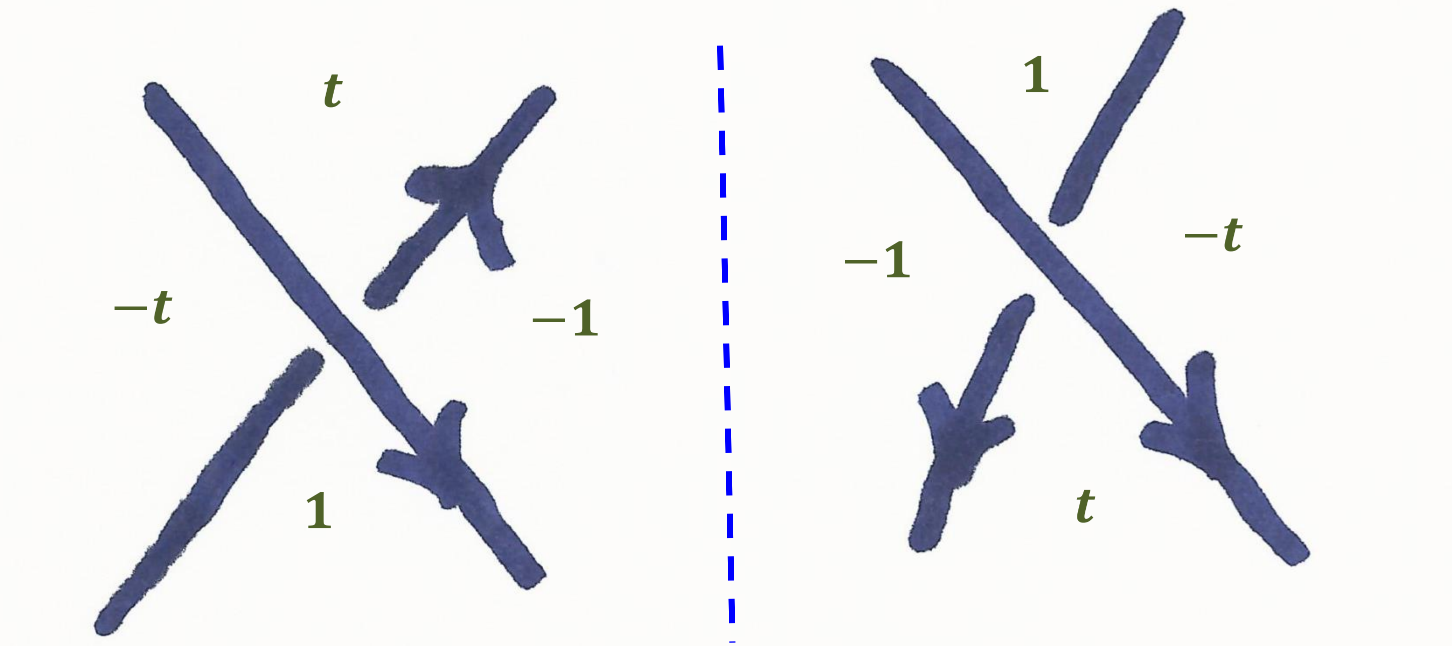

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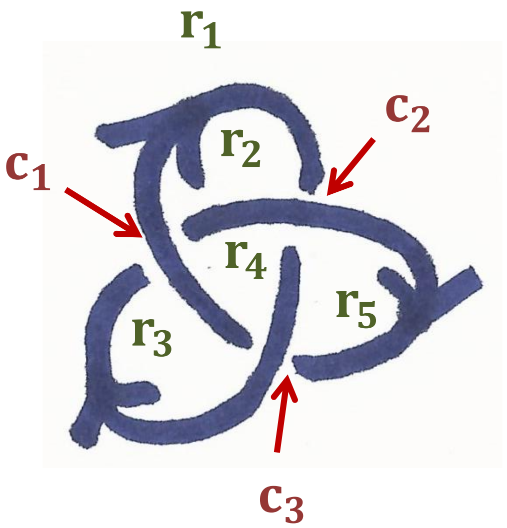

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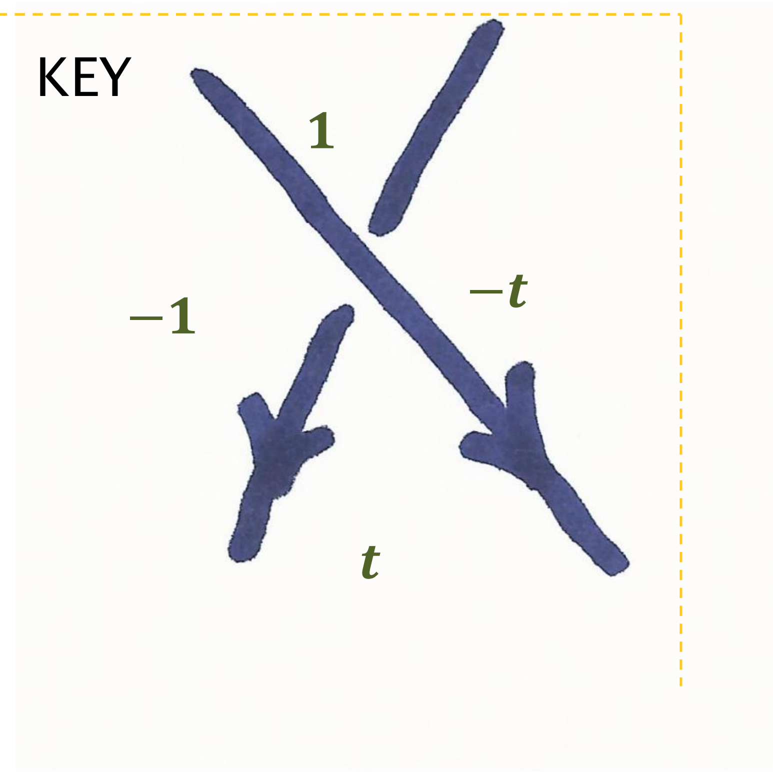

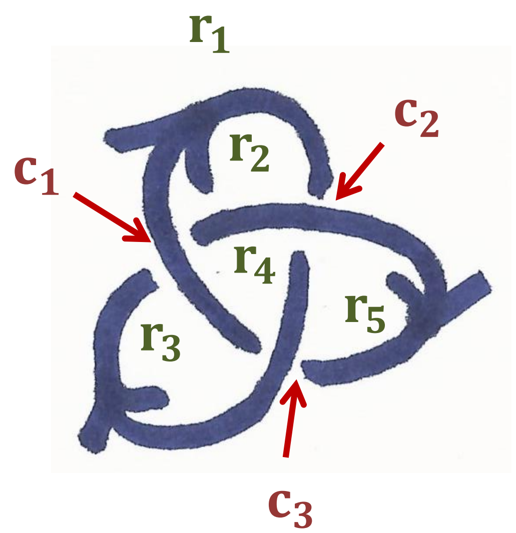

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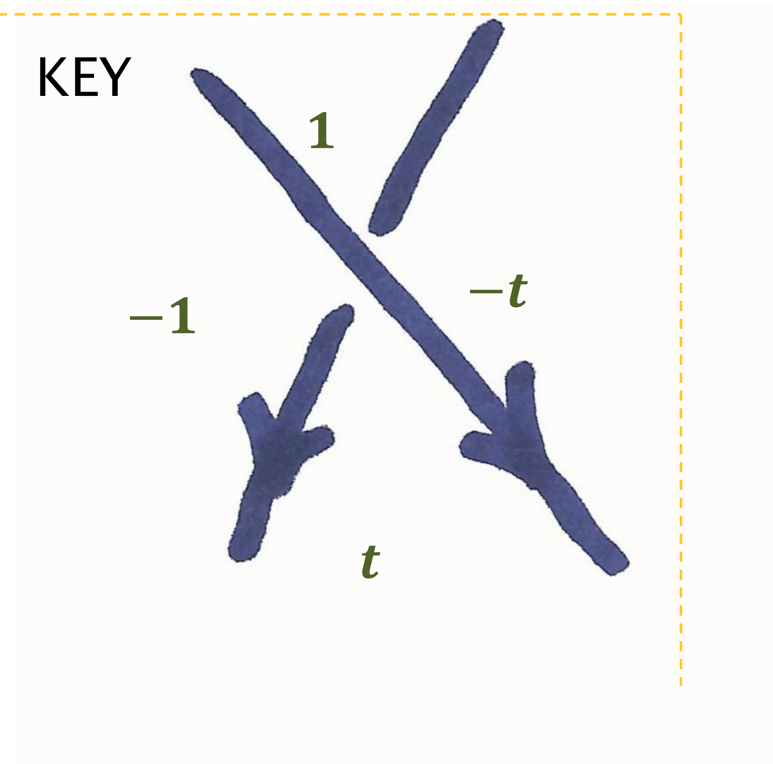
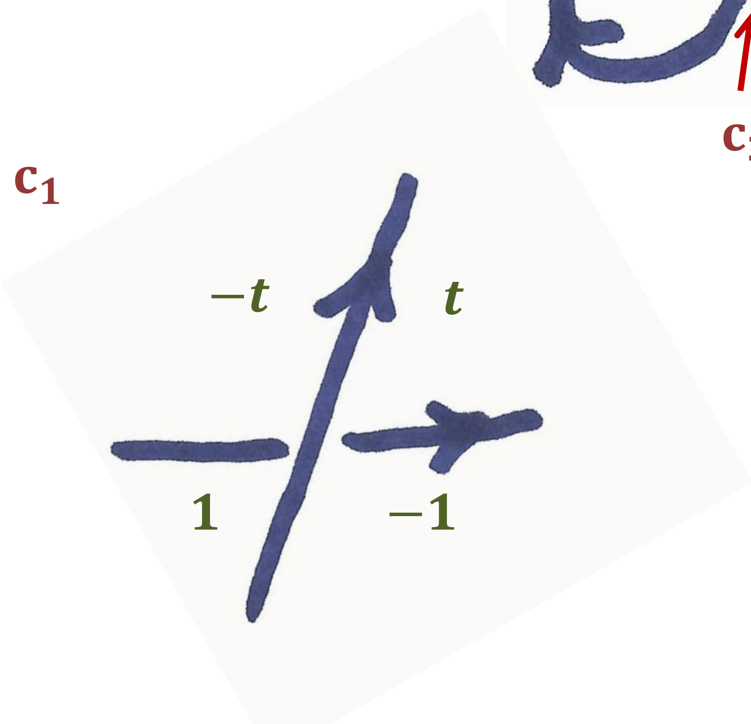
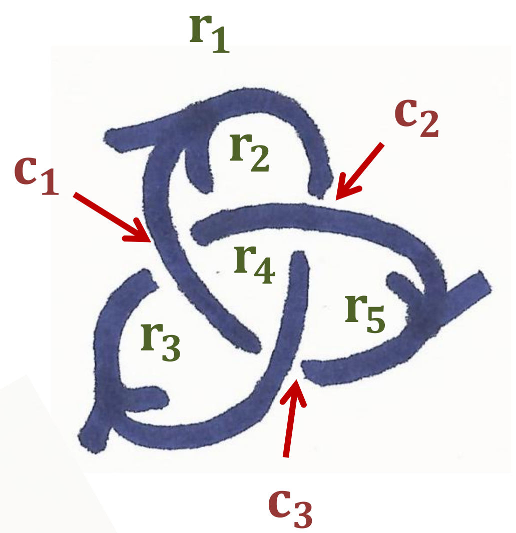
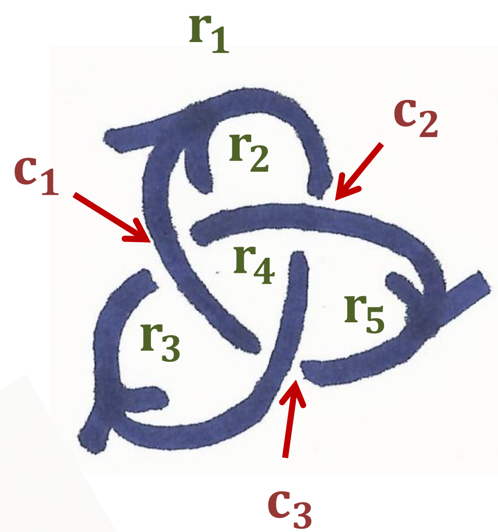






Alexander Polynomial

$$M = \begin{matrix} \\ c_1 \\ c_2 \\ c_3 \end{matrix} \begin{matrix} r_1 & r_2 & r_3 & r_4 & r_5 \\ \left(\right. & & & & \left. \right) \end{matrix}$$



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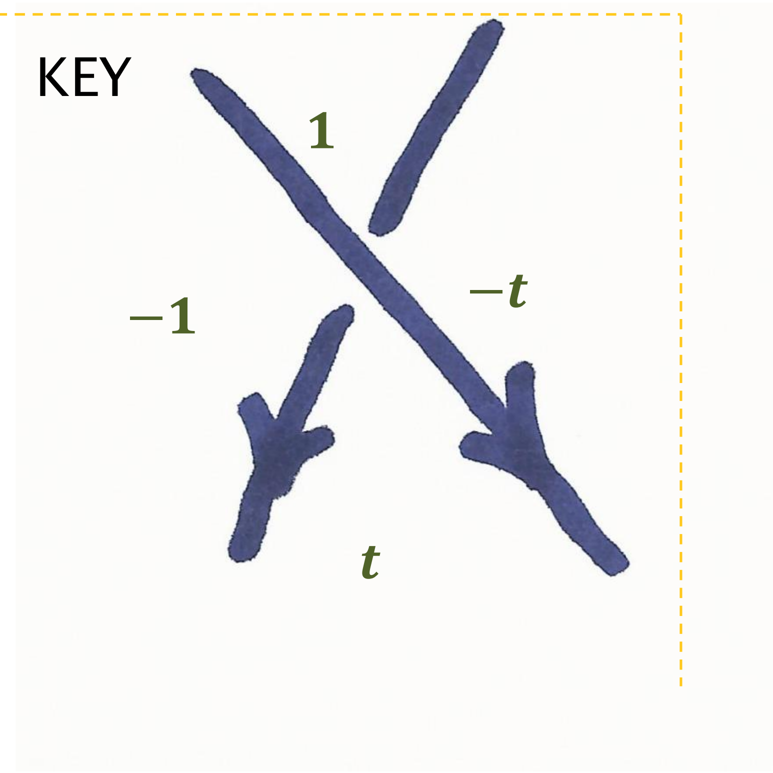

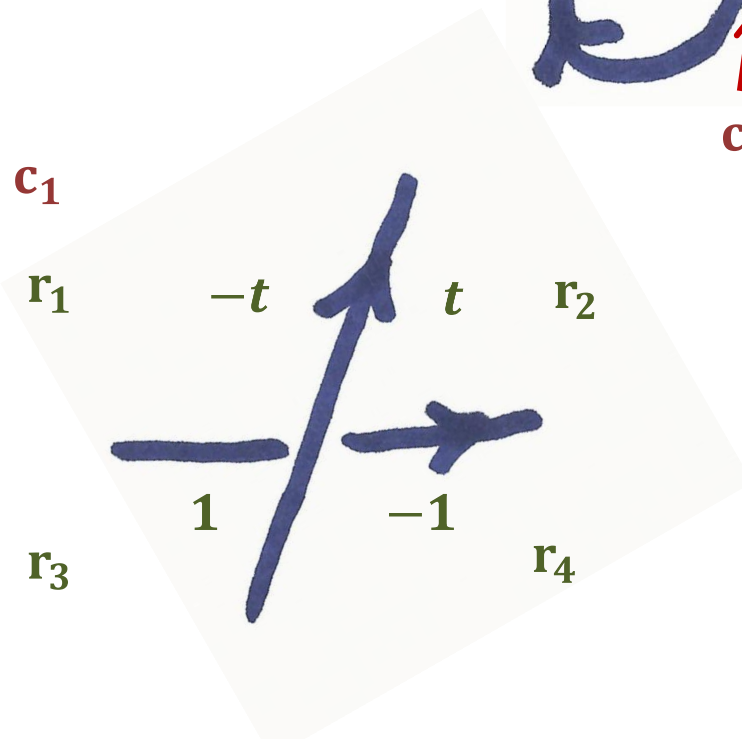

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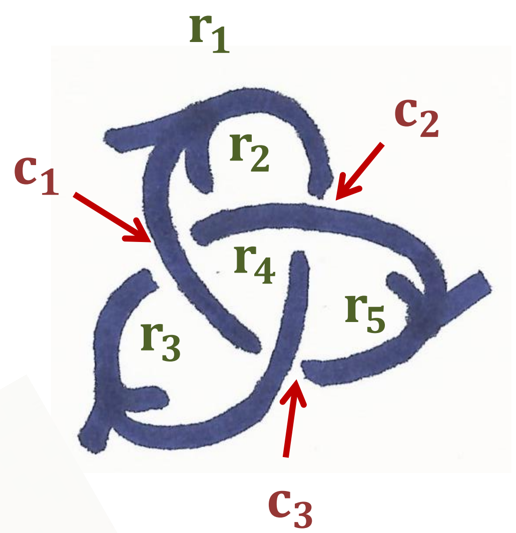
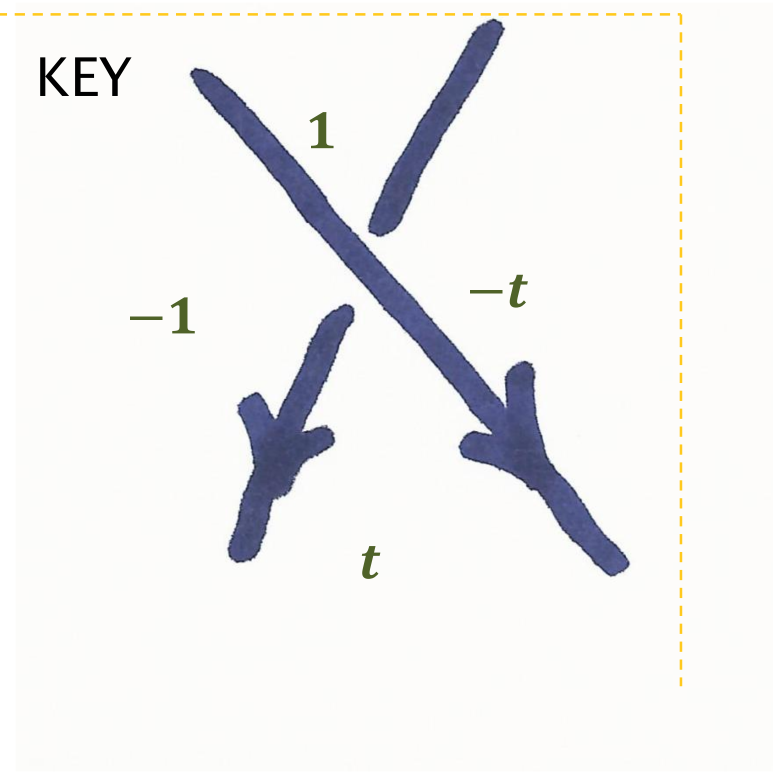
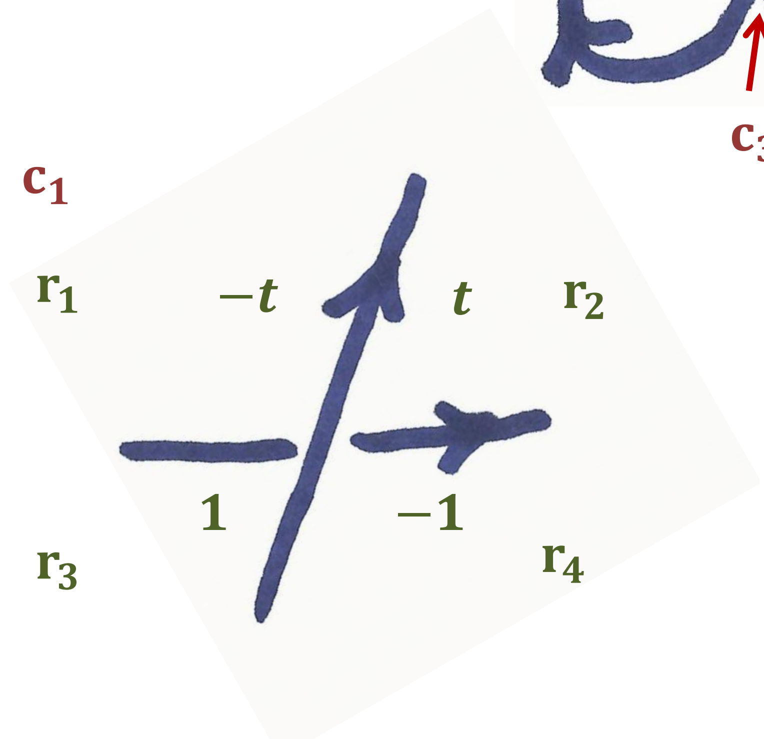
$$M = \begin{matrix} & r_1 & r_2 & r_3 & r_4 & r_5 \\ c_1 & & & & & 0 \\ c_2 & & & & & \\ c_3 & & & & & \end{matrix}$$







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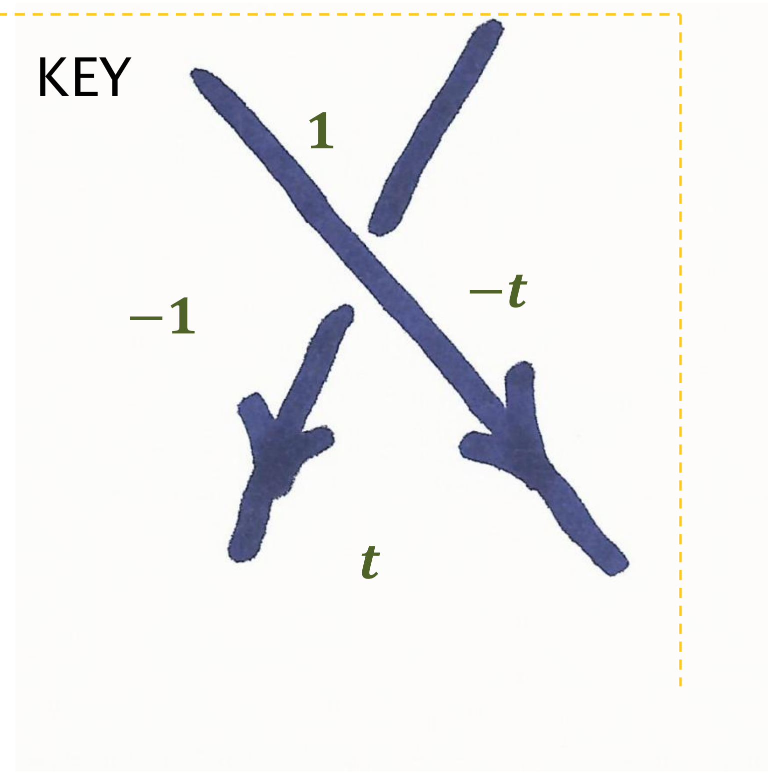
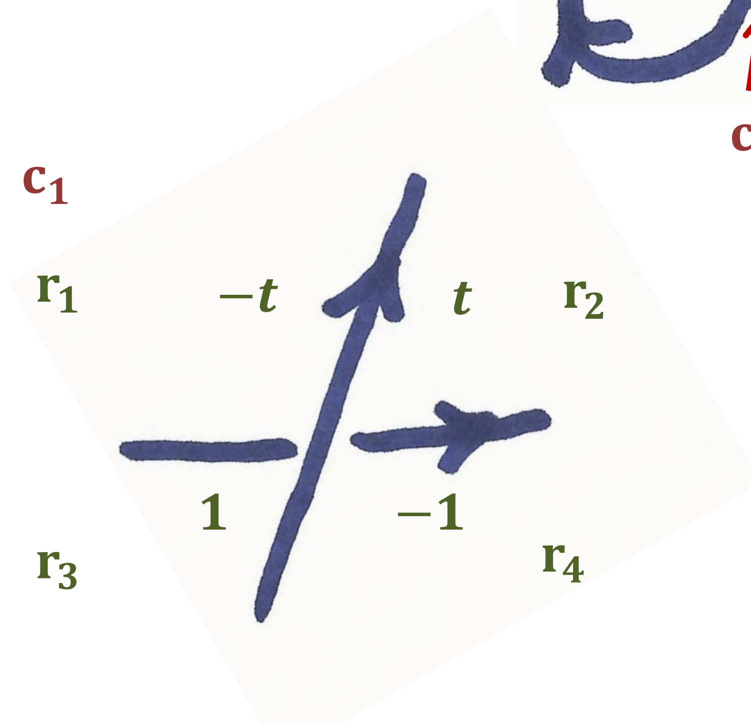
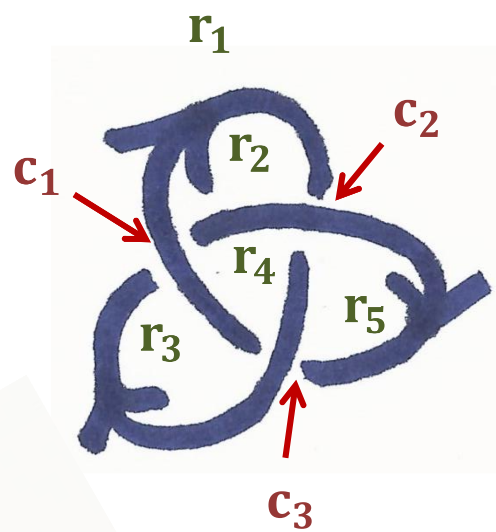


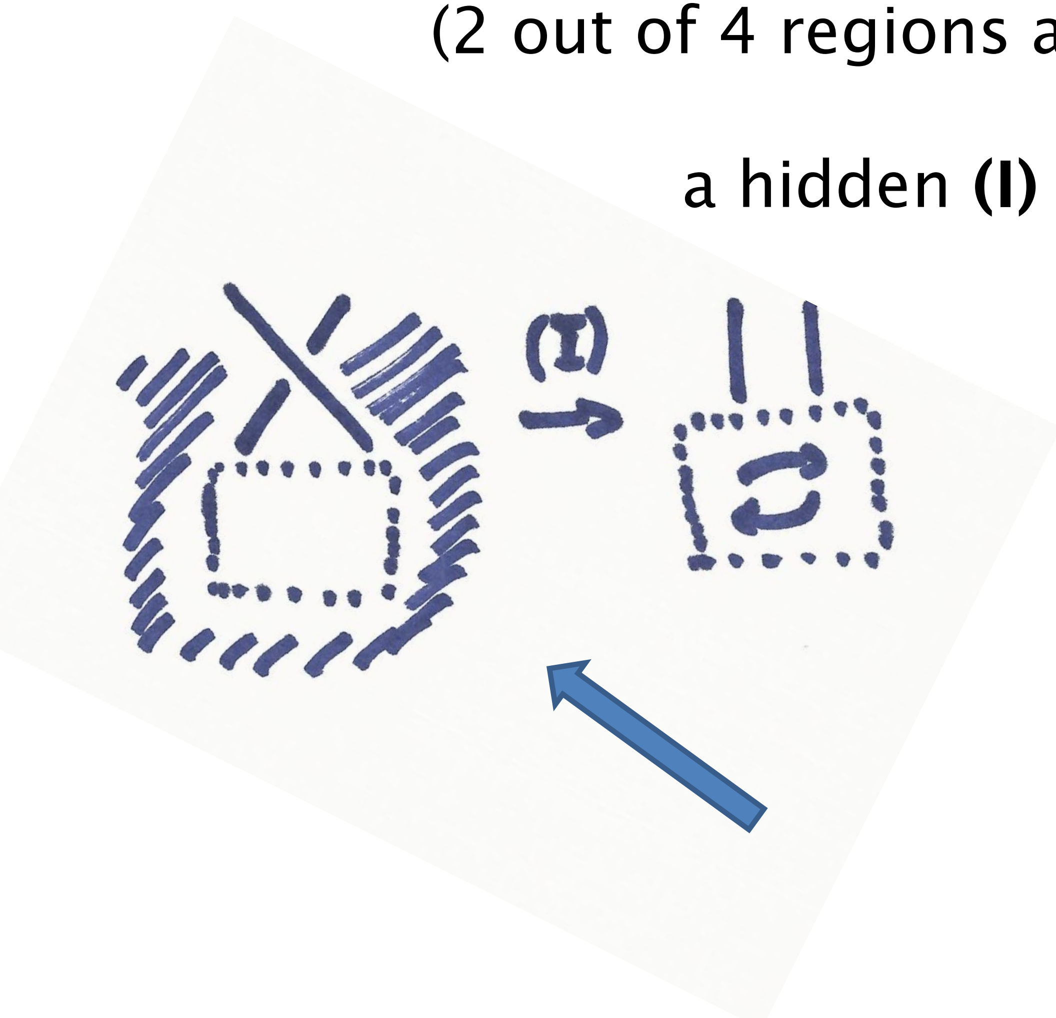




WARNING!

Crossing with 3 different regions
(2 out of 4 regions are connected)

a hidden **(I) move**:



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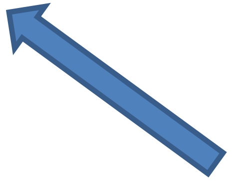

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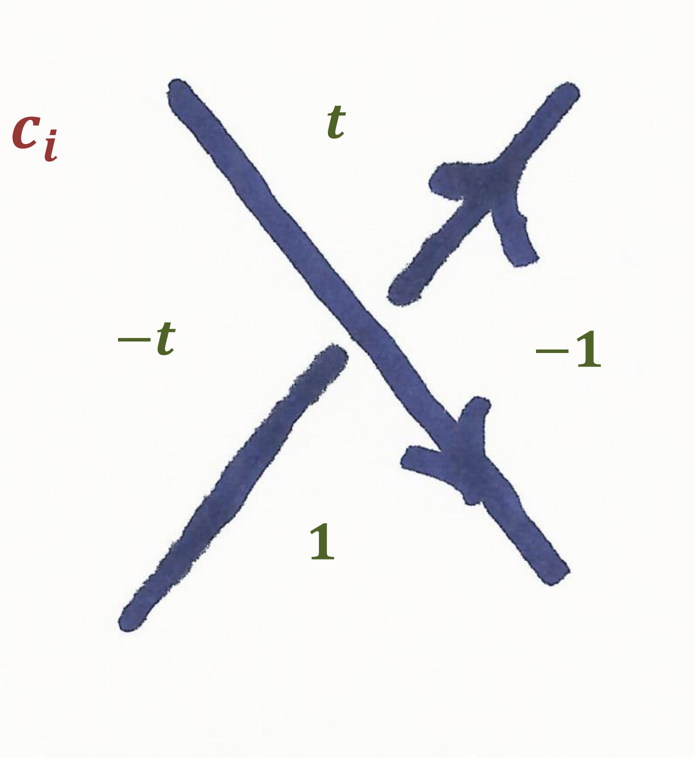

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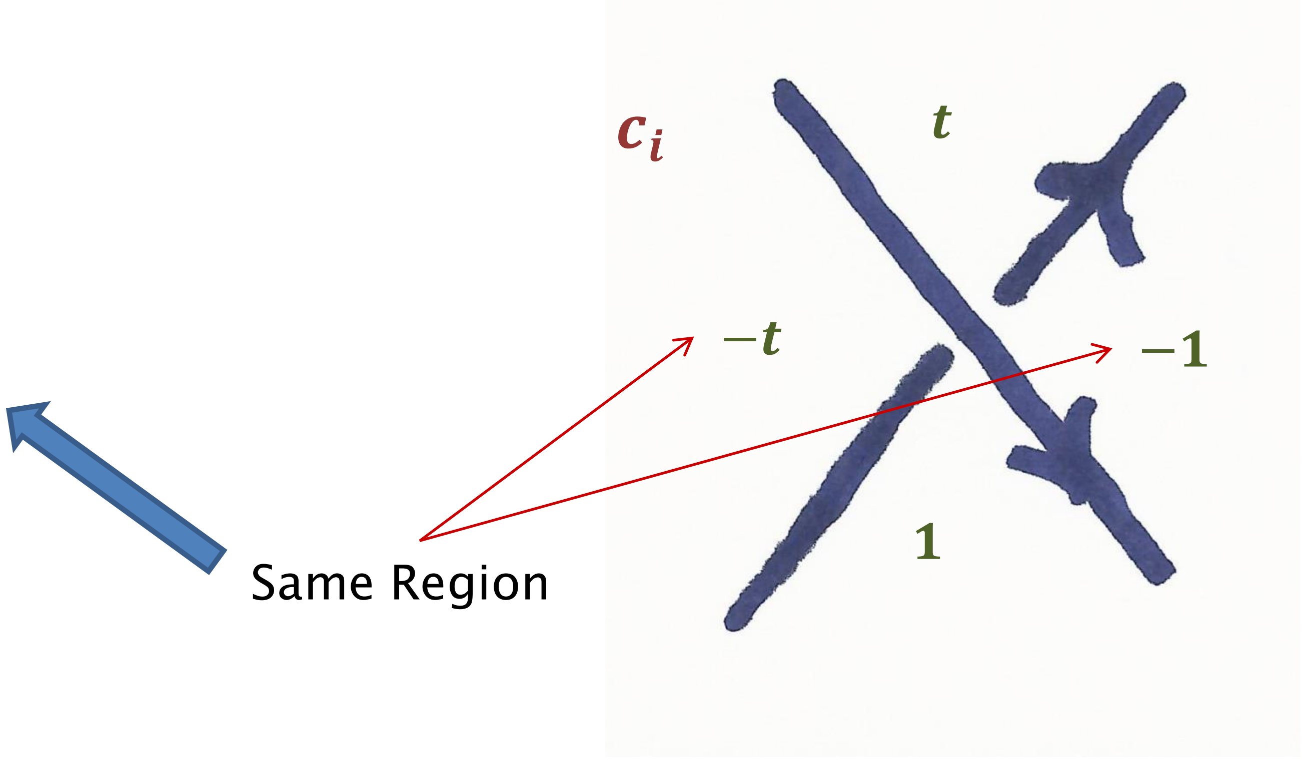

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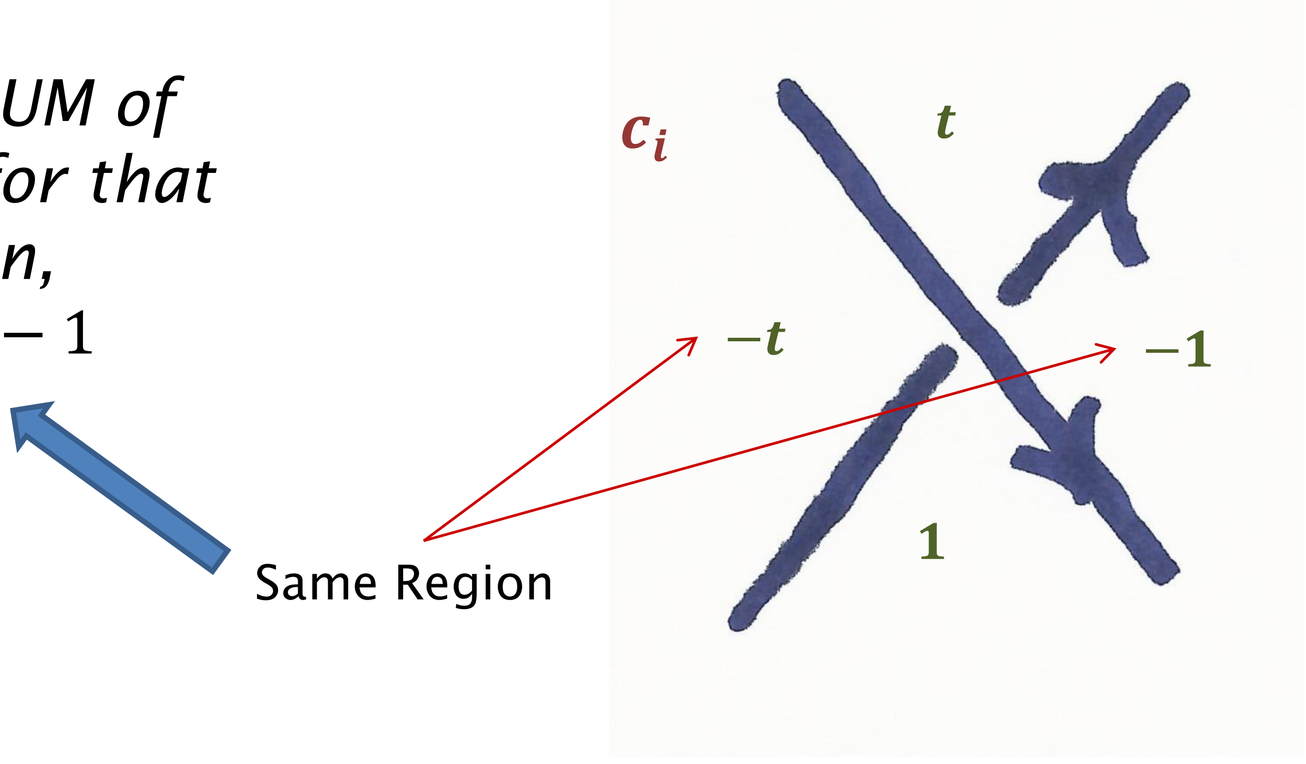
Put the SUM of numbers for that region,
i.e. $-t-1$



Alexander Polynomial

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- You should have a matrix like this:

$$M = \begin{pmatrix} -t & t & 1 & -1 & 0 \\ -t & 1 & 0 & -1 & t \\ -t & 0 & t & -1 & 1 \end{pmatrix}$$

- Choose two <u>neighboring regions</u>, for example r_4, r_5
- Delete their columns, to get a square matrix

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$$M = \begin{pmatrix} -t & t & 1 & -1 & 0 \\ -t & 1 & 0 & -1 & t \\ -t & 0 & t & -1 & 1 \end{pmatrix} \quad \Rightarrow \quad M' = \begin{pmatrix} -t & t & 1 \\ -t & 1 & 0 \\ -t & 0 & t \end{pmatrix}$$

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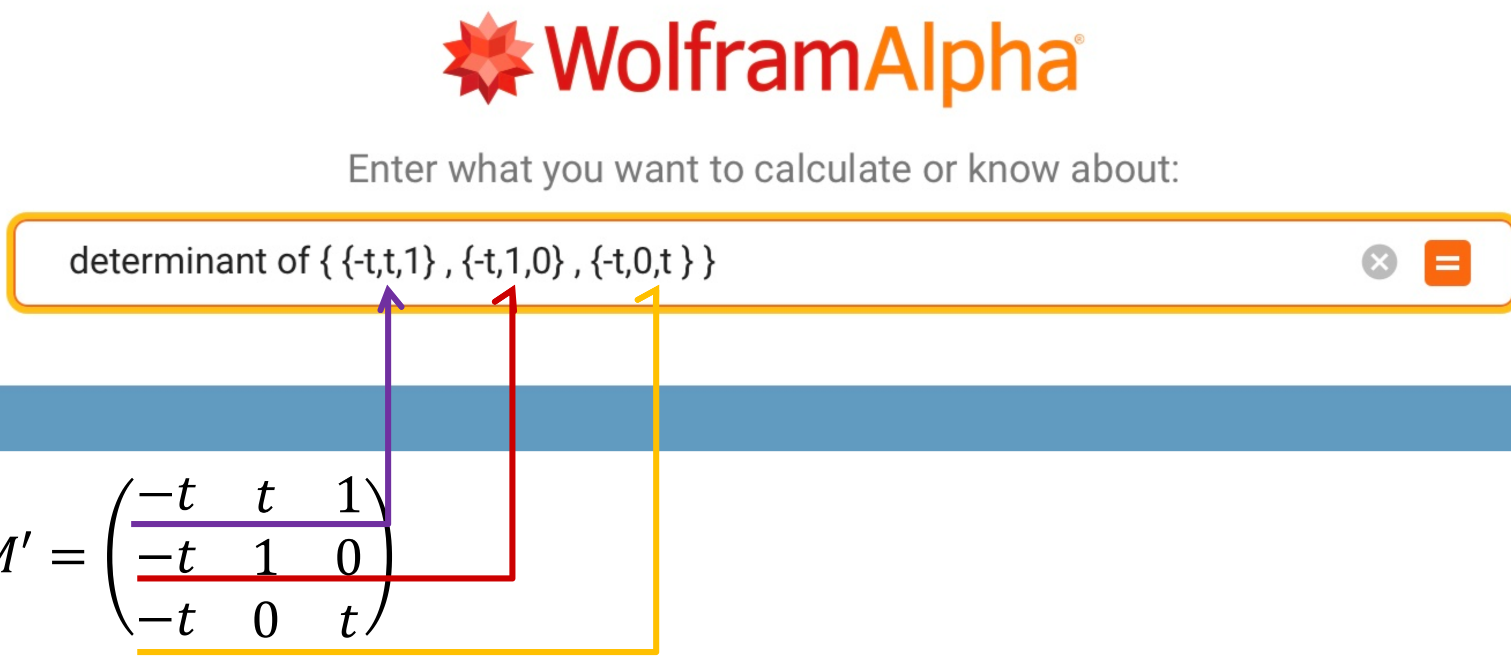

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  - Take powers of t until a constant appears

$$-7t^5 - 3t^3 + \mathbf{5t^2} = t^2(-7t^3 - 3t + 5)$$
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(Alexanders's Theorem, 1928)
The procedure described above gives Knot Invariants

Assumptions for
Alexander Polynomial

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- First Choose an *orientation*
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- Name your regions ($r_1, r_2, \cdots, r_{n+2}$) and crossings ($c_1, c_2, \cdots, c_n$)
- Draw a Matrix with *n* rows and *n+2* columns

$$
\begin{array}{c c}
 & \begin{array}{ccccc} r_1 & r_2 & r_3 & r_4 & r_5 \end{array} \\
M = \begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array} & \left(\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \end{array} \right)
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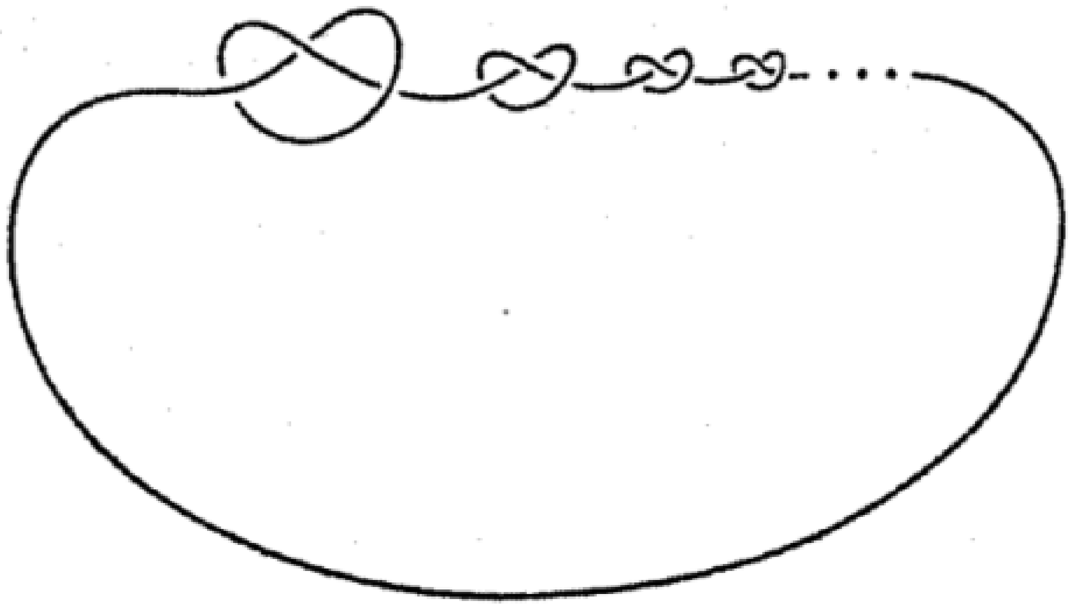

Invariants:

Assign a "number" to each Knot called it's <u>invariant</u> so that

- Equivalent knots get the same number

Then, if two knots get different numbers, then they're not equivalent.

- No guarantee that an assignment tells apart all Knots
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crossings	2	3	4	5	6	7	8	9	10	11
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- Tells knots of $n < 9$ crossings apart

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⬇

Need to check if Alexander Polynomial doesn't change after a Reidmeister move!

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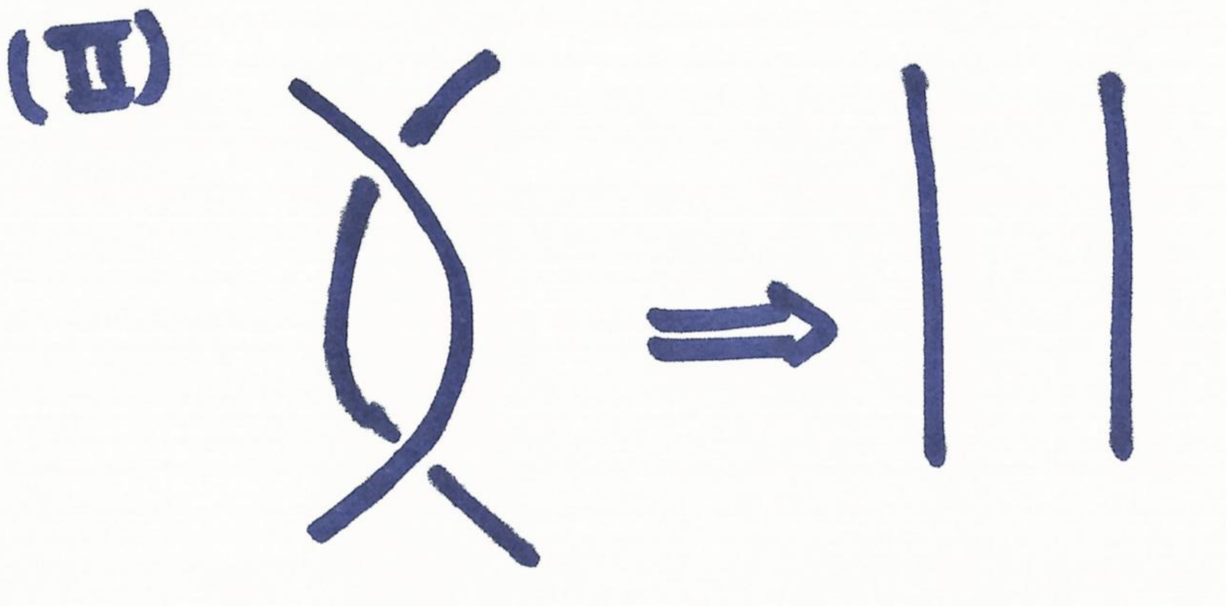

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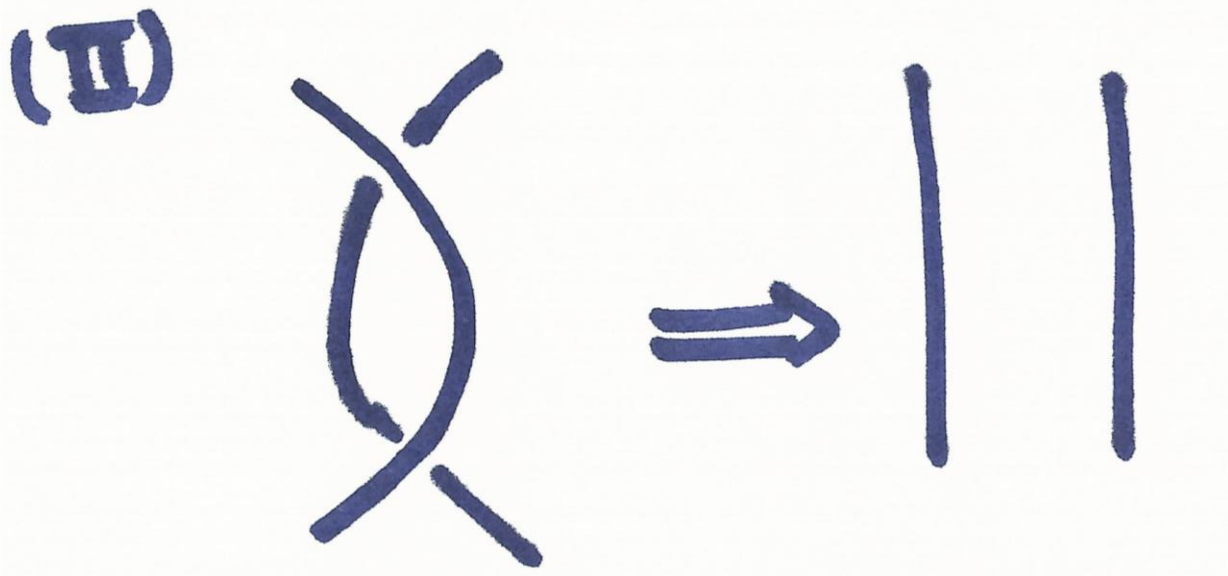

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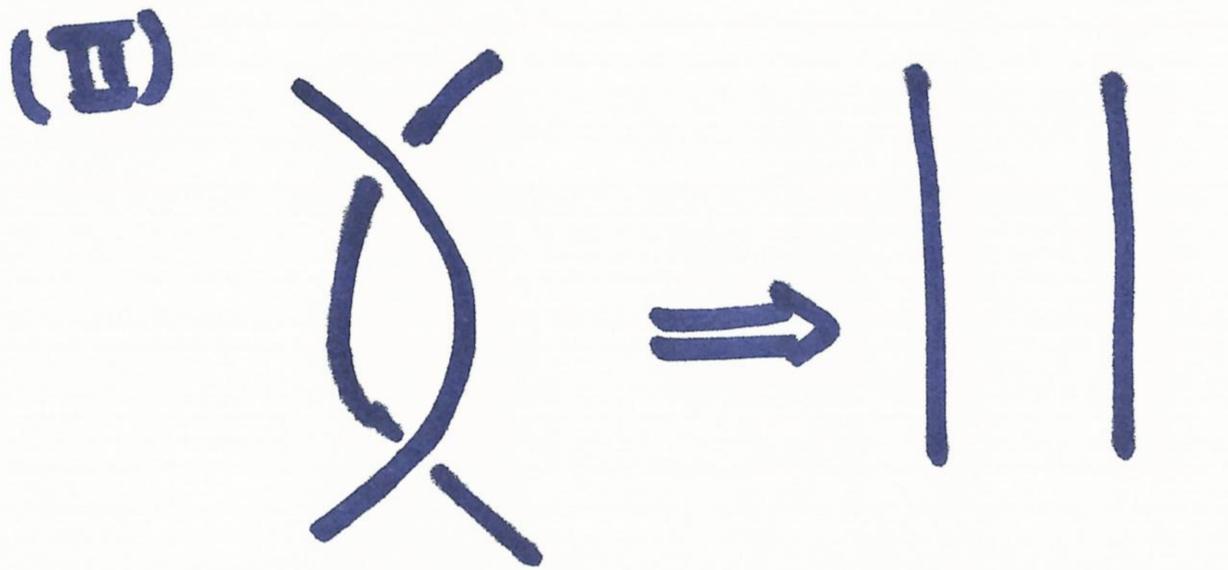

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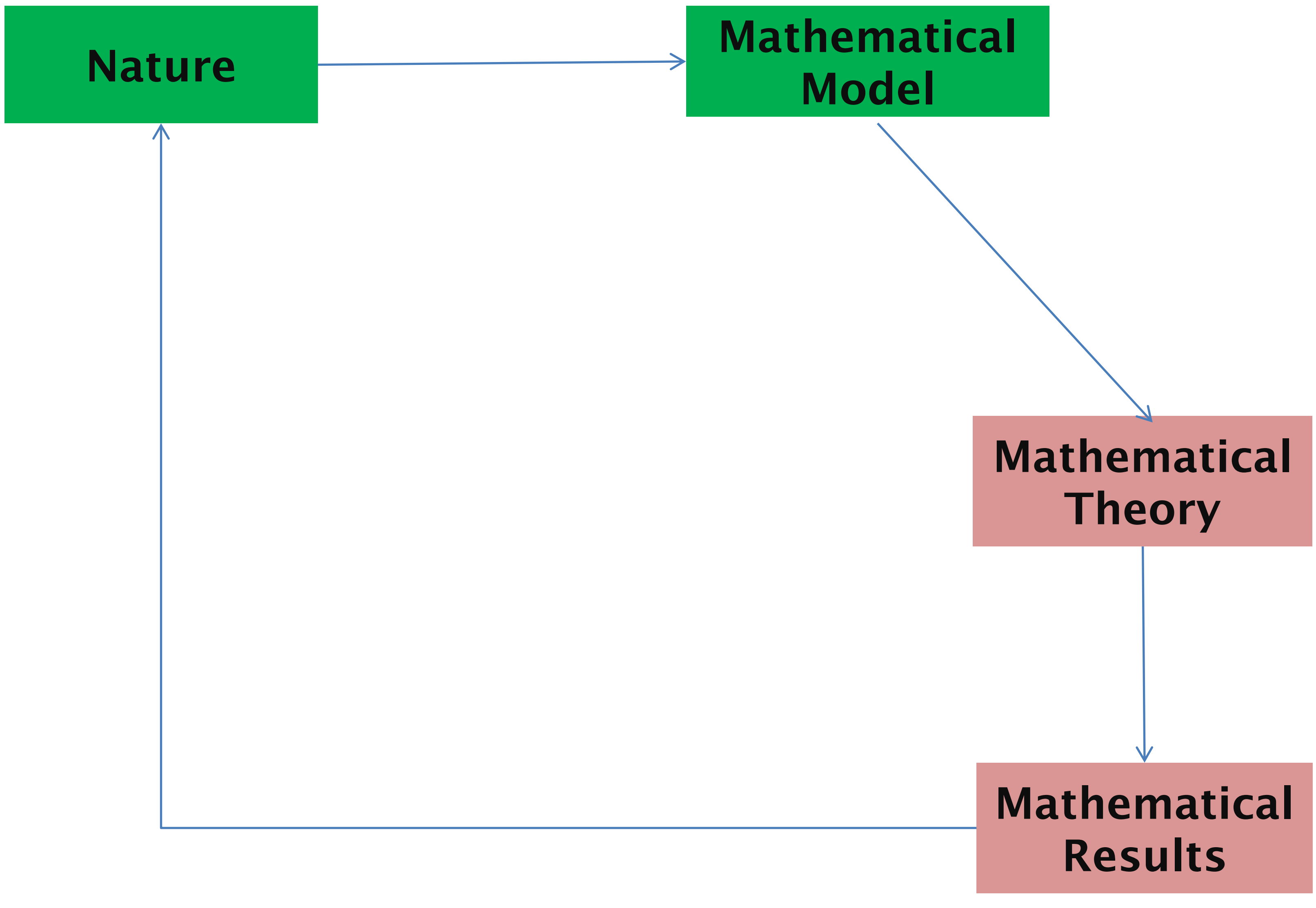
Nature
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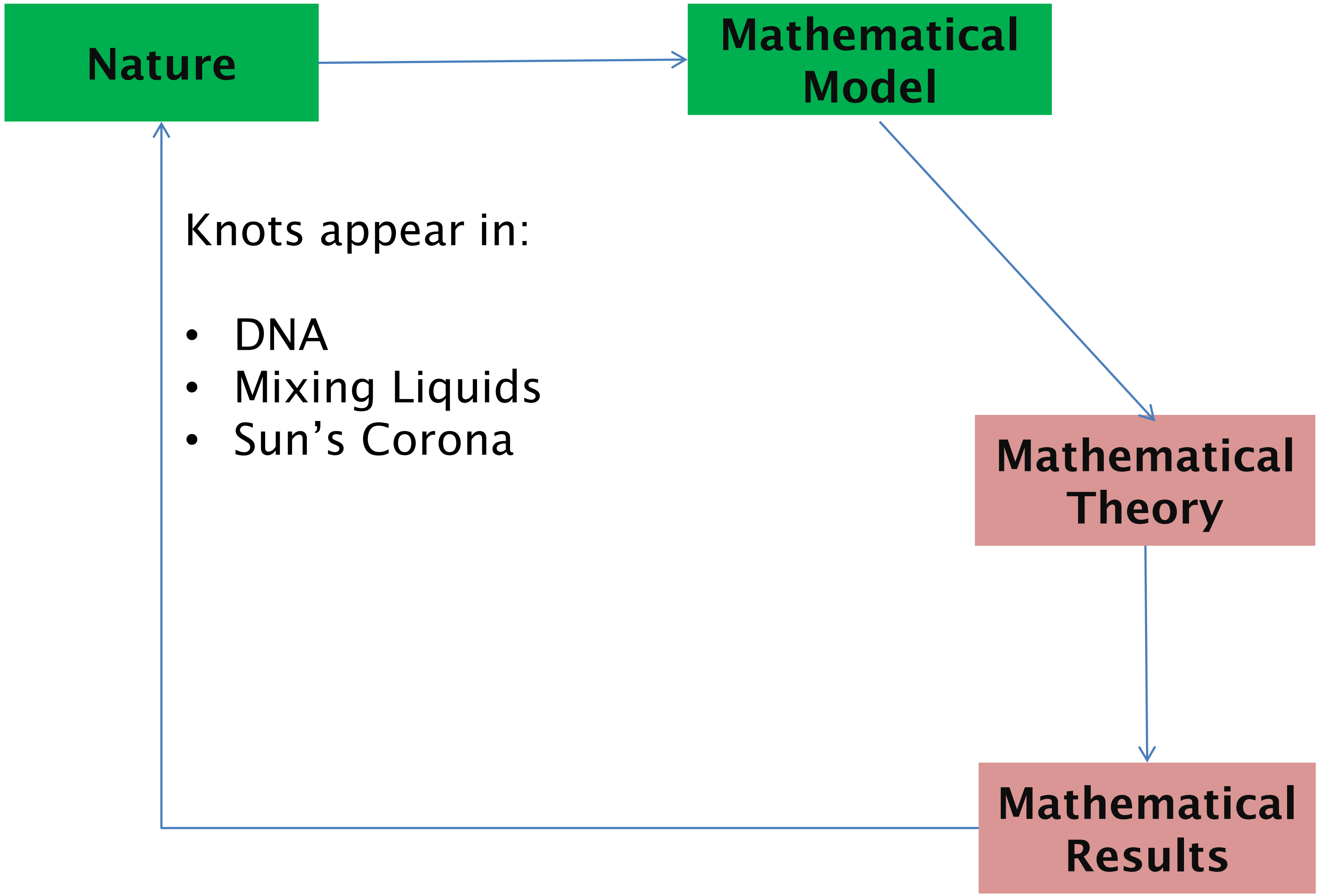
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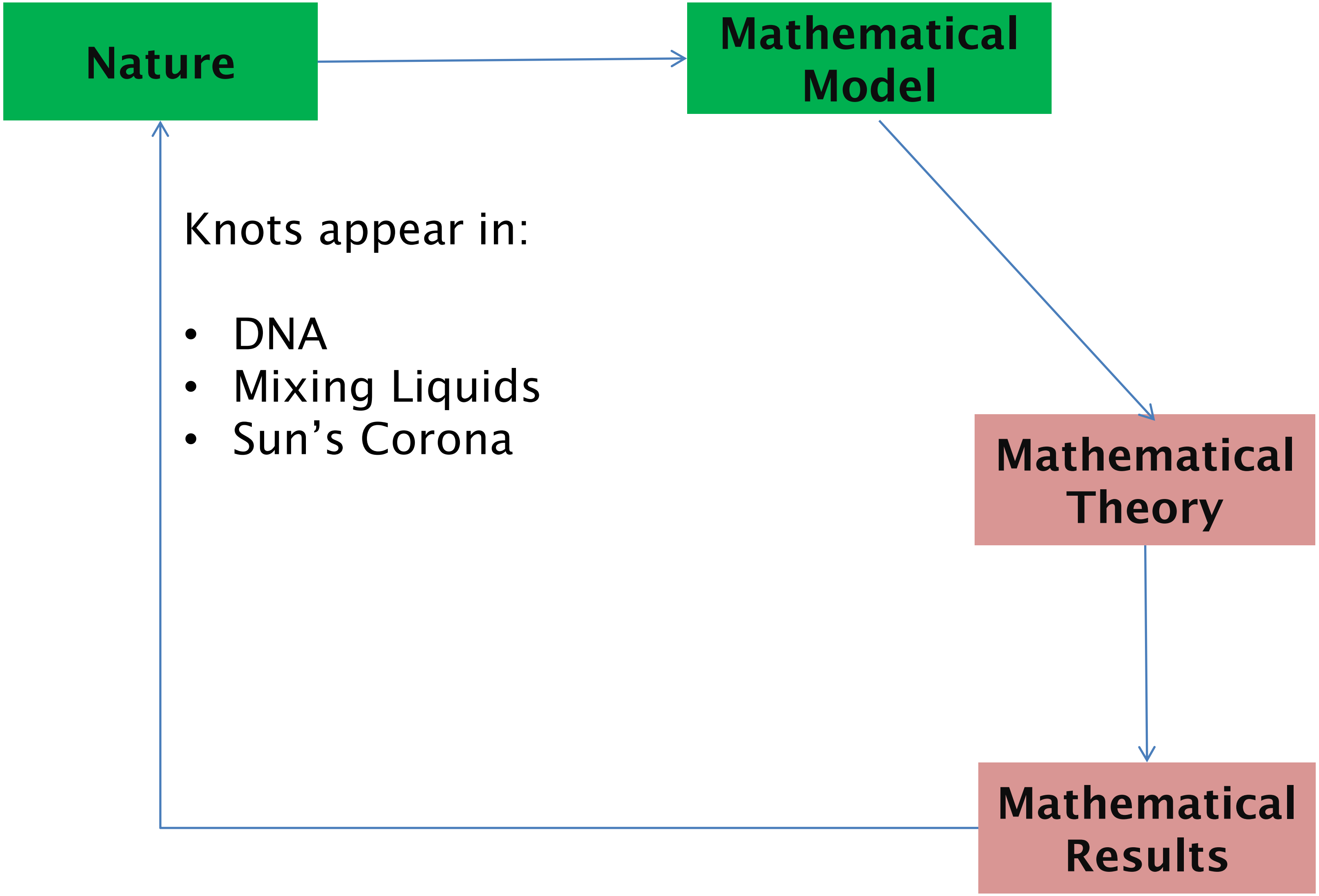
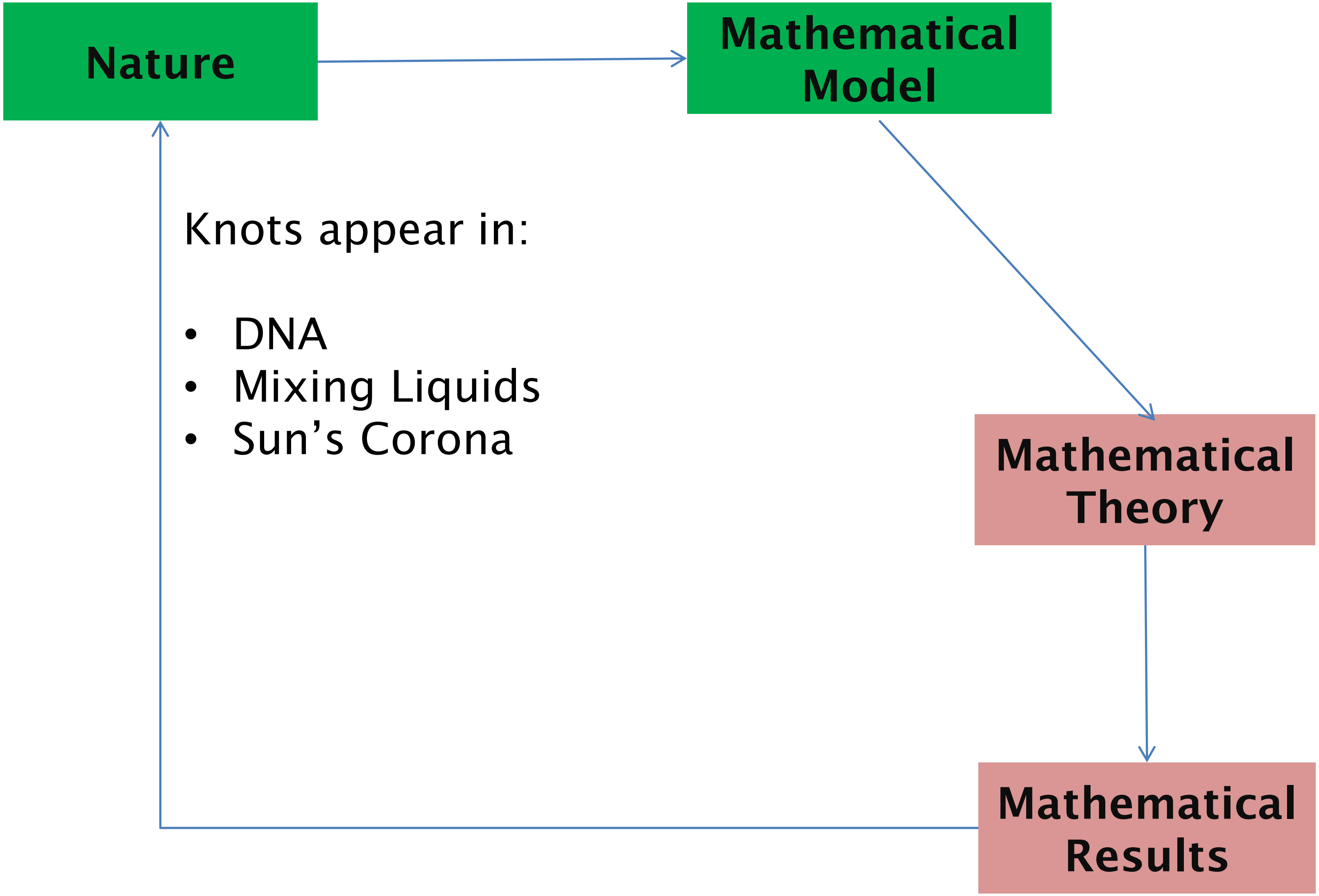
Mathematical Theory
Mathematical Results


Nature
Mathematical Model
Mathematical Theory
Mathematical Results
Knots appear in:
• DNA
• Mixing Liquids
• Sun’s Corona



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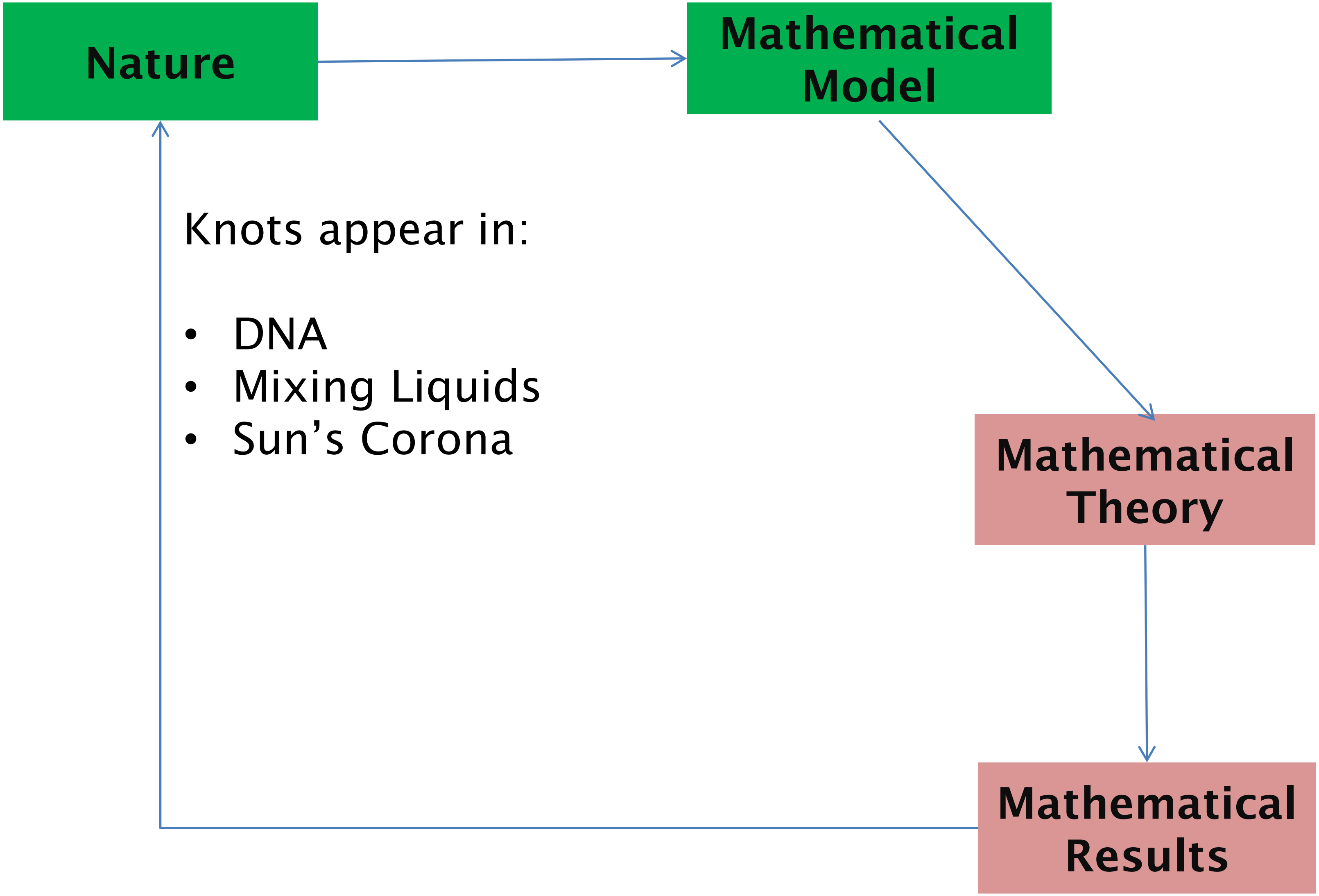

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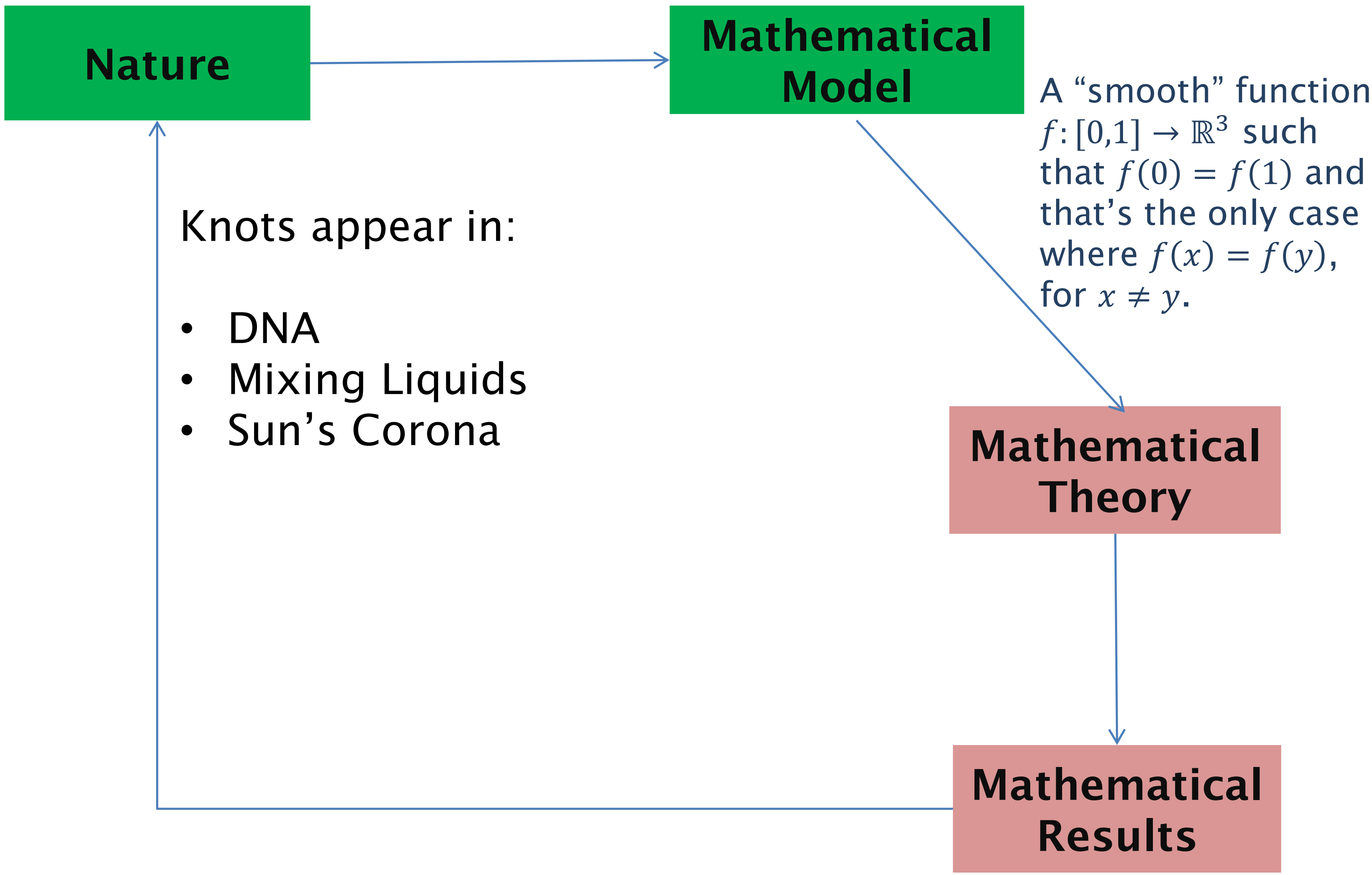

Nature
Mathematical Model
A "smooth" function $f:[0,1] \to \mathbb{R}^3$ such that $f(0) = f(1)$ and that's the only case where $f(x) = f(y)$, for $x \neq y$.
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Mathematical Theory
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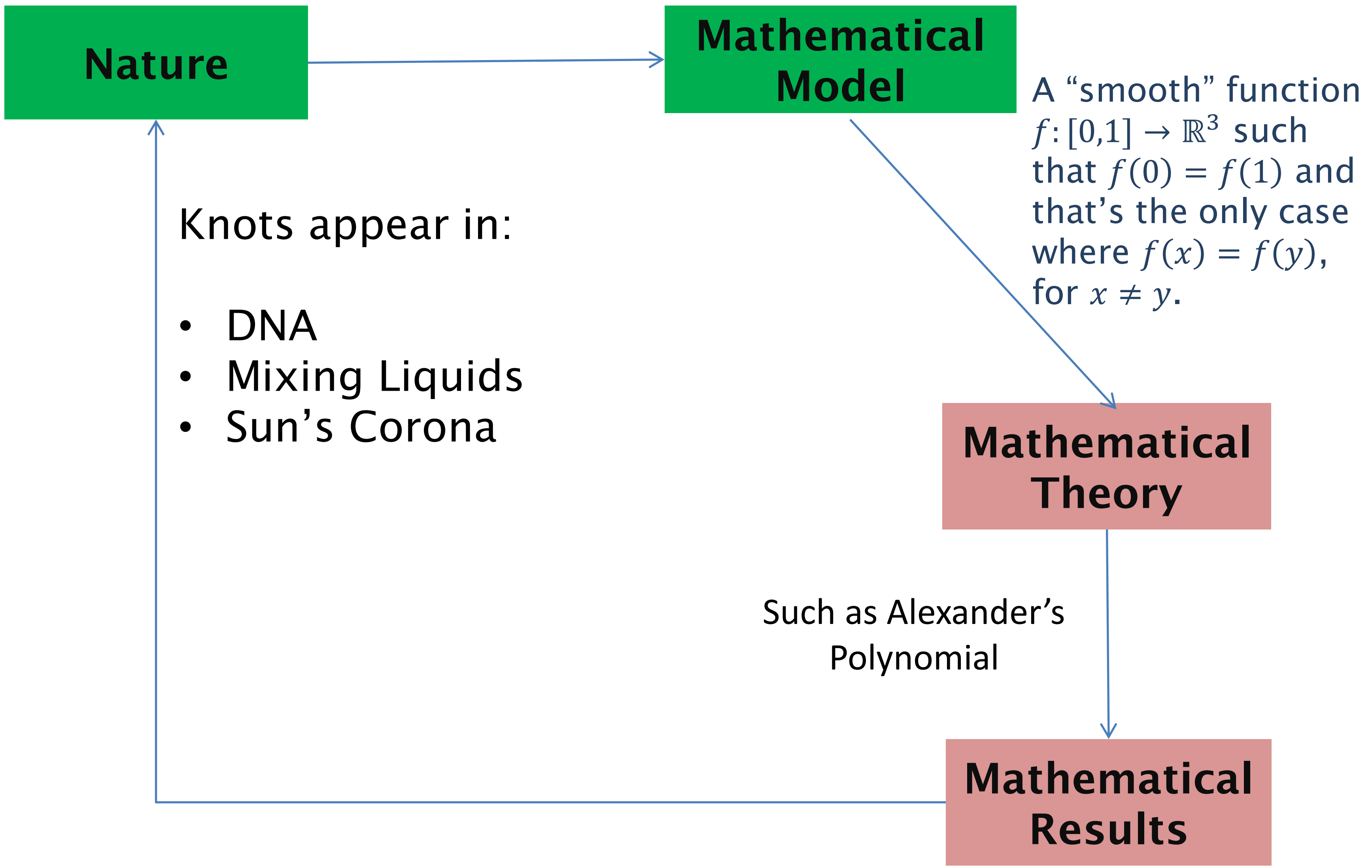

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Knots appear in:
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Mathematical Theory
Such as Alexander’s Polynomial
Mathematical Results

- *Good mathematics should connect with other good mathematics!*

Knots

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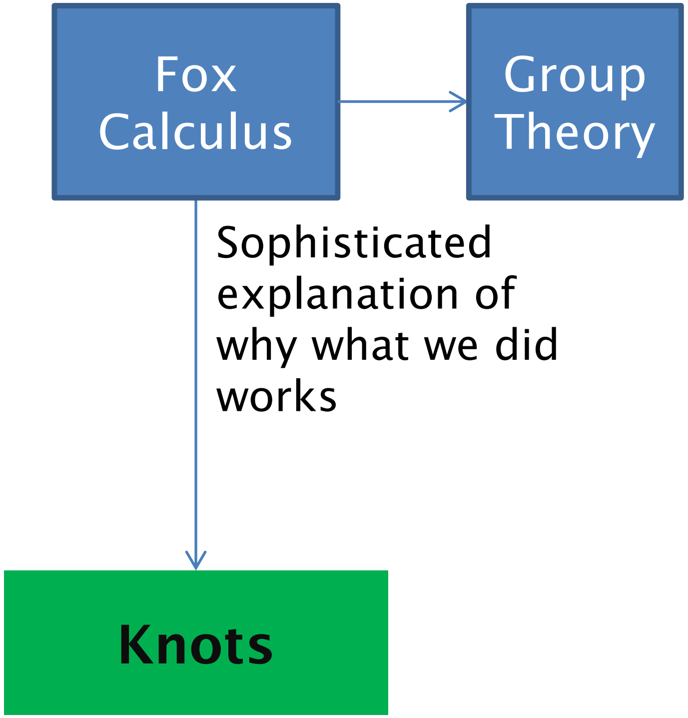

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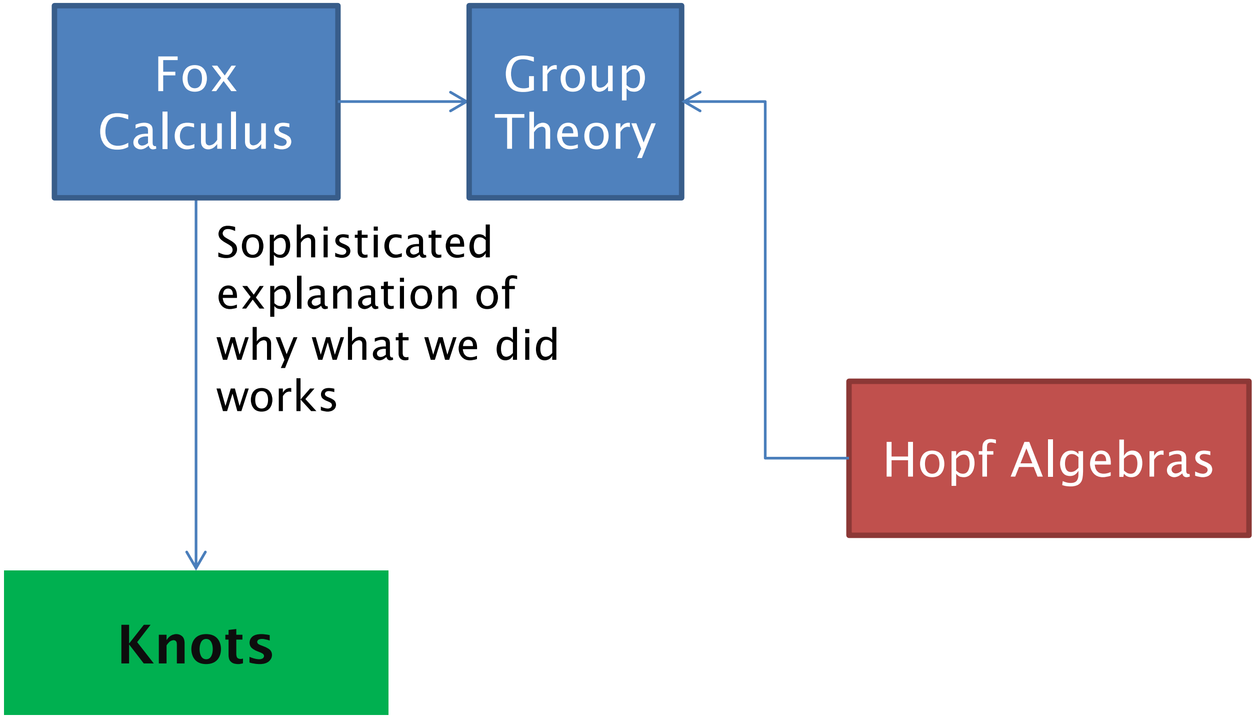

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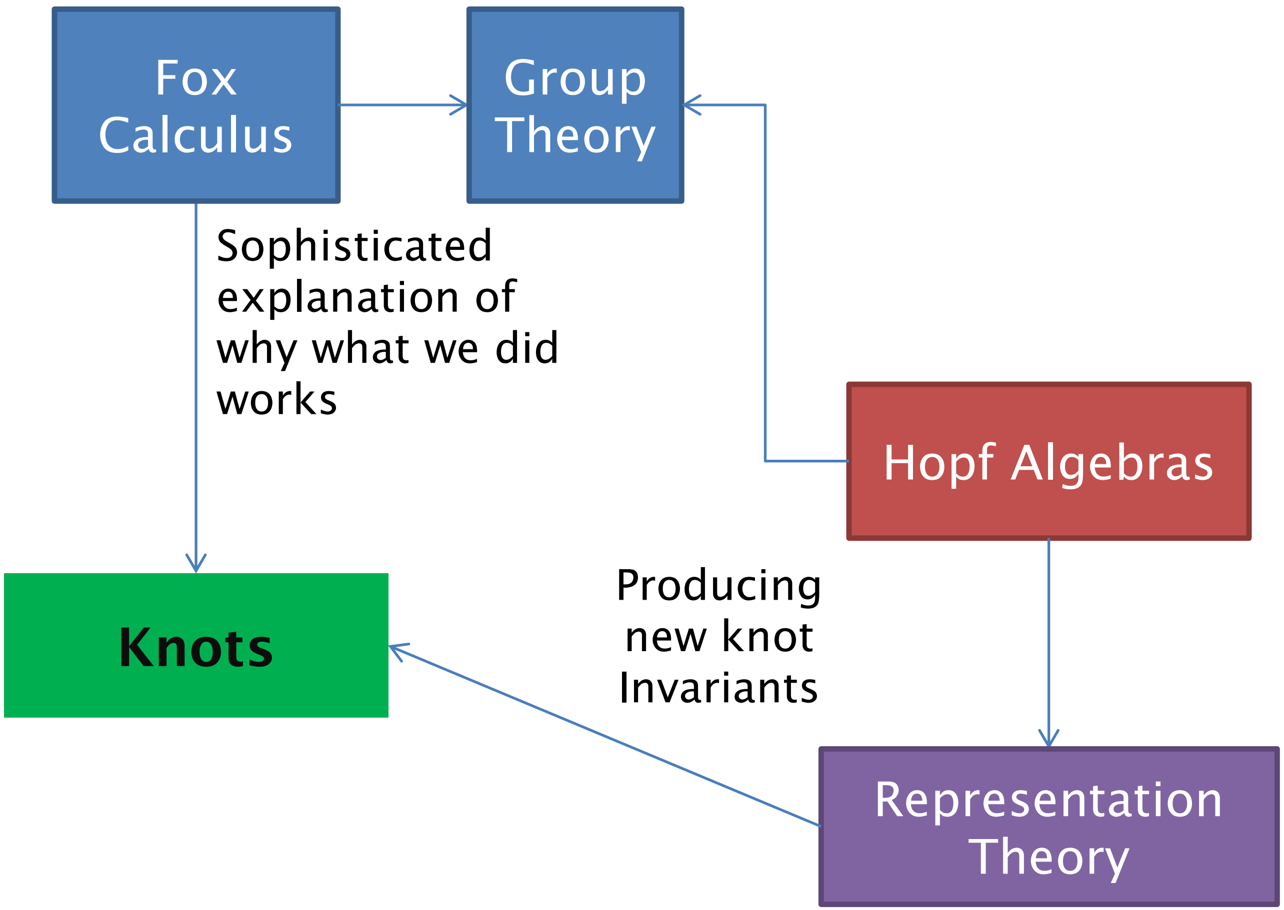

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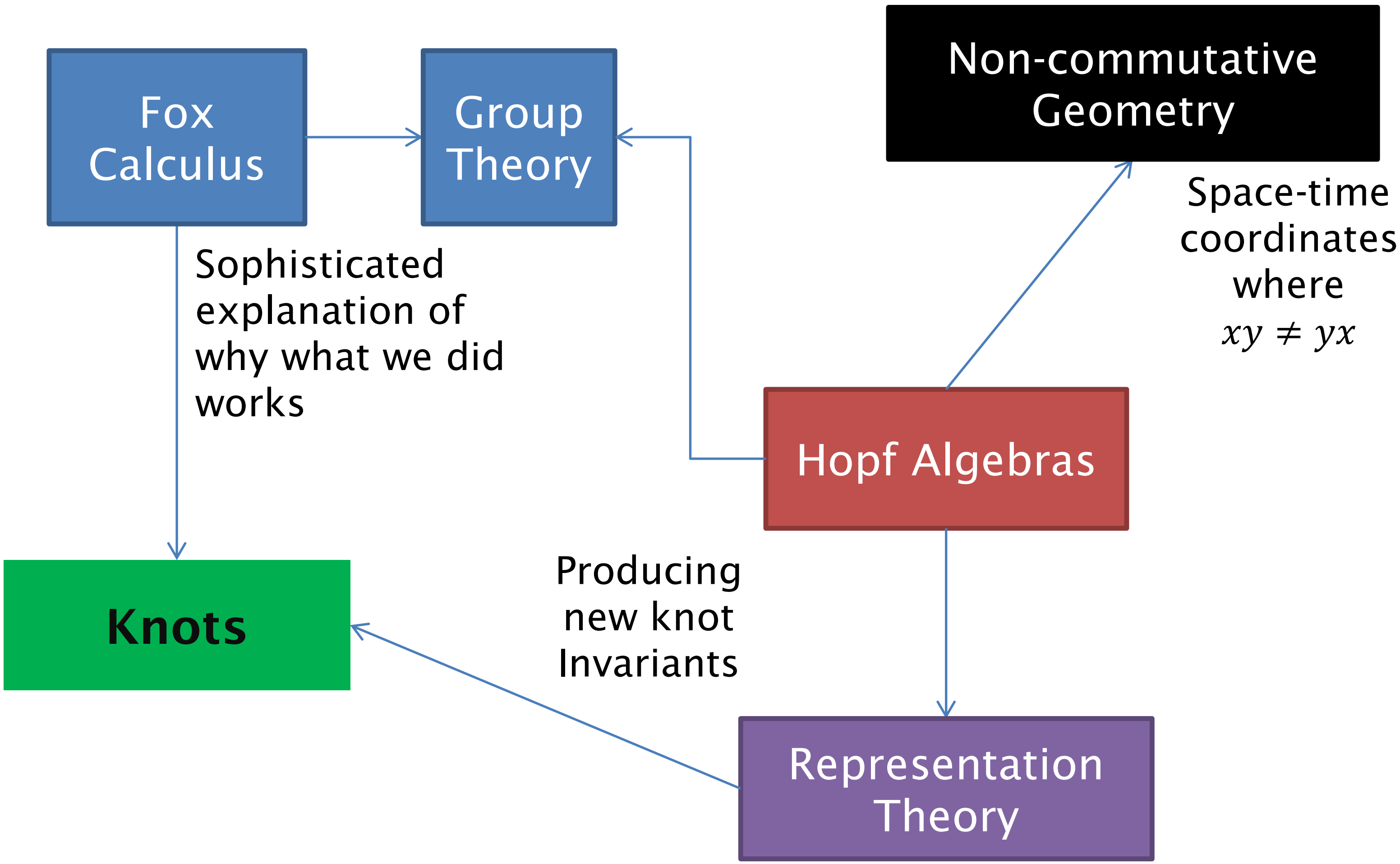

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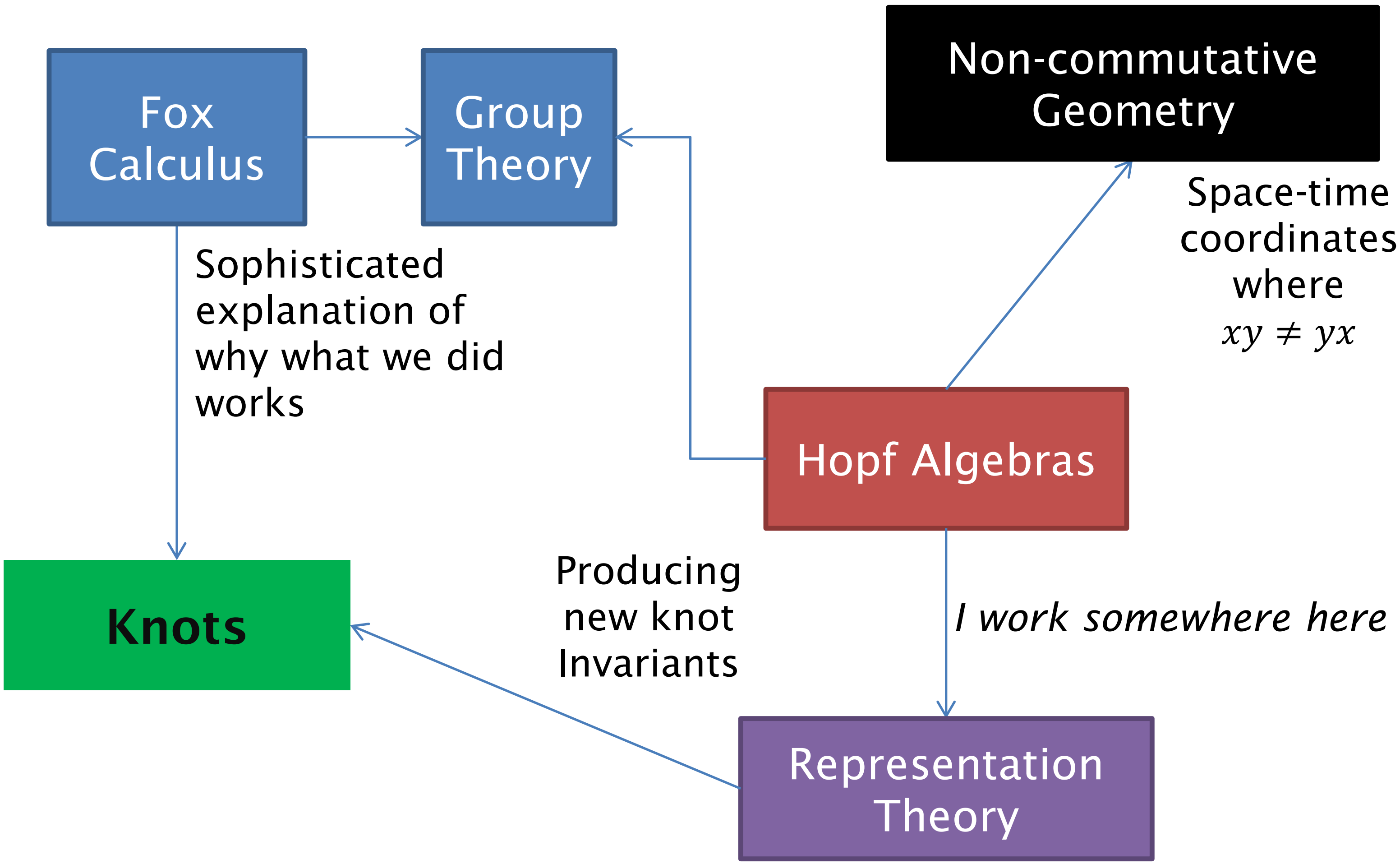

Slides will be in the "Outreach/Engagement" tab on my Website: http://maths.qmul.ac.uk/~ghobadi/welcome.html

Bibliography and Recommended Texts

- *J.W. Alexander, **Topological Invariants of Knots and Links**, Transactions of the AMS, Vol 30, 1928 p 275-306*

- *Colin C. Adams, **The Knot Book**,* American Mathematical Society,
  ISBN-13: 978-0821836781

- *Andrew Ranniki's Website has some great links*
- *Edward Long, **Topological Invariants of Knots: Three Routes to Alexander's Polynomial**, Manchester University, 2005*
- *Will Adkison, **An Overview of Knot Invariants***

Pictures used from

- *https://en.wikipedia.org/wiki/List_of_mathematical_knots_and_links*
- *http://math201s09.wikidot.com/richeson-knot*
- *http://blog.kleinproject.org/?p=2130*
- *https://wildandnoncompactknots.wordpress.com/*
- *https://math.stackexchange.com/questions/1436652/list-of-number-of-knots-distinguished-by-alexander-polynomials?rq=1*