LTCC Enumerative Combinatorics

Notes 2 | Alex Fink | Fall 2015

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2 Subsets, partitions, functions

We start with an examination of simple counting problems involving some of the most foundational objects of combinatorics, subsets and partitions and permutations.

Many combinatorial structures are founded on sets. If no further structure on the set is relevant, then answers to enumerative problems will not change if you replace the underlying set with another with which it is in bijection. As such combinatorialists have a habit of using one canonical finite set of each cardinality, namely

$$[n] = \{1,\dots,n\}$$

for each $n \geq 0$. (I'm not wholly fond of this choice; I would have preferred the standard set to be $\{0,\dots,n-1\}$. But the difference between the two is of no importance in most contexts, where at most the total order on the set is relevant, so I go along with the weight of convention.)

2.1 Subsets

Let us take *subsets* as our first structure. The number of k-element subsets of the set $[n]$ is the *binomial coefficient*

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

The "otherwise" case is clear; let's justify the first case. Here is a way to miscount the subsets. Let's call the subset $\{a_1,\dots,a_k\}$. The element a_1 can be any of the n elements of $[n]$. As for a_2, it must not be equal to a_1, but it can be any of the $n-1$ elements that remain. And so on, till we reach a_k, for which there are $n-k+1$ remaining options. By the multiplication principle, there are

$$\frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1) =: (n)_k$$

possibilities overall. The notation $(n)_k$ is read as the kth *falling power* of n (or *falling factorial*; the notation is called the *Pochhammer symbol*).

What we have counted here aren't subsets, though! We are distinguishing the order in which the elements are written out. This is worth recording for later, together with an important special case:

Proposition 2.1 *The number of ordered lists of length k of distinct elements of $[n]$ is $(n)_k$.*

Corollary 2.2 *The number of* permutations *of $[n]$ is $n!$.*

But there's another way to count the ordered lists of k distinct elements from $[n]$. First choose the underlying unordered set of k elements: momentarily say there are $C(n,k)$ ways to do this. Then choose its order, which there are $k!$ ways to do, by Corollary 2.2. Therefore

$$C(n,k) \cdot k! = \frac{n!}{(n-k)!}$$

and we recover $C(n,k) = \binom{n}{k}$.

2.1.1 Generating functions

The generating function for subsets is well known under another name:

Proposition 2.3 (Binomial Theorem)

$$\sum_{S \subseteq [n]} x^{|S|} = \sum_k \binom{n}{k} x^k = (1+x)^n.$$

My omissions of bounds on the k summation is intentional. If k is not between 0 and n inclusive it makes no contribution; it's needless to exclude these terms a second time.

The proof again uses the addition and multiplication principles: each monomial in the expansion of the right hand side arises from picking a subset S of $[n]$, and choosing from the ith factor of $(1+x)$ the x term if $i \in S$, and the 1 term otherwise.

Substituting $x = 1$ gives the total number of subsets,

$$\sum_k \binom{n}{k} = 2^n.$$

This substitution is legitimate, notwithstanding my cautions about convergence of power series, because the sums are finite!

Since the binomial coefficients have two indices, we could let n vary as well, and ask for a two-variable generating function:

$$\sum_{n\geq 0,k}\binom{n}{k}x^k y^n = \sum_{n\geq 0}(1+x)^n y^n$$
$$= \frac{1}{1-(1+x)y}.$$

And we get the other univariate generating function, where the size of the subset is fixed, by expanding in powers of x:

$$\frac{1}{1-y-xy} = \frac{1}{1-y}\cdot\frac{1}{1-\frac{y}{1-y}x}$$
$$= \sum_{n\geq 0}\frac{y^n}{(1-y)^{n+1}}x^n,$$

so that

$$\sum_{n\geq 0}\binom{n}{k}y^n = \frac{y^n}{(1-y)^{n+1}}.$$

2.1.2 Identities

The following identities can be proven through either algebraic grounds, manipulating the generating functions, or combinatorial ones, finding appropriate bijections. These cases are simple enough that the two proofs can be seen to mirror one another.

We single out the *recurrence relation*:

Proposition 2.4 *For* $n \geq 1$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

and $\binom{0}{0} = 1$ *while* $\binom{0}{k} = 0$ *for* $k \neq 1$.

The proofs are easy, so we only sketch the algebraic one: expanding the last factor in the factorisation $(1+x)^n = (1+x)^{n-1}(1+x)$, we get

$$\sum_k \binom{n}{k}x^k = \sum_k \binom{n-1}{k}x^k + \sum_k \binom{n-1}{k}x^{k+1},$$

from which extracting coefficients of x^k gives the proposition.

Other basic identities are the symmetry

$$\binom{n}{k} = \binom{n}{n-k},$$

which algebraically follows quickly after substituting x^{-1} for x, and the *Vandermonde convolution* (first recorded by Zhū Shìjié, 1303)

$$\binom{m+n}{k} = \sum_{i+j=k} \binom{m}{i}\binom{n}{j},$$

which corresponds to $(1+x)^{m+n} = (1+x)^m(1+x)^n$.

Examples of formulae of the general aspect of this last identity, expressing sums of products or ratios of binomial coefficients in terms of further binomial coefficients, can be multiplied endlessly. For instance, volume 4 of Henry Gould's tables available at `http://www.math.wvu.edu/~gould/` contains hundreds, and chapter 5 of Graham, Knuth and Patashnik, *Concrete Mathematics*, the material for thousands more. The theory underlying these is that of *hypergeometric series*, to which I can't do justice here; I'll only note only that they are quite amenable to algorithmic determination of the existence or otherwise of closed forms, the first such algorithm due to Bill Gosper in 1977.

2.2 Multisets and compositions

A *multiset* is like a set except that it may contain repeated elements; order is still insignificant. It is a multiset *on* a set S if all its elements are drawn from S. For example, $\{1,1,1,2,4,4\}$ is a multiset on [5] with six elements.

Proposition 2.5 *The number of k-element multisets on $[n]$ is $\binom{n+k-1}{k}$.*

We prove this bijectively. A *weak composition* of k is a list of natural numbers whose sum is k. If S is a k-element multiset on $[n]$, containing a_i occurrences of i for each $i \in [n]$, then $\sum_{i=1}^{n} a_i = k$, so the list $(a_1,\dots,a_n)$ is a weak composition of k. This gives a bijection; the existence of the inverse function is obvious.

Now, given this weak composition, replace each a_i by a string of a_i "balls" $\circ$, and separate these strings by commas. The result is a string of characters, k of which are balls and $n-1$ of which are commas. Again, every string of this makeup arises, so this operation is bijective. But the number of such strings is $\binom{n+k-1}{k}$, since the balls can be put in any k of the $n+k-1$ positions.

E.g. our multiset $\{1,1,1,2,4,4\}$ on [5] corresponds to the weak composition $(3,1,0,2,0)$ of 6, and to the string "$\circ\circ\circ,\circ,,\circ\circ,$" of $6+5-1$ characters, 6 of them $\circ$.

A *composition* of k is a list of *positive* integers whose sum is k. Incrementing each integer in a weak composition of $k-n$ of length n will yield a composition of k of the same length, and this is yet again bijective. We conclude:

Proposition 2.6 *The number of compositions of k of length n is $\binom{n-1}{k-n} = \binom{n-1}{k-1}$.*

The generating function for multisets on $[n]$ is

$$\sum_{S \text{ a multiset on } [n]} x^{|S|} = \sum_k \binom{n+k-1}{k} x^k = \left(\frac{1}{1-x}\right)^n.$$

Again this is proved by expanding the right hand side into monomials, where the ith factor of $1/(1-x) = 1+x+x^2+x^3+\cdots$ records the number of is in S.

Multisets also provide a first instance of a recurrent phenomenon in enumerative combinatorics, where an enumeration which is a priori only sensible for nonnegative values of the parameter turns out to solve a related counting problem (up to sign) when the parameter is set to negative values. This is known as *combinatorial reciprocity*.

Our definition above the binomial coefficients requires n to be natural, but it can be extended to all complex n by setting

$$\binom{n}{k} = \frac{(n)_k}{k!}.$$

This gives a reciprocity law between subsets and multisets,

$$\binom{-n}{k} = (-1)^n \binom{n+k-1}{k},$$

which can be proved by putting $-n$ for n and $-x$ for x in the binomial theorem.

The binomial theorem, Proposition 2.3, remains true for the above extended definition of the binomial coefficients, once the right side $(1+x)^n$ is made meaningful for $n \in \mathbb{C}$. This is done by invoking the power series for exp and log,

$$(1+x)^n = \exp(n \cdot \log(1+x)).$$

We omit the proof.

2.3 Partitions

A *(set) partition* of a set S is a collection of pairwise disjoint nonempty sets $S_1, \ldots, S_k$ (*parts*, or sometimes *blocks*) whose union is S.

The *Bell number* $B(n)$ is the number of partitions of the set $[n]$. We refine this count according to the number of parts: let $S(n,k)$ be the number of partitions of $[n]$ into k parts. These are called the *Stirling numbers* of the second kind. Much as for binomial coefficients, $S(n,k)$ is obviously zero if $k > n$ or $k \leq 0 < n$.

Here are the first few Stirling and Bell numbers:

$S(n,k)$ \ $k=$	0	1	2	3	4	5	6	$B(n)$
$n=0$	1	0	0	0	0	0	0	1
1	0	1	0	0	0	0	0	1
2	0	1	1	0	0	0	0	2
3	0	1	3	1	0	0	0	5
4	0	1	7	6	1	0	0	15
5	0	1	15	25	10	1	0	52
6	0	1	31	90	65	15	1	203

You can find more terms of the sequences in N J A Sloane's *Online Encyclopedia of Integer Sequences*, `https://oeis.org/`, which is an irreplaceable tool for the working enumerative combinatorialist. If you've got a new enumeration problem and you can answer the first several cases, a search in OEIS can hand you a wealth of hypotheses, generating functions and bijections and recurrences etc., to investigate.

The Bell numbers have their "unlabelled" counterpart $p(n)$, the *partition number*, which is the number of *integer partitions* of the number n, that is, lists in non-increasing order of positive integers with sum n. Thus, given any set partition, the list of sizes of its parts is an integer partition; and two set partitions are equivalent under *relabelling* the elements of the underlying set $[n]$ (that is, under the action of some permutation $\sigma : [n] \to [n]$) if and only if the corresponding integer partitions are equal. Similarly, to the Stirling numbers correspond $p_k(n)$, the number of partitions of the number n into k parts. We will not investigate these in detail in this section; the generating function of $p(n)$ is taken up in Section 3.4.1.

Proposition 2.7 *A recurrence relation for the Stirling numbers of the second kind is*

$$S(n,k) = kS(n-1,k) + S(n-1,k-1).$$

for $n \geq 1$. *Moreover* $S(0,0) = 1$ *while* $S(0,k) = 0$ *for* $k \neq 0$.

Proof We split the partitions of $[n]$ into two classes.

- Those for which $\{n\}$ is a single part arise from a partition of $[n]$ into $k-1$ parts, by adjoining $\{n\}$ as a new part.
- The remainder are obtained by taking a partition of $[n]$ into k parts, selecting one part, and inserting n into it.

2.3.1 Generating functions

The exponential generating function for the Stirling numbers, as n varies, is arguably the more useful one.

Proposition 2.8

$$\sum_n S(n,k)\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}.$$

Lemma 2.9 *The Stirling numbers satisfy the recurrence*

$$S(n,k) = \sum_i \binom{n-1}{i-1} S(n-i,k-1).$$

Proof Consider the part P containing n of an arbitrary partition with k parts; suppose that it has cardinality i. Then there are $\binom{n-1}{i-1}$ choices for the remaining $i-1$ elements in P, and $S(n-i,k-1)$ partitions of the remaining $n-i$ elements into the remaining $k-1$ parts.

Proof of Proposition 2.8 The proof is by induction on k. When $k=0$ the result is $1=1$, which is true. For $k \geq 1$ we have

$$\begin{aligned}
\frac{(e^x-1)^k}{k!} &= \frac{1}{k}(e^x-1)\frac{(e^x-1)^{k-1}}{(k-1)!} \\
&= \frac{1}{k}\left(\sum_{i\geq 1}\frac{x^j}{j!}\right)\left(\sum_m S(m,k-1)\frac{x^m}{m!}\right).
\end{aligned}$$

We wish to compare the coefficient of $x^n/n!$ in this sum to $S(n,k)$. The term concerned arises from the products of summands with $m=n-j$, and is

$$\begin{aligned}
\frac{n!}{k}\sum_{j\geq 1}\frac{S(n-j,k-1)}{j!(n-j)!} &= \frac{1}{k}\sum_{j\geq 1}\binom{n}{j}S(n-j,k-1) \\
&= \frac{1}{k}\sum_{j\geq 1}\binom{n}{j}S(n-j,k-1).
\end{aligned}$$

This is the right hand side of Proposition 2.9 for $S(n+1,k)$ with $i=j+1$, except for the omission of the $j=0$ term. So it equals

$$\frac{1}{k}\left(S(n+1,k)-S(n,k-1)\right)$$

which is $S(n,k)$ by Proposition 2.7.

I leave as an exercise the proof of the ordinary generating function; an induction along the same lines will do it.

Proposition 2.10

$$\sum_k S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

2.3.2 Identities

The most familiar vector space basis for the polynomial ring $\mathbb{Q}[x]$ is the one $\{x^n : n \geq 0\}$ consisting of the powers of x. In some contexts, however, it is convenient to use other bases. For instance, there is a theory of discrete difference equations which appears for instance in the umbral calculus, analogous to differential equations except that the derivative operator is replaced by the discrete analogue Δ, given by $(\Delta f)(x) = f(x+1) - f(x)$. Just as $\{x^n : n \geq 0\}$ is a Jordan chain of generalised eigenvectors for the derivative, the falling powers $\{(x)_n : n \geq 0\}$ give a Jordan chain of generalised eigenvectors for Δ.

The change of basis matrix from powers $\{x^n\}$ to falling powers $\{(x)_n\}$ is populated by the Stirling numbers of the second kind.

Proposition 2.11

$$x^n = \sum_k S(n,k)(x)_k.$$

Proof As this is an identity of polynomials (not of series), it is enough to prove their equality at infinitely many values of x; we prove it for $x \in \mathbb{N}$, by double-counting.

How many functions $f : [n] \to [x]$ are there? Clearly there are x^n. But alternatively, we can count according to the size of the image. Assume this is $k = |f([n])|$. The equivalence relation on $[n]$ defined by $i \equiv j$ if and only if $f(i) = f(j)$ partitions $[n]$ into k equivalence classes, and there are $S(n,k)$ choices of this partition. If $i_1, \ldots, i_k$ are representatives of each class, in order, then f is determined by the sequence $(f(i_1) \ldots, f(i_k))$. This is an ordered list of k elements of $[x]$, of which there are $(x)_k$ by Proposition 2.1. So the right side of the result also counts functions $f : [n] \to [x]$.

We will shortly encounter the inverse change of basis.

Another basis for polynomials often encountered in practice is $\{(x+1)^n\}$. By the binomial theorem, the coefficients in the change of basis from $\{x^n\}$ to this one are the binomial coefficients. As for the inverse change of basis matrix, its entries are also binomial coefficients with suitable signs, using the binomial theorem on $(1+(-x-1))^n$. Composing these changes of basis gives the identity

$$\sum_{k\geq 0}(-1)^{n-k}\binom{n}{k}\binom{k}{m} = \begin{cases} 1 & n=m \\ 0 & \text{otherwise} \end{cases}$$

for naturals m and n.

2.4 Functions

The name *the Twelvefold Way* is given to twelve closely related counting problems about functions. The idea to collect these problems together is due to Gian-Carlo Rota, the name to Joel Spencer.

Let x and n be naturals. Here are twelve variants of the problem of counting functions from $[n]$ to $[x]$:

- Shall we count all the functions, or just the injections, or just the surjections?
- Are the elements of $[n]$ "distinguishable"?
- Are the elements of $[x]$ "distinguishable"?

More formally, if the elements of $[n]$ are indistinguishable then we wish to identify two functions $f,g : [n] \to [x]$, and not count them separately, if there is a permutation $\sigma : [n] \to [n]$ with $\sigma \circ f = g$. If the elements of $[x]$ are indistinguishable, then we wish to identify f and g if there is a permutation $\tau : [x] \to [x]$ with $f \circ \tau = g$.

Less formally, our problem is to count ways of putting balls in boxes. We have n balls and x boxes, and we put ball i in box $f(i)$, for each i. Injections place at most one ball in each box, surjections at least one. The balls might be distinguishable, say each of a different colour, or indistinguishable, with no way to tell them apart; the same goes for the boxes. Thus of these three functions from [3] to [4]

i	1	2	3
$f_1(i)$	1	1	4
$f_2(i)$	1	4	1
$f_3(i)$	2	2	1

f_1 and f_2 are identified if the balls are indistinguishable, and f_1 and f_3 are identified if the boxes are.

The foregoing sections encompass most of the counting problems in Table 1 already. The functions counted in the proof of Proposition 2.11 are (a). The count

Elements of $[n]$ (balls)	Elements of $[x]$ (boxes)	functions	injections	surjections
distinguishable	distinguishable	a. x^n	b. $(x)_n$	c. $x!\,S(n,x)$
indistinguishable	distinguishable	d. $\binom{x+n-1}{n}$	e. $\binom{x}{n}$	f. $\binom{n-1}{x-1}$
distinguishable	indistinguishable	g. $\sum_{i\le x} S(n,i)$	h. $\begin{cases} 1 & n\le x \\ 0 & n>x \end{cases}$	i. $S(n,x)$
indistinguishable	indistinguishable	j. $\sum_{i\le x} p_i(n)$	k. $\begin{cases} 1 & n\le x \\ 0 & n>x \end{cases}$	l. $p_x(n)$

Table 1: The twelvefold way.

corresponding to the right hand side proceeds by factoring an arbitrary $f : [n] \to [x]$ as a surjection followed by an injection,

$$[n] \overset{f}{\twoheadrightarrow} f([n]) \hookrightarrow [x].$$

The count of injections was (b). The count of surjections, in which we didn't treat the elements of $f([n])$ as distinguishable, was (i); to get (c) from this we need only make them distinguishable by composing them with one of the $k!$ bijections from $f([n])$ to $[k]$, where $k = |f(n)|$ (and in the table $k = x$).

If the boxes are indistinguishable and we are asked for an arbitrary function f, not necessarily a surjection, we can again consider the surjection from f onto its image. The size of the image can be any natural $i \le x$, and indistinguishability of the boxes means that the size is the only data about the image present. Given i, there are $S(n,i)$ such surjections. This yields (g), and in this light (h) is trivial. Making the balls indistinguishable together with the boxes passes from set partitions to partitions of an integer, whereupon (l, j, k) are derived in the same way as (i, g, h).

Lastly, if the balls are indistinguishable the only data is the number of balls in each box. If the boxes are distinguishable, we have a sequence of x naturals summing to n. If the sequence is unrestricted this is a weak composition (d); if the terms must be positive, it is a composition (f); and if the terms must be zero or one, the positions of the ones are a subset of cardinality n (e).

2.5 Cycle types of permutations

We will have more to say about permutations in the sequel. At this juncture I mean only to address the following question: we've seen the Stirling numbers of the second kind; what are the first kind?

Recall that any permutation of a finite set N can be written as a product of cycles $(a_1 \ \cdots \ a_s)$ on disjoint sets, uniquely up to the order of the factors and the possibility to omit cycles of length 1. Choosing to include all the cycles of length 1 yields a map from permutations of N to partitions of N. Moreover, two permutations σ and τ of N are conjugate (that is, there exists π such that $\pi^{-1}\sigma\pi = \tau$) if any only if they map to the same partition. As a consequence, the theory we develop henceforth for permutations will all be for permutations of distinguishable objects, as the indistinguishable case coincides with the theory of integer partitions.

The *Stirling numbers of the first kind* are defined by the rule that $s(n,k)$ is $(-1)^{n-k}$ times the number of permutations of $[n]$ having k cycles. Sometimes the number of such permutations is referred to as the *unsigned Stirling number*.

Proposition 2.12 *A recurrence relation for the Stirling numbers of the first kind is*

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k)$$

and $s(0,0) = 1$*,* $s(0,k) = 0$ *for* $k \neq 0$*.*

Proof We split the permutations of $[n]$ into two classes.

- Those for which (n) is a single cycle arise by adjoining this cycle to a permutation of $[n-1]$ with $k-1$ cycles.
- The remainder are obtained by taking a permutation of $[n-1]$ with k cycles and interpolating n at some position in one of the cycles, for which there are $n-1$ choices.

The second construction changes the parity of $n-k$, accounting for the minus sign in the recurrence.

From here we get the generating function:

Proposition 2.13

$$\sum_k s(n,k)x^k = (x)_n.$$

Proof This is a routine induction. The case $n = 0$ is clear. If the result holds at $n-1$, then

$$\begin{aligned}\sum_k s(n,k)x^k &= \sum_k s(n-1,k-1)x^k - \sum_k (n-1)s(n-1,k)x^k \\ &= (x-n+1)(x)_{n-1} \\ &= (x)_n.\end{aligned}$$

This proposition can also be read to say that the Stirling numbers of the first kind populate the change of basis matrix from falling powers $\{(x)_n\}$ to powers x^n. This is the inverse change of basis to that in Proposition 2.11, provided by the Stirling numbers of the second kind. In other words, we have the identities

$$\sum_{k\geq 0} S(n,k)s(k,m) = \sum_{k\geq 0} s(n,k)S(k,m) = \begin{cases} 1 & n = m \\ 0 & \text{otherwise} \end{cases}$$

coming from composing these two changes of basis, for naturals m and n.