

## POLYMATROID SUBDIVISION: COMPUTATIONAL APPENDIX

ALEX FINK

In the *Polymatroid subdivision* notes to which these are an addendum, we showed that the three-dimensional set of matrices

$$\begin{pmatrix} 12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \\ 1 & 1 & 0 & 0 & 1 & u & v & tu - u & -tv + v & 0 \\ 1 & 0 & 1 & 0 & t & 1 & v & -tu + t & 0 & uv - v \\ 1 & 0 & 0 & 1 & t & u & 1 & 0 & tv - t & -uv + u \end{pmatrix}$$

with  $t, u, v \in \mathbb{C}$  make up an open set in the realisation space of the Desargues matroid. We then selected the subset of configurations with  $(t, u, v) = (t, t^2, t^4)$ , giving the realisation

$$A(t) = \begin{pmatrix} 12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \\ 1 & 1 & 0 & 0 & 1 & t^2 & t^4 & t^3 - t^2 & -t^5 + t^4 & 0 \\ 1 & 0 & 1 & 0 & t & 1 & t^4 & -t^3 + t & 0 & t^6 - t^4 \\ 1 & 0 & 0 & 1 & t & t^2 & 1 & 0 & t^5 - t & -t^6 + t^2 \end{pmatrix},$$

and used this to build a subdivision of the base polytope. This document is dedicated to stepping through how to get one's hands on it, concretely, in the software packages Macaulay2 and polymake.

We begin in Macaulay2; I am using version 1.6. The first task is to compute the valuations of the minors of  $A(t)$ . We begin by instantiating the ring  $\mathbb{Q}[t]$ , which is sufficient to define  $A(t)$ , and keying the realisation in.

```
i1 : R = QQ[t]

o1 = R

o1 : PolynomialRing

i2 : A = matrix {{1,1,0,0,1,t^2,t^4,t^3-t^2,-t^5+t^4,0},
                {1,0,1,0,t,1,t^4,-t^3+t,0,t^6-t^4},
                {1,0,0,1,t,t^2,1,0,t^5-t,-t^6+t^2}}

o2 = | 1 1 0 0 1 t2 t4 t3-t2 -t5+t4 0      |
      | 1 0 1 0 t 1  t4 -t3+t  0      t6-t4  |
      | 1 0 0 1 t t2 1  0      t5-t   -t6+t2 |

          3      10
o2 : Matrix R <--- R
```

The built-in function `minors` returns the ideal generated by given-sized minors of a matrix. The minors are iterated over in reverse lexicographic order: e.g. the

3-minors appear in the order

$$012, 013, 023, 123, 014, 024, 124, 034, 134, 234, 015 \dots,$$

following Macaulay2's zero-based conventions. Minors which are equal to zero are silently omitted from the list of generators, so there are  $\binom{10}{3} - 10 = 110$  generators here.

i3 : minors(3,A)

```

o3 = ideal (1, -1, 1, 1, t - 1, t, - t + 1, -t, 1, - t + 1, t, t - 1, -1, t,
-----
3 2 3 3 2 3 4 4 4 4
t - t - t + 1, t - t, t - t, - t + 1, t - 1, - t + 1, 1, -t, t, -
-----
5 4 5 5 5 4 6 4 2 6
t + t + t - 1, - t + t, t - 1, - t + t, t - t - t + 1, - t + 1,
-----
6 2 6 4 7 6 5 3 3 3 2 3 2
t - t, t - t, 2t - t - t - t + 1, - t + t, - t + t, 2t - t -
-----
3 3 2 4 3 2 4 2 4 3 4 5 4
t, t - t, t - t, t - t - t + t, t - t, t - t, - t + t, t - t -
-----
3 2 5 3 5 4 5 2 7 6 5 3 2 3
t + t, t - t, t - t, - t + t, - 2t + t + t + 2t - t - t, t -
-----
3 2 7 6 5 8 7 6 4 9 8 7 5
t, t - t, - 2t + t + t, - 2t + t + t + t - t, - 2t + t + t + t
-----
2 5 5 4 5 5 4 5 4 6 5 2
- t, - t + t, 2t - t - t, t - t, - t + t, - t + t, t - t - t +
-----
6 2 6 6 5 7 6 5 4 3 5
t, t - t, - t + t, t - t, - 2t + t + 2t - t + t - t, t - t, -
-----
7 6 3 5 4 8 7 6 4 9 8 5 4 9
2t + t + t, t - t, - 2t + t + t + t - t, t - t - t + t, t -
-----
5 9 4 9 8 11 10 9 7 4 8 7 6 5
t, - t + t, t - t, 2t - t - t - t + t, - 3t + 2t + 2t - t +
-----
4 3 2 8 6 4 2 8 7 4 3 8 7 6
2t - t - t, - t + t + t - t, - t + t + t - t, - t + t + t -
-----
5 9 8 7 5 2 10 9 8 6 3 12 11
t, - 2t + t + t + t - t, - 2t + t + t + t - t, - 2t + t +
-----
10 8 5 6 4 2 6 2 6 2 6 4 6 4

```

$$\begin{array}{l}
t^7 + t^6 - t^5, 2t^4 - t^3 - t^2, -t^7 + t^5, -t^6 + t^3, -t^5 + t^2, -t^4 + t^3, - \\
\hline
2t^7 + 2t^6 + t^5 - t^4 + t^3 - t^2, -2t^7 + t^5 + t^3, t^6 - t^5, t^4 - t^3, t^2 - t^3 - \\
\hline
4t^4 + t^2, -t^8 + t^2, t^8 - t^4, t^8 - t^6, 2t^6 - t^8 - t^7 - t^5 + t^2, t^8 - t^6 - \\
\hline
6t^6 + t^4, -t^{10} + t^4, t^{10} - t^6, t^{10} - t^8, 2t^8 - t^{11} - t^{10} - t^9 + t^7, 3t^9 - \\
\hline
8t^8 - 2t^7 + t^6 - 2t^5 + t^4 + t^3, t^9 - t^7 - t^5 + t^3, t^8 - t^7 - t^5 + t^3, t^7 - \\
\hline
8t^7 - t^6 + t^5, 2t^{10} - t^9 - t^8 - t^6 + t^3, 2t^{11} - t^{10} - t^9 - t^7 + t^4, 2t^{13} - \\
\hline
12t^{12} - t^{11} - t^9 + t^6, -3t^{11} + 2t^{10} + 2t^9 - t^8 + 2t^7 - t^6 - t^5, -t^{11} + t^9 + \\
\hline
7t^7 - t^5, -t^{11} + t^{10} + t^7 - t^6, -t^{11} + t^{10} + t^9 - t^8, -2t^{12} + t^{11} + t^{10} + \\
\hline
8t^8 - t^5, -2t^{13} + t^{12} + t^{11} + t^9 - t^6, -2t^{15} + t^{14} + t^{13} + t^{11} - t^8)
\end{array}$$

o3 : Ideal of R

The valuation is computed by the built-in function `weightRange`, which returns a pair of the lowest and highest degrees figuring in its input polynomial; we want only the 0th element of this pair. The syntax with `/` is a way of applying a function to every element of a list; `flatten entries gens` extracts the list from the ideal. The next step will be to copy and paste these valuations into `polymake`, so we call `toString` on the result to avoid the result being pretty-printed in a way that would interfere with copying and pasting.

i4 : toString(flatten entries gens o3 / (f -> (weightRange f)#0))

```
o4 = {0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 2, 0, 0, 2, 0, 1, 2, 0, 0, 0, 0, 4, 4, 0, 1,
      0, 4, 0, 0, 2, 4, 0, 1, 2, 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 2, 1, 1, 2, 5, 1,
      2, 1, 1, 1, 4, 4, 1, 2, 1, 5, 1, 1, 3, 4, 1, 4, 5, 4, 8, 4, 2, 2, 3, 5, 2,
      3, 5, 2, 2, 2, 4, 4, 2, 3, 2, 4, 2, 2, 4, 6, 2, 4, 4, 6, 8, 4, 3, 3, 4, 6,
      3, 4, 6, 5, 5, 6, 8, 5, 6, 8}
```

One other thing we will need is the list of positions of minors which are zero, so that we can tell `polymake` which triples of columns don't figure in the above list. We encode this data as a zero-one vector. The function `toString` generates the subsets of a given set of a given size, in the same order as `minors` uses them.

i5 : toString(subsets(10,3) / (J -> if det A\_J == 0 then 0 else 1))

```
o5 = {1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0,
      1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
      0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
      1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1,
      1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0}
```

We now turn to polymake; I am using version 2.14. Built in to polymake is a routine `matroid_from_characteristic_vector` (within application `matroid`) for constructing a matroid, given a zero-one vector of which sets index bases. The catch is that polymake iterates over sets in the opposite order as Macaulay2 does: the list of 3-element supsets of  $\{0, \dots, 9\}$  begins

```
012, 013, 014, 015, 016, 017, 018, 019, 023, 024, 025, ... ,
```

so that the zero-one vector above appears to be in a useless order. Luckily there is a workaround! Polymake's list of 7-element subsets of  $[10]$  is the list of complements of Macaulay2's list of 3-element subsets. So we will construct the dual of the Desargues matroid if we tell polymake the vector above belongs to a rank-7 matroid. For tidiness, we dualise it to get back to the Desargues matroid itself.

```
polytope > $m = matroid::dual(matroid::matroid_from_characteristic_vector(
[1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1,
1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0], 7, 10));
```

We store the heights in a vector of rationals as well.

```
polytope > $p = new Vector<Rational>([0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 2, 0, 0,
2, 0, 1, 2, 0, 0, 0, 0, 4, 4, 0, 1, 0, 4, 0, 0, 2, 4, 0, 1, 2, 1, 1, 2, 1, 2,
3, 1, 2, 3, 4, 2, 1, 1, 2, 5, 1, 2, 1, 1, 1, 4, 4, 1, 2, 1, 5, 1, 1, 3, 4, 1,
4, 5, 4, 8, 4, 2, 2, 3, 5, 2, 3, 5, 2, 2, 2, 4, 4, 2, 3, 2, 4, 2, 2, 4, 6, 2,
4, 4, 6, 8, 4, 3, 3, 4, 6, 3, 4, 6, 5, 5, 6, 8, 5, 6, 8]);
```

Regular subdivisions in polymake are created from a point configuration with its lifting heights rather than a polytope per se, so we extract the `VERTICES` of the `POLYTOPE` of the matroid.

```
polytope > $subdiv = regular_subdivision($m->POLYTOPE->VERTICES, $p);
polymake: used package cdd
  cddlib
  Implementation of the double description method of Motzkin et al.
  Copyright by Komei Fukuda.
  http://www.ifor.math.ethz.ch/~fukuda/cdd_home/cdd.html
```

```
polymake: used package ppl
  The Parma Polyhedra Library (PPL): A C++ library for convex polyhedra
  and other numerical abstractions.
  http://www.cs.unipr.it/ppl/
```

Let's see the output! A regular subdivision is returned as an array of sets of indices of the vertices, each set specifying one maximal-dimensional face. We see there are 14 such faces.

```

polytope > print $subdiv;
{2 3 8 11 12 17 19 20 25 27 28 31 34 35 36 40 44 45 46 47 49 50 52 53 55 58
60 61 63 64 67 69 70 71 72 73 74 75 76 78 79 80 81 84 85 86 87 90 91 92 95 96
97 98 99 100 101 102 103 104 105 106 107 108 109}
{2 3 8 11 12 13 17 19 20 22 25 27 28 29 30 31 36 44 47 50 55 63 67 69 78 79
80 81 84 85 86 87 88 89 90 91 92 93 94 95 98 99 101 102 105 106 108 109}
{1 2 3 6 8 11 12 17 18 19 20 23 25 27 28 31 34 35 40 45 46 49 51 52 53 54 55
56 58 60 61 63 64 65 67 69 70 71 73 74 76 80 81 85 91 92 95 103 104 106 107
109}
{1 3 6 7 8 12 17 18 20 21 23 24 25 26 28 31 35 40 46 49 51 53 54 55 56 57 58
59 61 63 64 65 66 67 68 69 71 73 74 76 81 85 92 95 104 106 107 109}
{2 3 8 11 12 17 19 20 22 25 27 28 30 31 34 35 36 40 44 45 46 47 48 49 50 55
63 67 69 73 76 80 81 85 91 92 94 95 99 102 106 109}
{1 2 3 6 8 11 12 17 18 19 20 21 22 23 25 26 27 28 30 31 34 35 40 45 46 48 49
54 55 63 65 67 68 69 73 76 80 81 85 91 92 94 95 106 109}
{0 2 3 4 8 9 10 11 12 13 14 16 17 19 20 25 27 28 29 31 33 36 41 43 44 47 50
52 53 58 60 61 62 64 72 75 78 79 84 86 87 88 90 98 101}
{0 1 2 3 4 6 8 9 11 12 14 17 18 19 20 23 25 27 28 31 32 34 35 37 40 45 46 49
51 52 53 56 58 60 61 64 70 71 74 77 78 79 82 84 86 87 90 96 97 100}
{0 2 3 4 8 9 11 12 14 17 19 20 25 27 28 31 32 33 34 35 36 37 40 41 44 45 46
47 49 50 52 53 58 60 61 64 70 71 72 74 75 78 79 84 86 87 90 96 97 98 100 101}
{1 3 6 7 8 12 17 18 20 23 24 25 28 31 35 40 46 49 51 53 56 57 58 61 64 71 74
77 79 81 82 83 84 85 87 90 92 95 97 100 104 107}
{2 3 8 11 12 13 17 19 20 25 27 28 29 31 36 44 47 50 52 53 55 58 60 61 62 63
64 67 69 72 75 78 79 84 86 87 88 90 98 101 105 108}
{0 1 3 4 5 6 7 8 9 12 14 15 17 18 20 23 24 25 28 31 32 35 37 38 40 46 49 51
53 56 57 58 61 64 71 74 77 79 82 83 84 87 90 97 100}
{0 3 4 5 8 9 10 12 14 15 16 17 20 25 28 31 32 33 35 36 37 38 39 40 41 42 43
44 46 47 49 50 53 58 61 64 71 72 74 75 79 84 87 90 97 98 100 101}
{1 2 3 6 8 11 12 17 18 19 20 23 25 27 28 31 34 35 40 45 46 49 51 52 53 56 58
60 61 64 70 71 74 77 78 79 80 81 82 84 85 86 87 90 91 92 95 96 97 100 103 104
107}

```

This is somewhat ungainly to work with as is; converting the data into a `PolyhedralComplex` object allows more to be done with it directly.

```

polytope > $c = new fan::PolyhedralComplex(POINTS => $m->POLYTOPE->VERTICES,
INPUT_POLYTOPES => $subdiv);

```

For instance, suppose we want to describe the matroids of the top-dimensional cells in terms of their circuits, so that we can draw point and line arrangement pictures of them. For this, we convert each of the top-dimensional polytopes (accessed using `polytope(i)`, where  $i = 0, 1, \dots$  is the index number) back to a matroid using `matroid_from_matroid_polytope`, and then ask for its `CIRCUITS`. The `for` loop does this for each of the fourteen cells.

We learn that, in fact, each of the cells has several pairs of parallel points. The points have of course been renumbered from 0 to 9: our earlier 12 is now “0”, etc.

```

polytope > for(my $i=0; $i<14; $i++) {
polytope (2)> print matroid::matroid_from_matroid_polytope($c->polytope($i))->CIRCUITS, "\n";
polytope (3)> }

```

{3 6}  
{4 7}  
{5 8}  
{5 9}  
{8 9}  
{0 1 2}  
{0 3 4}  
{0 3 7}  
{0 4 6}  
{0 6 7}  
{1 3 5}  
{1 3 8}  
{1 3 9}  
{1 5 6}  
{1 6 8}  
{1 6 9}  
{2 4 5}  
{2 4 8}  
{2 4 9}  
{2 5 7}  
{2 7 8}  
{2 7 9}  
{0 1 4 5}  
{0 1 4 8}  
{0 1 4 9}  
{0 1 5 7}  
{0 1 7 8}  
{0 1 7 9}  
{0 2 3 5}  
{0 2 3 8}  
{0 2 3 9}  
{0 2 5 6}  
{0 2 6 8}  
{0 2 6 9}  
{1 2 3 4}  
{1 2 3 7}  
{1 2 4 6}  
{1 2 6 7}

{1 2}  
{1 5}  
{1 8}  
{1 9}  
{2 5}  
{2 8}  
{2 9}  
{5 8}  
{5 9}

{8 9}  
{0 3 4}  
{0 6 7}  
{1 3 6}  
{1 4 7}  
{2 3 6}  
{2 4 7}  
{3 5 6}  
{3 6 8}  
{3 6 9}  
{4 5 7}  
{4 7 8}  
{4 7 9}  
{0 1 3 7}  
{0 1 4 6}  
{0 2 3 7}  
{0 2 4 6}  
{0 3 5 7}  
{0 3 7 8}  
{0 3 7 9}  
{0 4 5 6}  
{0 4 6 8}  
{0 4 6 9}  
{3 4 6 7}

{0 2}  
{0 4}  
{0 7}  
{2 4}  
{2 7}  
{3 6}  
{4 7}  
{5 8}  
{0 5 9}  
{0 8 9}  
{1 3 5}  
{1 3 8}  
{1 5 6}  
{1 6 8}  
{2 5 9}  
{2 8 9}  
{4 5 9}  
{4 8 9}  
{5 7 9}  
{7 8 9}  
{0 1 3 9}  
{0 1 6 9}  
{1 2 3 9}

{1 2 6 9}  
{1 3 4 9}  
{1 3 7 9}  
{1 4 6 9}  
{1 6 7 9}

{0 2}  
{0 4}  
{0 7}  
{0 9}  
{2 4}  
{2 7}  
{2 9}  
{4 7}  
{4 9}  
{7 9}  
{0 3 6}  
{0 5 8}  
{1 3 5}  
{1 6 8}  
{2 3 6}  
{2 5 8}  
{3 4 6}  
{3 6 7}  
{3 6 9}  
{4 5 8}  
{5 7 8}  
{5 8 9}  
{0 1 3 8}  
{0 1 5 6}  
{1 2 3 8}  
{1 2 5 6}  
{1 3 4 8}  
{1 3 7 8}  
{1 3 8 9}  
{1 4 5 6}  
{1 5 6 7}  
{1 5 6 9}  
{3 5 6 8}

{0 4}  
{0 7}  
{1 5}  
{1 8}  
{1 9}  
{4 7}  
{5 8}  
{5 9}



{8 9}  
{0 1 2}  
{0 2 5}  
{0 2 8}  
{0 2 9}  
{1 2 4}  
{1 2 7}  
{1 3 6}  
{2 4 5}  
{2 4 8}  
{2 4 9}  
{2 5 7}  
{2 7 8}  
{2 7 9}  
{3 5 6}  
{3 6 8}  
{3 6 9}  
{0 2 3 6}  
{2 3 4 6}  
{2 3 6 7}

{0 2}  
{0 4}  
{0 7}  
{1 5}  
{1 8}  
{2 4}  
{2 7}  
{4 7}  
{5 8}  
{0 1 9}  
{0 5 9}  
{0 8 9}  
{1 2 9}  
{1 4 9}  
{1 7 9}  
{2 5 9}  
{2 8 9}  
{3 6 9}  
{4 5 9}  
{4 8 9}  
{5 7 9}  
{7 8 9}  
{0 1 3 6}  
{0 3 5 6}  
{0 3 6 8}  
{1 2 3 6}  
{1 3 4 6}

{1 3 6 7}  
{2 3 5 6}  
{2 3 6 8}  
{3 4 5 6}  
{3 4 6 8}  
{3 5 6 7}  
{3 6 7 8}

{0 1}  
{0 3}  
{0 6}  
{1 3}  
{1 6}  
{2 5}  
{2 8}  
{3 6}  
{5 8}  
{0 2 9}  
{0 5 9}  
{0 8 9}  
{1 2 9}  
{1 5 9}  
{1 8 9}  
{2 3 9}  
{2 6 9}  
{3 5 9}  
{3 8 9}  
{4 7 9}  
{5 6 9}  
{6 8 9}  
{0 2 4 7}  
{0 4 5 7}  
{0 4 7 8}  
{1 2 4 7}  
{1 4 5 7}  
{1 4 7 8}  
{2 3 4 7}  
{2 4 6 7}  
{3 4 5 7}  
{3 4 7 8}  
{4 5 6 7}  
{4 6 7 8}

{0 1}  
{0 3}  
{0 6}  
{1 3}  
{1 6}

{2 4}  
{2 7}  
{3 6}  
{4 7}  
{5 8}  
{0 2 5 9}  
{0 2 8 9}  
{0 4 5 9}  
{0 4 8 9}  
{0 5 7 9}  
{0 7 8 9}  
{1 2 5 9}  
{1 2 8 9}  
{1 4 5 9}  
{1 4 8 9}  
{1 5 7 9}  
{1 7 8 9}  
{2 3 5 9}  
{2 3 8 9}  
{2 5 6 9}  
{2 6 8 9}  
{3 4 5 9}  
{3 4 8 9}  
{3 5 7 9}  
{3 7 8 9}  
{4 5 6 9}  
{4 6 8 9}  
{5 6 7 9}  
{6 7 8 9}

{0 1}  
{0 3}  
{0 6}  
{1 3}  
{1 6}  
{3 6}  
{4 7}  
{5 8}  
{0 5 9}  
{0 8 9}  
{1 5 9}  
{1 8 9}  
{2 4 5}  
{2 4 8}  
{2 5 7}  
{2 7 8}  
{3 5 9}  
{3 8 9}

{5 6 9}  
{6 8 9}  
{0 2 4 9}  
{0 2 7 9}  
{1 2 4 9}  
{1 2 7 9}  
{2 3 4 9}  
{2 3 7 9}  
{2 4 6 9}  
{2 6 7 9}

{1 3}  
{1 6}  
{2 4}  
{2 7}  
{2 9}  
{3 6}  
{4 7}  
{4 9}  
{7 9}  
{0 1 2}  
{0 1 4}  
{0 1 7}  
{0 1 9}  
{0 2 3}  
{0 2 6}  
{0 3 4}  
{0 3 7}  
{0 3 9}  
{0 4 6}  
{0 6 7}  
{0 6 9}  
{2 5 8}  
{4 5 8}  
{5 7 8}  
{5 8 9}  
{0 1 5 8}  
{0 3 5 8}  
{0 5 6 8}

{0 3}  
{0 6}  
{2 5}  
{2 8}  
{2 9}  
{3 6}  
{5 8}  
{5 9}

{8 9}  
{0 1 2}  
{0 1 5}  
{0 1 8}  
{0 1 9}  
{1 2 3}  
{1 2 6}  
{1 3 5}  
{1 3 8}  
{1 3 9}  
{1 5 6}  
{1 6 8}  
{1 6 9}  
{2 4 7}  
{4 5 7}  
{4 7 8}  
{4 7 9}  
{0 1 4 7}  
{1 3 4 7}  
{1 4 6 7}

{0 1}  
{0 3}  
{0 6}  
{1 3}  
{1 6}  
{2 4}  
{2 7}  
{3 6}  
{4 7}  
{0 2 9}  
{0 4 9}  
{0 7 9}  
{1 2 9}  
{1 4 9}  
{1 7 9}  
{2 3 9}  
{2 6 9}  
{3 4 9}  
{3 7 9}  
{4 6 9}  
{5 8 9}  
{6 7 9}  
{0 2 5 8}  
{0 4 5 8}  
{0 5 7 8}  
{1 2 5 8}  
{1 4 5 8}

{1 5 7 8}  
{2 3 5 8}  
{2 5 6 8}  
{3 4 5 8}  
{3 5 7 8}  
{4 5 6 8}  
{5 6 7 8}

{0 1}  
{0 3}  
{0 6}  
{0 9}  
{1 3}  
{1 6}  
{1 9}  
{3 6}  
{3 9}  
{6 9}  
{0 4 7}  
{0 5 8}  
{1 4 7}  
{1 5 8}  
{2 4 5}  
{2 7 8}  
{3 4 7}  
{3 5 8}  
{4 6 7}  
{4 7 9}  
{5 6 8}  
{5 8 9}  
{0 2 4 8}  
{0 2 5 7}  
{1 2 4 8}  
{1 2 5 7}  
{2 3 4 8}  
{2 3 5 7}  
{2 4 6 8}  
{2 4 8 9}  
{2 5 6 7}  
{2 5 7 9}  
{4 5 7 8}

{1 3}  
{1 6}  
{2 4}  
{2 7}  
{3 6}  
{4 7}

{5 8}  
{0 1 2}  
{0 1 4}  
{0 1 7}  
{0 2 3}  
{0 2 6}  
{0 3 4}  
{0 3 7}  
{0 4 6}  
{0 6 7}  
{2 5 9}  
{2 8 9}  
{4 5 9}  
{4 8 9}  
{5 7 9}  
{7 8 9}  
{0 1 5 9}  
{0 1 8 9}  
{0 3 5 9}  
{0 3 8 9}  
{0 5 6 9}  
{0 6 8 9}