LTCC Enumerative Combinatorics

Notes 3 Alex Fink Fall 2015

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3 Operations on (ordinary) generating functions

Formal power series support a number of operations. This section is dedicated to introducing these operations, and then describing their combinatorial meaning for ordinary generating functions. For exponential generating functions the interpretation is not dissimilar, but the theories are each individually rich enough that they deserve separate treatment.

3.1 More on the ring of formal power series

We have introduced the ring $R[[x]]$ in Section 1, defining its ring operations. Here we say more about its properties.

Proposition 3.1 *A formal power series is invertible if and only if its constant term is invertible.*

Proof Suppose that $f = \sum r_n x^n$ and $g = \sum s_n x^n$ satisfy $fg = 1$. Considering the term of degree zero, we see that $r_0 s_0 = 1$, so that r_0 is invertible.

Conversely, suppose that $r_0 s_0 = 1$, where $f = \sum r_n x^n$. The inverse $g = \sum s_n x^n$ must satisfy

$$\sum_{k=0}^{n} r_k s_{n-k} = 0$$

for all $n > 0$. These equations constitute a linear recurrence which can be solved recursively for the s_n: as the coefficient of s_n is r_0, we have

$$s_n = -s_0 \sum_{k=1}^{n} r_k s_{n-k}.$$

In consequence, we see that if R is an integral domain, so is $R[[x]]$. Similarly if R is a field, then $R[[x]]$ is a discrete valuation ring, and if R is local, so is $R[[x]]$.

We emphasise that knowledge of the inverse of a formal power series $f = \sum r_n x^n$ is equivalent to knowledge of a linear recurrence relation for the r_i. This recurrence relation might have infinitely many terms, though. We discuss examples below, in Section 3.4. There will only exist a recurrence relation of finite length if f is a rational function. The previous proof shows this in the case where

$f = 1/p$, for a polynomial $p = p_0 + p_1 x + \cdots$, and the initial conditions can be taken to be $r_0 = 1/p_0$, $r_i = 0$ for $i < 0$, with the recurrence relation in force from r_1 onward. Different initial conditions can be imposed by changing the numerator.

The ring $R[[x]]$ is a differential algebra, with the derivative operator defined formally:

$$\left(\sum_{n\geq 0} r_n x^n\right)' = \sum_{n\geq 0} (n+1) f_{n+1} x^n.$$

This means that the usual rules of calculus for differentiating sums and products are valid.

Also, $R[[x]]$ bears the structure of a topological ring. In fact, it is the completion of $R[x]$ with respect to the *I-adic topology*, where I is the maximal ideal $\langle x \rangle$: this is the topology whose basic open sets are the sets $f + I^n$, for $f \in R[x]$ and $n \in \mathbb{N}$, and whose Cauchy sequences are therefore the sequences the differences between whose terms eventually lie in I^n, for each n. If R itself bears a topology, then replacing the I-adic topology on $R[x] = R + Rx + Rx^2 + \cdots \cong R^{\mathbb{N}}$ with the product topology on $R^{\mathbb{N}}$ yields a topology on $R[[x]]$ which respects that on R.

In the case $R = \mathbb{C}$, for any $r > 0$ we have a continuous injection to $\mathbb{C}[[x]]$ from the differential algebra of power series converging on the disc $\{x \in \mathbb{C} : |x| < r\}$. This lets us do analysis on power series: any identity that holds analytically between formal power series converging on a disc also holds formally.

For example, we know analytically that $e^x e^{-x} = 1$, and as such we deduce the identity of coefficients

$$\sum_k \frac{1}{k!} \frac{(-1)^{n-k}}{(n-k)!} = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1. \end{cases}$$

This particular identity is also easy to prove algebraically, using the binomial theorem expansion of $(1 + (-1))^n$.

Lastly, let f and g be formal power series in which the constant term of g is zero. Then the composition $f \circ g$ is defined: if $f(x) = \sum r_n x^n$, then $(f \circ g)(x) = \sum r_n g^n$. Like the formula for the product, this expression only has finitely many contributions to the coefficient of x^n for each n. The chain rule for differentiation is valid as well.

3.2 Ordinary versus exponential

Ordinary and exponential generating functions are advantageous in different contexts. The general rule of thumb is this. Exponential generating functions are good when the combinatorial structures being counted are *labelled*. Labelled structures are structures on the set $[n]$, whose elements are treated as distinguishable; the S_n

action on $[n]$ induces a nontrivial S_n action on the structures, unless the structures are trivial themselves. By contrast, ordinary generating functions are good when there is no obvious meaningful group action on the structures of parameter n (except perhaps by a group of cardinality $O(1)$). For example, the structures might be sequences of a sort whose terms can't be scrambled willy-nilly; n might not even be the length of the sequence.

For "unlabelled" structures which come from structures labelled in the above sense by "erasing the labels" and making the elements of $[n]$ undistinguished, an approach that sometimes succeeds is to work with the labelled structures and count S_n-orbits thereof. We discuss this together with labelled structures in general in the next section.

3.3 Operations on ordinary generating functions

Let us say that a *combinatorial class* $\mathscr{A}$ is a set bearing the data of a *size function* $|\cdot| : \mathscr{A} \to \mathbb{N}$, so that the fibre $\mathscr{A}_n = (|\cdot|)^{-1}(n)$ is finite for each n. Let $a_n = \#\mathscr{A}_n$, and associate the generating function $A(x) = \sum_{n\geq 0} a_n x^n$. An isomorphism between combinatorial classes is a set isomorphism between them preserving the size function.

For example, in the combinatorial class of binary words

$$\mathscr{W} = \{\varepsilon, 0, 1, 00, 01, 10, 11, \ldots\},$$

where ε denotes the empty word, the sets $\mathscr{W}_n$ are the words of length n, $|\cdot|$ is the length function, and $w_n = 2^n$, making

$$W(x) = \sum_n 2^n x^n = \frac{1}{1-2x}.$$

We will also want to refer to certain small finite classes, as building blocks. We define one of these now: $\zeta = \{\circ\}$ is the singleton class whose only element has size $|\circ| = 1$. Versions of this class where the element is renamed will appear as well.

Let $\mathscr{A}$ and $\mathscr{B}$ be combinatorial classes. The next propositions are enrichments of our addition and multiplication principles from Section 1.; the proofs follow easily from expanding the stated generating functions. Let $\mathscr{A} + \mathscr{B}$ be the disjoint union of $\mathscr{A}$ and $\mathscr{B}$, with the size function extended from $\mathscr{A}$ and $\mathscr{B}$ in the natural way.

Proposition 3.2 *The generating function of* $\mathscr{A} + \mathscr{B}$ *is* $A(x) + B(x)$.

Let $\mathscr{A} \times \mathscr{B}$ be the Cartesian product of $\mathscr{A}$ and $\mathscr{B}$,

$$\mathscr{A} \times \mathscr{B} = \{(\alpha, \beta) : \alpha \in \mathscr{A}, \beta \in \mathscr{B}\},$$

with the size function $|(\alpha, \beta)| = |\alpha| + |\beta|$.

Proposition 3.3 *The generating function of* $\mathscr{A} \times \mathscr{B}$ *is* $A(x)B(x)$.

Special cases of the product are the powers $\mathscr{A}^k = \underbrace{\mathscr{A} \times \cdots \times \mathscr{A}}_{k}$, for $k \geq 1$. It is the only defensible convention, if not strictly a special case, to set $\mathscr{A}^0 = \mathbf{1}$, where $\mathbf{1}$ is the combinatorial class with a single element ε, whose size is 0. Suppose $\mathscr{A}_0 = \emptyset$. Let $\mathscr{A}^* = \sum_{k \geq 0} \mathscr{A}^k$ be the free monoid on $\mathscr{A}$, i.e. the class of sequences of elements of $\mathscr{A}$.

Corollary 3.4 *The generating function of* $\mathscr{A}^*$ *is* $1/(1 - A(x))$.

This follows from the propositions by expanding $1/(1 - A(x))$ as a geometric series. The sum converges in the I-adic topology, i.e. there are only finitely many summands in each power of x, because $\mathscr{A}_0 = \emptyset$.

We recognise our example class $\mathscr{W}$ above of binary words as $\{0, 1\}^*$, where $|0| = |1| = 1$. Several examples from Section 2 are also easy to handle in this mould, with judicious choices of size functions for the components:

(1) The class of all compositions is $(\mathbb{N}_+)^*$, where $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ has the identity weight function $|n| = n$.

Here $\mathbb{N}_+$ is itself nearly a free monoid class: it is obtained by deleting the size-zero element from $\mathbb{N} = \zeta^*$. So $\mathbb{N}_+$ has generating function $1/(1-x) - 1 = x/(1-x)$, and the class of compositions has generating function

$$\frac{1}{1 - \frac{x}{1-x}} = \frac{1-x}{1-2x} = 1 + \sum_{n \geq 1} 2^{n-1} x^n.$$

So we recover the fact that there are 2^{n-1} compositions of a positive integer n. This fact also has an elementary proof (exercise) using the balls-and-commas bijection of Section 2.2.

There is no class of all weak compositions according to our formalism, for the size function would have infinite fibres.

(2) The class of multisets on a finite set $[n]$ is $\prod_{i\in[n]}\{i\}^*$, where each $\{i\}$ is isomorphic to ζ, i.e. is a singleton class with $|i| = 1$. So the generating function for multisets on $[n]$ is

$$\prod_{i\in[n]} \frac{1}{1-x} = (1-x)^{-n}.$$

Expanding with the binomial theorem, this equals

$$\sum_{k\geq 0} (-1)^k \binom{-n}{k} x^k,$$

reproducing our reciprocity between multisets and subsets.

(3) The class of integer partitions can't quite be built using these tools, but with a suitable extension of the Cartesian product to an infinite family of classes, it would be $\prod_{n\geq 1}(\zeta^n)^*$, yielding the generating function

$$\prod_{n\geq 1} \frac{1}{1-x^n}$$

of Section 3.4.1.

(4) Here's one way to tackle set partitions using these tools. The combinatorial class of set partitions of sets of the form $[n]$ into k parts is $\prod_{i=1}^{k}\{i\}\times[i]^*$, where in each class $\{i\}$ or $[i]$, each element has size 1. Indeed, if we name the k parts of such a set partition $S_1,\dots,S_k$ sorted in increasing order of their least element, then we can encode a set partition of $[n]$ by the list $(i_1,\dots,i_n)$ where $j\in S_{i_j}$ for each j, and these lists are what the above class is constructed to contain. This yields the generating function from Section 2,

$$\prod_{i=1}^{k} \frac{x}{1-ix}.$$

3.4 Linear recurrences

The combinatorial setting that most directly gives rise to linear recurrences is that of free monoids. Let $\mathscr{A}$ be a combinatorial class with $\mathscr{A}_0 = \emptyset$. Then we have the relation

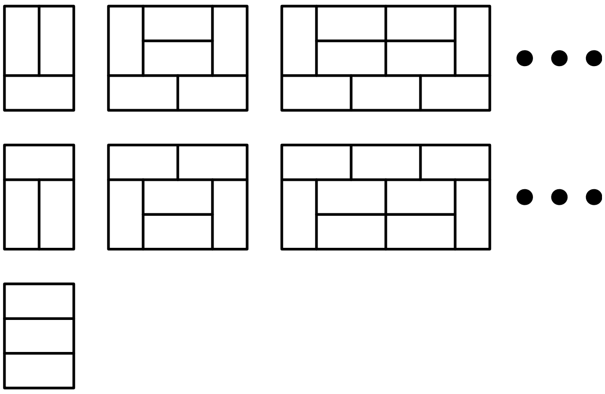
$$\mathscr{A}^* = 1 + \mathscr{A}\times\mathscr{A}^* :$$

that is, every element of $\mathscr{A}^*$, aside from the empty one, is an element of $\mathscr{A}$ followed by an element of $\mathscr{A}^*$, and this is a bijection. As we have seen, in Section 3.1

and our opening example from Section 1, both a linear recurrence and a rational ordinary generating function follow directly.

The example in Section 1 concerned domino tilings of the $2 \times n$ rectangle, and analysed these tilings as the free monoid over {,}, whose elements' sizes 1 and 2 are given by their widths.

Let us analyse domino tilings of the $3 \times n$ rectangles in the same fashion. These will form the free monoid over the class of nonempty tiled $3 \times n$ rectangles with at least one domino lying over each horizontal gridline: let me call these *faultfree*. The faultfree tilings are easy enough to count by hand experimentation, once we observe that no vertical domino can occur except up against a short side of the rectangle. There are infinitely many faultfree tilings, but only two of each (large enough) even size:



So the generating function for the nonempty faultfree rectangles is

$$3x^2 + 2x^4 + 2x^6 + \cdots = \frac{3x^2 - x^4}{1 - x^2},$$

from which we get directly both a recurrence relation

$$a_n = 3a_{n-2} + 2a_{n-4} + 2a_{n-6} + \cdots \tag{1}$$

for the number a_n of $3 \times n$ domino tilings overall (omitting mention of the base case), as well as a generating function

$$\sum a_n x^n = \frac{1}{1 - \frac{3x^2 - x^4}{1 - x^2}} = \frac{1 - x^2}{1 - 4x^2 + x^4}.$$

From this denominator, or from subtracting from (1) the same equation with $n-2$ substituted for n, we extract another, finite, recurrence relation

$$a_n = 4a_{n-2} - a_{n-4}.$$

The latter recurrence relation, with its negative coefficient, of course does not encode directly any other free monoid decomposition of our tilings, but it is possible to give it a bijective meaning: the reader may wish to construct a bijection between $\mathscr{A}_n \dot\cup \mathscr{A}_{n-4}$ and $\dot\bigcup_{i=1}^{4} \mathscr{A}_{n-2}$.

Exercise Count the permutations $\sigma : [n] \to [n]$ such that $|\sigma(i) - i| \leq 2$ for all i.

3.4.1 The infinite recurrence for integer partitions

Recall that the partition number $p(n)$ is the number of partitions of n indistinguishable objects, that is, the number of ways to write n as a sum of a nonincreasing sequence of positive integers. Its generating function is

$$\sum_{n\geq 0} p(n)x^n = \prod_{k\geq 1} \frac{1}{1-x^k}.$$

For $(1-x^k)^{-1} = 1 + x^k + x^{2k} + \cdots$. Thus a term in x^n in the product, with coefficient 1, arises from every expression $n = \sum c_k k$, where the c_k are non-negative integers, all but finitely many equal to zero. This structure is an integer partition, the number of which is $p(n)$.

Thus, to get a recurrence relation for $p(n)$, we have to understand the coefficients a_n of its inverse,

$$\sum_{n\geq 0} a_n x^n = \prod_{k\geq 1} (1-x^k).$$

Now a term in x^n on the right arises from each expression for n as the sum of *distinct* positive integers; its coefficient is $(-1)^k$, where k is the number of terms in the sum. Interpreting this as a weighted counting problem, the coefficient a_n we seek is the total weight of the integer partitions of n into distinct parts, where partitions with evenly many parts have weight $+1$ and those with oddly many parts have weight -1.

This number is evaluated by Euler's Pentagonal Numbers Theorem:

Proposition 3.5

$$a_n = \begin{cases} (-1)^k & \textit{if } n = \frac{1}{2}k(3k-1) \textit{ for } k \in \mathbb{Z} \\ 0 & \textit{otherwise.} \end{cases}$$

That is,

$$\prod_{k\geq 1}(1-x^k) = \cdots + x^{26} - x^{15} + x^7 - x^2 + 1 - x + x^5 - x^{12} + x^{22} - \cdots$$

The exponents appearing here are the *pentagonal numbers*; they are one of the sequences of *figurate numbers* generalising the more familiar triangular and square numbers. The Ferrers diagrams of the crucial partitions in the proof below are the pentagons from which the name derives (except that two of the five sides of the pentagon have degenerated into a single side twice as long).

Proof Our proof is bijective, using the method of a *sign-reversing involution*. That is, we describe a partial involution f on the set of partitions of n into distinct parts, so that λ and $f(\lambda)$ have opposite weight whenever the latter is defined. This way, the contributions of λ and $f(\lambda)$ to the sum a_n will cancel, and we will only need to (weightedly) count the λ for which $f(\lambda)$ is undefined.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n, with $\lambda_1 > \cdots > \lambda_k > 0$ and $\sum \lambda_i = n$, define two statistics:

- $d(\lambda)$ is the largest i such that $\lambda_i = \lambda_1 - i + 1$: that is, the first $d(\lambda)$ parts of λ successively decrease by only one, but the next decreases by more.

- $e(\lambda) = \lambda_k$ is the smallest part.

Define f according to how these statistics compare:

- If $d(\lambda) < e(\lambda)$, let
$$f(\lambda) = (\lambda_1 - 1, \ldots, \lambda_{d(\lambda)} - 1, \lambda_{d(\lambda)+1}, \ldots, \lambda_k, d(\lambda)).$$

- If $d(\lambda) \geq e(\lambda)$, let
$$f(\lambda) = (\lambda_1 + 1, \ldots, \lambda_{e(\lambda)} + 1, \lambda_{e(\lambda)+1}, \ldots, \lambda_{k-1}).$$
Note that the final omitted part is $\lambda_k = e(\lambda)$.

We make several observations. Firstly, by the definition of $d(\lambda)$ and $e(\lambda)$, the sequences $f(\lambda)$ remain sequences of positive integers with sum n, and these are strictly decreasing sequences, i.e. partitions of n into distinct parts, in *nearly* every case. We mean to exclude the cases where they are not from the domain of f. Secondly, the two operations in the definition of f are inverses of one another, and whichever case λ is in, $f(\lambda)$ will be in the other when it is defined; this makes f a partial involution. Thirdly, $f(\lambda)$ has either one part more or one part fewer than λ, so that the two have opposite weight. Thus our claims in the first paragraph are vindicated.

It remains just to characterise the λ for which $f(\lambda)$ is undefined. This happens only when the parts involved in defining $d(\lambda)$ and $e(\lambda)$ "interfere" with each other, i.e. λ is of the shape $(\ell + k - 1, \ldots, \ell + 1, \ell)$ for some ℓ. Even in this case the problem arises only when $\ell = k$ or $\ell = k+1$, i.e. the only partitions outside the domain of f are

$$(2k-1, 2k-2, \cdots, k+1, k) \quad \text{and} \quad (2k, 2k-1, \cdots, k+2, k+1).$$

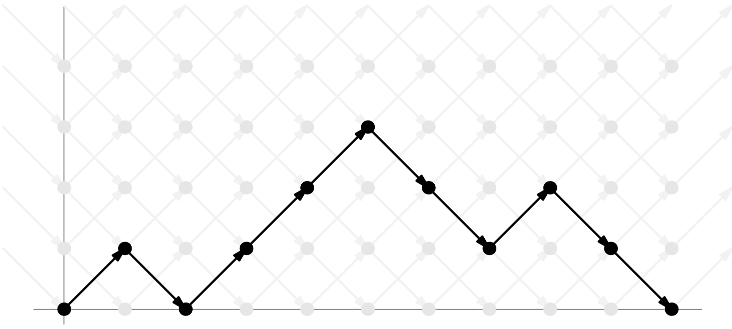
These are partitions of $\frac{1}{2}k(3k-1)$ and $\frac{1}{2}(-k)(3(-k)-1)$ respectively, and their weights are $(-1)^k$.

The number of terms that must be evaluated in the recurrence issuing from Proposition 3.5 grows with n, but only as $O(\sqrt{n})$. So evaluating $p(n)$ for all $n \leq N$ requires only $O(N^{3/2})$ additions and subtractions. In practice, if you had the task of computing a table of partition numbers, this recurrence is the most efficient way I am aware of to do so.

3.5 Catalan objects

One particular sequence of naturals, the *Catalan numbers*, deserves exposition because of the extraordinary number of counting problems it solves: over two hundred, by Stanley's count (`http://www-math.mit.edu/~rstan/ec/catadd.pdf`).

A *Dyck path* of size n is a path from $(0,0)$ to $(2n,0)$ in the directed graph whose vertices are the upper half-plane $\mathbb{Z} \times \mathbb{N}$ and which contains all possible edges of the forms $(i,j) \to (i+1, j+1)$ and $(i,j) \to (i+1, j-1)$. The figure depicts a Dyck path of size 5.



The edges are often called *steps*.

Dyck paths are in easy bijection with a more typographically convenient object, strings of *matched* pairs of parentheses. These are strings of "(" and ")" whose characters can be paired off, each pair consisting of a "(" and a ")" somewhere to its right, so that no two pairs interweave: that is, the subconfiguration $\cdots (_1 \cdots (_2 \cdots)_1 \cdots)_2 \cdots$ does not occur, using the subscripts (and colours) to indicate the pairing. The bijection reads the steps of a Dyck path in order, turning each *up* step $(i,j) \to (i+1,j+1)$ to a "(" and each *down* step $(i,j) \to (i+1,j-1)$ to a ")". For example, the above picture corresponds to

$$()((())()).$$

Let $\mathscr{D}$ be the class of Dyck paths. We recognise this as a free monoid class, $\mathscr{D} = \mathscr{I}^*$, where $\mathscr{I}$ is the class of those *irreducible* Dyck paths of positive size which include no vertex $(i,0)$ aside from their start- and end-points. On the other hand, every irreducible Dyck path is simply a general Dyck path flanked by an initial up step and a final down step, implying that $\mathscr{I} = \zeta \times \mathscr{D}$. The generating functions $D(x)$ and $I(x)$ therefore satisfy

$$D(x) = \frac{1}{1-I(x)} = \frac{1}{1-xD(x)},$$

i.e.

$$xD^2(x) - D(x) + 1 = 0$$

or

$$D(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

In solving the quadratic we must take the negative sign, as the positive would produce a nonzero coefficient of x^{-1} in $D(x)$.

Definition 3.6 The *Catalan number* C_n is the number of Dyck paths of size n.

Here is a related way the structure of a Dyck path, or a string of matched parentheses, could have been unrolled. Every such string is either empty, or starts with a "(". In the latter case the string is composed of this "(", a string of matched parentheses, the ")" to match the first "(", and then another string of matched parentheses. This gives

$$\mathscr{D} = \mathbf{1} + \zeta \times \mathscr{D}^2$$

which translates to the same equation for $D(x)$.

Attempting to read the above equation directly gives us another Catalan combinatorial class. Every structure is either empty or consists of a $\circ$, of size 1, and two whole structures of the same sort. If we denote the empty case by an ε and draw edges from the $\circ$ to the two substructures in the nonempty case, what we end up drawing are *binary trees*, with $\circ$s on the internal nodes and εs on the leaf nodes. We have proved

Proposition 3.7 *The number of binary trees with n non-leaf nodes is the Catalan number* C_n.

The proof is easy to render bijective.

Exercise State a functional equation for the generating function of the class of binary trees wherein the two children of a node are undistinguished, i.e. (The coefficients are known as the *Wedderburn-Etherington numbers*.)

We would like a closed form for the Catalan numbers. Thankfully, the generating function is one we can expand with the binomial theorem:

$$D(x) = \frac{1}{2}\left(1 - \sum_n \binom{1/2}{n}(-4)^n x^{n-1}\right).$$

Hence

$$\begin{aligned} C_n &= -\frac{1}{2}\binom{1/2}{n+1}(-4)^{n+1} \\ &= (-1)^n \frac{1}{2} \cdot \left(\frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdots \frac{2n-1}{2}\right) \cdot \frac{2^{2n+2}}{(n+1)!} \\ &= \frac{1}{2^{n+2}} \frac{(2n)!}{2^n \cdot n!} \cdot \frac{2^{2n+2}}{(n+1) \cdot n!} \\ &= \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

In the flesh, beginning from C_0, the Catalan numbers are

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots$$

We can read coarse asymptotics of the Catalan numbers directly from the generating function. The nearest singularity of $D(x)$ to the origin is a branchpoint at $1/4$, so with the ratio test in mind, C_n grows "like" 4^n. If more precision be desired, our closed form together with Stirling's approximation gives the asymptotic statement

$$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}.$$

Finally, here is a sketch of a bijective proof of the formula for C_n. Let $\mathscr{P}$ be the class of paths, in the graph on vertices $\mathbb{Z} \times \mathbb{Z}$ with analogous up and down edges, from $(0,0)$ to $(2n,0)$: informally, these are Dyck paths without the restriction that the second coordinate stay positive. Clearly $|\mathscr{P}_n| = \binom{2n}{n}$, since n of the $2n$ steps in a path of size n must be up, and the others down. If P is a path in $\mathscr{P}_n$ that is not a Dyck path, then it contains some vertex (i, j) with $j < 0$: select the one with j minimal, and then with i maximal for that choice of j. Then P can be decomposed as some path P_1 from $(0,0)$ to (i, j), followed by an up step, followed by a path P_2 from $(i+1, j+1)$ to $(2n, 0)$.

The reader may verify that the new path made of P_2 translated to begin at $(0,0)$, followed by an up step, followed by P_1 translated to end at $(2n,0)$ is a Dyck path. Call it D. Moreover, if it's known which is the special up step separating P_2 from P_1 in D, P can be recovered; and any of its n up steps can be this special one. Therefore, we have an n-to-one map from $\mathscr{P}_n \setminus \mathscr{D}_n$ to $\mathscr{D}_n$, implying that $|\mathscr{D}_n| = \binom{2n}{n}/(n+1)$.