K-classes for matroids and equivariant localization

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Overview

If you remember one thing. . .

You can get the Tutte polynomial of an arbitrary matroid via algebraic geometry.

Outline:

- Setup: matroids and torus orbits on the Grassmannian; valuations and K-theory
- A K-theoretic matroid invariant
- Invariants that factor through it, incl. Tutte
- Equivariant localization
- Some ingredients of proofs

Matroids as polytopes

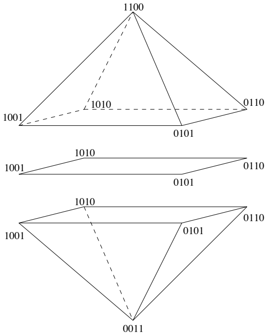
Definition (Edmonds; Gelfand-Goresky-MacPherson-Serganova)

A matroid M on the ground set $[n]$ is a polytope in $\mathbb{R}^n$ such that

- every vertex (basis) of M lies in $\{0,1\}^n$;
- every edge of M is parallel to $e_j - e_i$ for some $i, j \in [n]$.

The edges are the exchanges between the bases.

M lies in $\{\sum_{i=1}^n x_i = r\}$ for some r, the rank.



Matroid toric varieties on the Grassmannian

The Grassmannian is

$$G(r,n) = \{\text{configs of } n \text{ vectors spanning } \mathbb{C}^r\}/GL_r.$$

$T := (\mathbb{C}^*)^n \curvearrowright G(r,n)$ by scaling the vectors.

If $x_M \in G(r,n)$ represents M,
the orbit closure $\overline{Tx_M} \subseteq G(r,n)$ is the toric variety of M.

Toric degenerations $\mathcal{D}$ of $\overline{Tx_M} \longleftrightarrow$ certain matroid subdivisions Σ.

Components of $\mathcal{D}$ are toric varieties of facets of Σ.
Ditto intersections.

Schematic example

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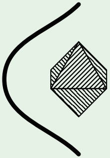
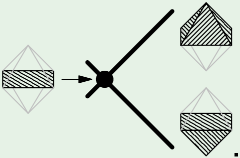

degenerates to

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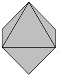
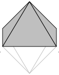
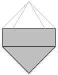
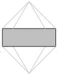
Matroid valuations

A matroid valuation f is a function that is additive with inclusion-exclusion in matroid subdivisions.

E.g. $f($  $) = f($  $) + f($  $) - f($  $)$.

Examples

- Lattice point count.
- the Tutte polynomial, $M \mapsto T_M \in \mathbb{Z}[x, y]$. (Not obvious!)

The Tutte polynomial is

$$T_M = \sum_{S \subseteq [n]} (x-1)^{\mathrm{corank}(S)} (y-1)^{\mathrm{nullity}(S)}.$$

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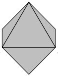
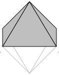
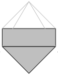
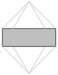
E.g. $f($  $) = f($  $) + f($  $) - f($  $)$.

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K-theory: a valuation from algebraic geometry

We use the *K-theory* ring $K_0(X)$
and the *T-equivariant K-theory* ring $K_0^T(X)$.

The class of $Y \subseteq X$ is denoted $[Y] \in K_0(X)$, resp. $[Y]^T \in K_0^T(X)$.

$[Y]^T$ determines $[Y]$.

Facts

- $[\cdot]^T$ is additive with inclusion-exclusion over components.
- $[\cdot]^T$ is unchanged by toric degenerations.

Theorem 1 (Speyer)

There is a valuation $Y : \{matroids\} \to K_0^T(G(r, n))$ *such that* $Y(M) = [\overline{Tx_M}]^T$ *for M representable.*

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Invariants that factor through K-theory

Theorem 2 (FS)

The Tutte polynomial factors through Y.
So do the Ehrhart polynomial, and Speyer's invariant h.

Speyer: How many faces can a matroid subdivision have?

Construct $h : \{\text{matroids}\} \to \mathbb{Z}[t]$:

- valuative (proved)
- positive ... (open in general)

Then h(uniform matroid) is an upper bound for the f-vector.

$h(\prod$ of k series-parallels$) = (-t)^k$.

Example

 are products of series-parallels, so $h($$) = -2t + t^2$.

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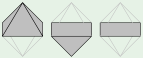
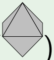
 are products of series-parallels, so $h($$) = -2t + t^2$.

Equivariant localization

Technique: Equivariant localization (Goresky-Kottwitz-MacPherson)

If X is nice & has an action of a big enough torus T, e.g. $G(r,n)$, $K_0^T(X)$ can be constructed from its moment graph Γ.

- $V(\Gamma) = \{T\text{-fixed points of } X\}$.
- $E(\Gamma) = \{1\text{-dimensional } T\text{-orbits of } X\}$.
  Their closures $\cong \mathbb{P}^1 = T \cup \underbrace{\{0,\infty\}}_{\text{endpoints}}$.

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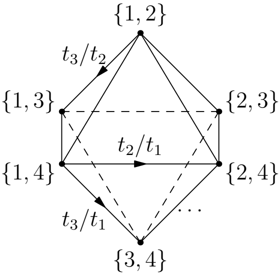
For $G(r, n)$, Γ is the union of all 1-skeleta of matroids.

- $V(\Gamma) \longleftrightarrow r$-subsets of n.
- $E(\Gamma) \longleftrightarrow$ exchanges $(S, S \setminus \{i\} \cup \{j\})$, with labels t_j/t_i.



Equivariant localization: the K-theory ring

$$K_0^T(\text{point}) = \mathbb{Z}[\text{Char } T] = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}].$$

Theorem (GKM, . . .)

$K_0^T(X)$ equals $\{$functions $V(\Gamma) \to K_0^T(\text{pt})$:
$f(v) \cong f(w)$ (mod $1 - \chi$) for $v \overset{\chi}{\text{———}} w$ an edge of $\Gamma\}$.

Example

There's a class $[\mathcal{O}(1)]$ on $G(r, n)$.

$$[\mathcal{O}(1)]^T(x_S) = t^S := \prod_{i \in S} t_i.$$

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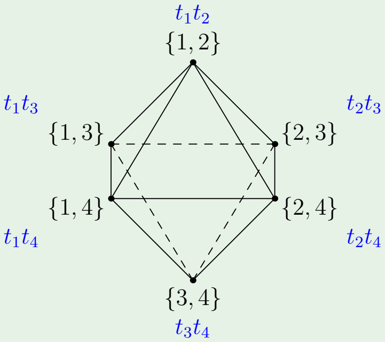
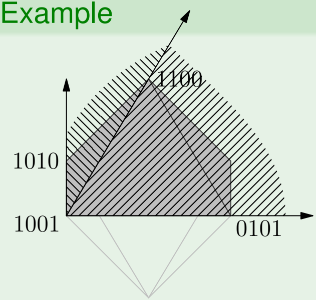


Classes of toric varieties on the moment graph

Near fixed points,
the toric variety of $M \cong$ toric varieties of tangent cones $\mathrm{Cone}_S(M)$.

The K-classes at points record multigraded Hilbert series, i.e. lattice point g.f.s of $\mathrm{Cone}_S(M)$.

Example



$\mathrm{Cone}_{\{1,4\}}(M)$ is simplicial, with g.f.

$$\frac{1}{(1-t_2/t_1)(1-t_2/t_4)(1-t_3/t_4)}$$

The denominator always divides $\prod_{i \in S, j \notin S}(1 - t_j/t_i)$.

Proof of Theorem 1

Proposition

The equivariant K-class of $\overline{Tx_M}$ is

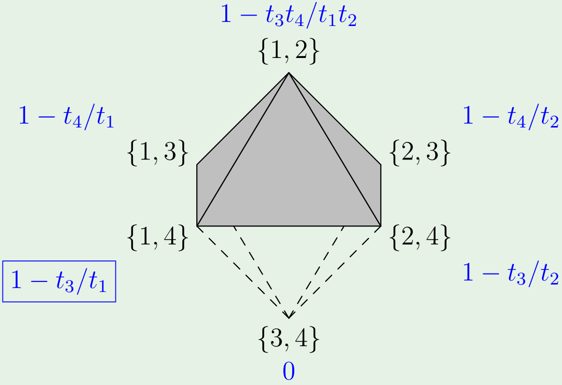
$$[\overline{Tx_M}]^T(x_S) = \underbrace{\left(\sum_{p \in \mathrm{Cone}_S(M) \cap \mathbb{Z}^n} t^p\right)}_{\text{lattice point g.f. of } \mathrm{Cone}_M(S)} \cdot \underbrace{\left(\prod_{i \in S, j \notin S} (1 - t_j/t_i)\right)}_{\text{denominator}}$$

Proof of Theorem 1.

The above is just polyhedral geometry. Do it for any matroid to construct $Y(M)$. □

Classes of toric varieties: example

Example



Closer relations between our invariants

Theorem 2

The Tutte polynomial, the Ehrhart polynomial, and Speyer's invariant h factor through Y.

Theorem 2′

There is a linear map $f : K_0^T(G(r,n)) \to \mathbb{Z}[x,y]$ *such that*

$$f(Y(M)) = h_M(1-(1-x)(1-y))$$
$$f(Y(M)\cdot[\mathcal{O}(1)]) = T_M(x,y)$$
$$f(Y(M)\cdot[\mathcal{O}(1)]^m)(0,0) = \mathit{Ehrhart}_M(m).$$

f is a pullback then a pushforward.

$$G(r,n) \leftarrow \mathcal{F}\ell(1,r,n-1;n) \to \mathbb{P}^{n-1}\times(\mathbb{P}^{n-1})^*$$

What happens equivariantly

There is an f^T, which becomes f non-equivariantly, such that

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$$f^T(Y(M)) = h_M(1-(1-x)(1-y))$$

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Note: f^T is **not** the equivariant pullback and pushforward!

Idea of proof: Rewrite in terms of the $\mathrm{Cone}_S(M)$.

To get at coefficients, flip all the cones' rays into a halfspace.

e.g. $\sum_{i\geq 0} t^i = \frac{1}{1-t} = -\sum_{i<0} t^i$

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Question

Can we use geometric positivity on combinatorial conjectures? For instance:

- Speyer's conjecture for h?
- for Tutte: Merino-Welsh type conjectures on convexity?
- de Loera's conjectures on the Ehrhart polynomial?

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