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On weight three blocks of symmetric groups in characteristic three

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Contents

1 Introduction

In [2], the author constructed the ordinary quivers of two symmetric group blocks of weight three in characteristic three. By results of Scopes, there are only finitely many such blocks (up to Morita equivalence), and here we complete the list of quivers for these blocks.

The books of James [3] and James and Kerber [4] give an excellent introduction to the representation theory of the symmetric group $\mathfrak{S}_n$; we summarise the important points.

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For any partition λ of n, one defines a *Specht module* S^λ over any field k; if k has characteristic zero, these modules are irreducible and pairwise non-isomorphic, and give a complete set of irreducibles for $k\mathfrak{S}_n$; in characteristic $p > 0$, the Specht modules are not irreducible in general; for each p-regular partition λ, S^λ has an irreducible cosocle D^λ; the modules D^λ are pairwise non-isomorphic and give a complete set of irreducibles for $k\mathfrak{S}_n$. The decomposition matrix for a symmetric group block records the composition multiplicities $[S^\lambda : D^\mu]$; the matrices for the blocks we consider in this paper are reproduced in the appendix.

The p-blocks of $\mathfrak{S}_n$ are determined by the Nakayama Conjecture [4, 6.1.21]; we may thus use James's abacus [4, pp. 78–80] to represent the partitions in a given block. We also employ the Branching Rule [3, Theorem 9.3], and the Carter-Lusztig theorem on semistandard homomorphisms [3, Theorem 13.13].

The *ordinary quiver* or *Ext-quiver* of an algebra A over a field k is a quiver with vertices indexed by simple A-modules, and with a number of arrows from vertex S to vertex T equal to $\dim_k \mathrm{Ext}^1_A(S,T)$. Simple modules for symmetric group blocks are self-dual, and so we may draw an undirected edge in our quiver to indicate an arrow in each direction.

1.1 Notation for modules

- Given any module M and any simple module S, we write $[M : S]$ for the multiplicity of S as a composition factor of M.

- If M has composition factors $S_1, \dots, S_r$, we write
$$M \sim S_1 + \ldots + S_r.$$

- If M lies in a block A of a symmetric group, and B is a block of a larger (resp. smaller) symmetric group, we write $M{\uparrow}^B$ (resp. $M{\downarrow}_B$) to denote the module M induced (resp. restricted) to B.

- We write $P(M)$ for the projective cover of M.

- If the submodule lattice of M is distributive, that is, if M does not have the direct sum of two isomorphic modules as a subquotient, we frequently encode the submodule lattice of M by drawing the (Hasse diagram of the) poset of submodules of M which have simple cosocle, labelling each by (the isomorphism class of) its cosocle. If the structure of M is particularly simple, we frequently omit the edges from this diagram.

1.2 Schaper's formula

Schaper's formula (see for example [1]) is a useful tool for calculating decomposition numbers. In addition, if the decomposition numbers are known, the formula can provide useful information about the structure of Specht modules. Specifically, the Specht module S^λ has a filtration
$$S^\lambda = S^\lambda_{(0)} \geqslant S^\lambda_{(1)} \geqslant S^\lambda_{(2)} \geqslant \ldots,$$

and Schaper's formula gives, for each simple module D^μ, the sum

$$\sum_{i=1}^{\infty}[S^\lambda_{(i)} : D^\mu].$$

Moreover, each quotient $S^\lambda_{(i)}/S^\lambda_{(i+1)}$ is self-dual (although not always semi-simple). $S^\lambda_{(1)}$ is the radical of S^λ when λ is p-regular, while $S^\lambda_{(1)}$ equals S^λ when λ is p-singular.

In fact, $S^\lambda_{(i)}$ is the reduction modulo p of the submodule

$$\{x \mid \langle x, y\rangle \equiv 0 \pmod{p^i}\ \forall\ y\}$$

of the integral Specht module $S^\lambda_{\mathbb{Z}}$ with its usual inner product $\langle , \rangle$.

2 The Scopes equivalence

Here we use the abacus notation extensively. Let B and C be symmetric group blocks of weight ω. We say that B and C form an $[\omega : \kappa]$-pair if there exist abacuses for B and C such that

- for some i, the abacus for B has κ more beads on runner $i-1$ than on runner i, and
- the abacus for C is obtained from that for B by moving κ beads from runner $i-1$ to runner i.

Scopes [5] then proves the following.

Theorem 2.1 (Scopes)**.** *If $\omega \leqslant \kappa$, then two symmetric group blocks forming an $[\omega : \kappa]$-pair are Morita equivalent. Hence there are only finitely many Morita equivalence classes of symmetric group blocks with a given weight in a given characteristic.*

We may represent the Scopes equivalence in characteristic p as follows: we draw a graph with vertices indexed by p-cores, with an edge labelled κ between cores λ and μ if the blocks of weight ω with cores λ and μ form an $[\omega : \kappa]$-pair. In characteristic three, part of this infinite graph is given in Figure 1; for each block, its conjugate block (that is, the block found by tensoring with the signature representation) is found by reflection in a central vertical axis.

In this paper, we shall concern ourselves with the case $p = 3$, $\omega = 3$. The Scopes classes of blocks may then be represented as in Figure 2, where each vertex represents an equivalence class and is labelled by the core of the 'smallest' member of that class, and where there is an edge labelled with an integer $\kappa < 3$ between classes $\mathfrak{B}$ and $\mathfrak{C}$ if B and C form a $[3 : \kappa]$ pair for some $B \in \mathfrak{B}$, $C \in \mathfrak{C}$. Two blocks are Scopes equivalent if and only if their conjugates are, and again conjugate blocks can be found by reflecting the diagram left-to-right.

Some of these blocks have already been studied; Tan [6, 7] found the Ext-quivers of the principal blocks of $k\mathfrak{S}_9$, $k\mathfrak{S}_{10}$ and $k\mathfrak{S}_{11}$, where k is a field of characteristic three, and the present author [2] found the quivers for the blocks of $k\mathfrak{S}_{13}$ and $k\mathfrak{S}_{14}$ with cores $(3, 1)$ and $(3, 1^2)$ respectively. Of course, conjugate blocks have the same quivers, so only three quivers remain to be found; we can find these by analysing three $[3 : 2]$-pairs, which turn out to behave very similarly.

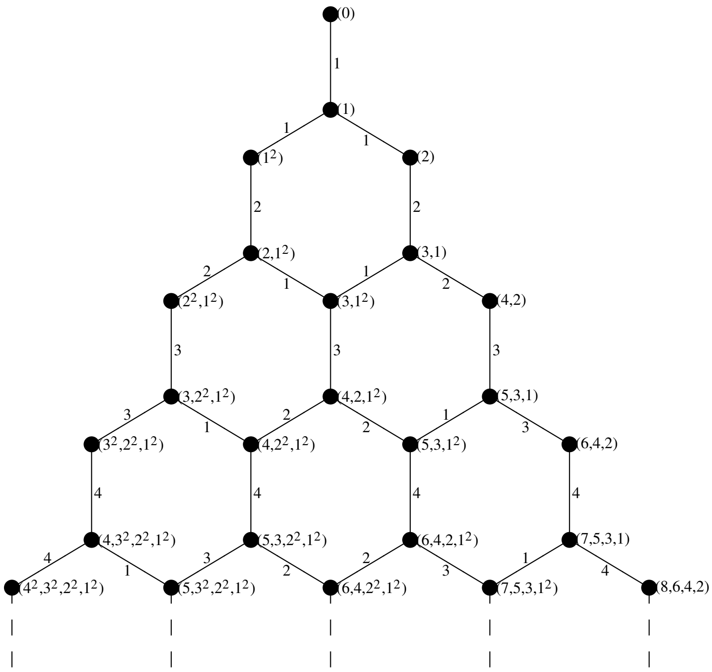

Figure 1: The Scopes equivalence in characteristic three

3 The Ext-quivers of weight three blocks in characteristic three

Given a weight three block B of a symmetric group in characteristic three, suppose we have an abacus for B with a beads on the first runner, b on the second, and c on the third. We may then denote the partitions of B using the $\langle a, b, c\rangle$ abacus notation:

- $\langle i\rangle$ denotes a partition with a bead of weight three on runner i;
- $\langle i, j\rangle$ denotes a partition with a bead of weight two on runner i and a bead of weight one on runner j;
- $\langle i, j, k\rangle$ denotes a partition with beads of weight one on runners i, j and k.

Let B_1 denote the block of $k\mathfrak{S}_{13}$ with core $(3,1)$, with the $\langle 3,5,3\rangle$ abacus notation, and C_1 the block of $k\mathfrak{S}_{15}$ with core $(4,2)$, with the $\langle 3,3,5\rangle$ abacus notation. The Ext-quiver of B_1 is found in [2], and is reproduced in Figure 3.

Let B_2 denote the block of $k\mathfrak{S}_{17}$ with core $(4,2,1^2)$, with the $\langle 3,6,4\rangle$ notation, and C_2 the block of $k\mathfrak{S}_{19}$ with core $(5,3,1^2)$, with the $\langle 3,4,6\rangle$ notation. The quiver of B_2 is the same as that of the block of $k\mathfrak{S}_{14}$ with core $(3,1^2)$, with which it forms a $[3:3]$-pair. The latter quiver is also found in [2]. By re-labelling the vertices, we obtain the quiver of B_2, also shown in Figure 3.

Let B_3 denote the block of $k\mathfrak{S}_{23}$ with core $(5,3,2^2,1^2)$, with the $\langle 3,7,5\rangle$ notation, and C_3 the block of $k\mathfrak{S}_{25}$ with core $(6,4,2^2,1^2)$, with the $\langle 3,5,7\rangle$-notation. The quiver of B_3 is the same as that of its conjugate block, whose core is $(6,4,2,1^2)$; the latter forms a $[3:4]$-pair with C_2, and so the quiver of B_3 will follow from the analysis of C_2.

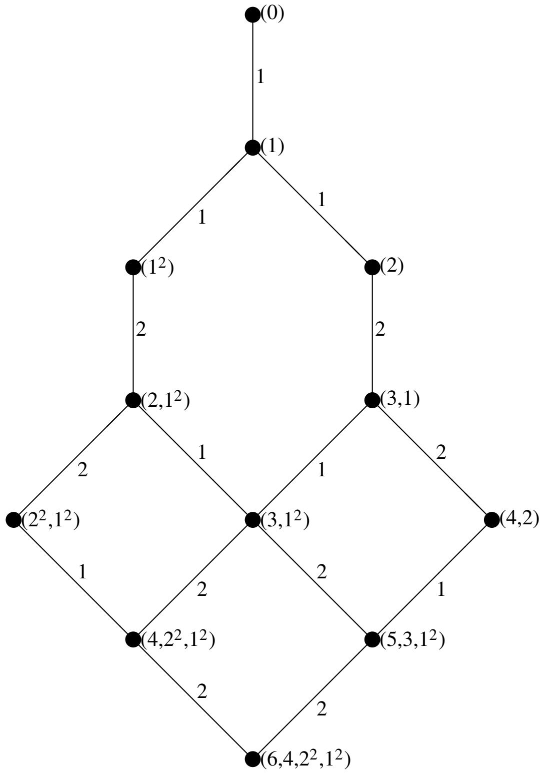

Figure 2: The Scopes classes of weight three blocks in characteristic three

First we examine induction and restriction between B_i and C_i.

Proposition 3.1.

1. *$D_{C_i}^{\langle 3,3,3\rangle}$ does not self-extend.*

2. *For $i = 1, 2, 3$ we have*

$$
\begin{aligned}
D_{C_i}^{\langle 3\rangle}{\downarrow_{B_i}} &\cong D_{B_i}^{\langle 2\rangle} \oplus D_{B_i}^{\langle 2\rangle}, &\qquad D_{B_i}^{\langle 2\rangle}{\uparrow^{C_i}} &\cong D_{C_i}^{\langle 3\rangle} \oplus D_{C_i}^{\langle 3\rangle};\\
D_{C_i}^{\langle 3,3\rangle}{\downarrow_{B_i}} &\cong D_{B_i}^{\langle 2,2\rangle} \oplus D_{B_i}^{\langle 2,2\rangle}, & D_{B_i}^{\langle 2,2\rangle}{\uparrow^{C_i}} &\cong D_{C_i}^{\langle 3,3\rangle} \oplus D_{C_i}^{\langle 3,3\rangle};\\
D_{C_i}^{\langle 3,2\rangle}{\downarrow_{B_i}} &\cong D_{B_i}^{\langle 2,3\rangle} \oplus D_{B_i}^{\langle 2,3\rangle}, & D_{B_i}^{\langle 2,3\rangle}{\uparrow^{C_i}} &\cong D_{C_i}^{\langle 3,2\rangle} \oplus D_{C_i}^{\langle 3,2\rangle};\\
D_{C_i}^{\langle 2,3,3\rangle}{\downarrow_{B_i}} &\cong D_{B_i}^{\langle 2,2,2\rangle} \oplus D_{B_i}^{\langle 2,2,2\rangle}, & D_{B_i}^{\langle 2,2,2\rangle}{\uparrow^{C_i}} &\cong D_{C_i}^{\langle 2,3,3\rangle} \oplus D_{C_i}^{\langle 2,3,3\rangle};\\
D_{C_i}^{\langle 2,3\rangle}{\downarrow_{B_i}} &\cong D_{B_i}^{\langle 2,2,3\rangle} \oplus D_{B_i}^{\langle 2,2,3\rangle}, & D_{B_i}^{\langle 2,2,3\rangle}{\uparrow^{C_i}} &\cong D_{C_i}^{\langle 2,3\rangle} \oplus D_{C_i}^{\langle 2,3\rangle};\\
D_{C_i}^{\langle 2\rangle}{\downarrow_{B_i}} &\cong D_{B_i}^{\langle 3,2\rangle} \oplus D_{B_i}^{\langle 3,2\rangle}, & D_{B_i}^{\langle 3,2\rangle}{\uparrow^{C_i}} &\cong D_{C_i}^{\langle 2\rangle} \oplus D_{C_i}^{\langle 2\rangle};
\end{aligned}
$$

for $i = 2, 3$ we have

$$
\begin{aligned}
D_{C_i}^{\langle 2,2,3\rangle}{\downarrow_{B_i}} &\cong D_{B_i}^{\langle 2,3,3\rangle} \oplus D_{B_i}^{\langle 2,3,3\rangle}, &\qquad D_{B_i}^{\langle 2,3,3\rangle}{\uparrow^{C_i}} &\cong D_{C_i}^{\langle 2,2,3\rangle} \oplus D_{C_i}^{\langle 2,2,3\rangle};\\
D_{C_i}^{\langle 2,2\rangle}{\downarrow_{B_i}} &\cong D_{B_i}^{\langle 3,3\rangle} \oplus D_{B_i}^{\langle 3,3\rangle}, & D_{B_i}^{\langle 3,3\rangle}{\uparrow^{C_i}} &\cong D_{C_i}^{\langle 2,2\rangle} \oplus D_{C_i}^{\langle 2,2\rangle};
\end{aligned}
$$

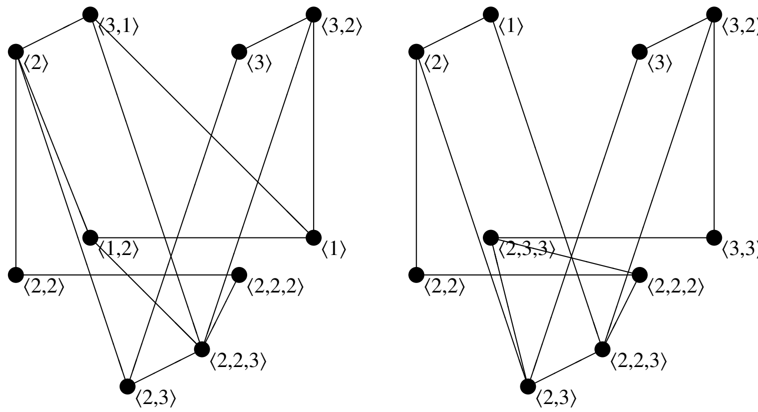

Figure 3: The Ext-quivers of B_1 and B_2

for $i = 1, 2$ we have

$$D^{\langle 1\rangle}_{C_i}{\downarrow}_{B_i} \cong D^{\langle 1\rangle}_{B_i} \oplus D^{\langle 1\rangle}_{B_i}, \qquad D^{\langle 1\rangle}_{B_i}{\uparrow}^{C_i} \cong D^{\langle 1\rangle}_{C_i} \oplus D^{\langle 1\rangle}_{C_i};$$

we also have

$$\begin{aligned} D^{\langle 2,2,2\rangle}_{C_3}{\downarrow}_{B_3} &\cong D^{\langle 3,3,3\rangle}_{B_3} \oplus D^{\langle 3,3,3\rangle}_{B_3}, & D^{\langle 3,3,3\rangle}_{B_3}{\uparrow}^{C_3} &\cong D^{\langle 2,2,2\rangle}_{C_3} \oplus D^{\langle 2,2,2\rangle}_{C_3};\\ D^{\langle 1,3\rangle}_{C_1}{\downarrow}_{B_1} &\cong D^{\langle 1,2\rangle}_{B_1} \oplus D^{\langle 1,2\rangle}_{B_1}, & D^{\langle 1,2\rangle}_{B_1}{\uparrow}^{C_1} &\cong D^{\langle 1,3\rangle}_{C_1} \oplus D^{\langle 1,3\rangle}_{C_1};\\ D^{\langle 2,1\rangle}_{C_1}{\downarrow}_{B_1} &\cong D^{\langle 3,1\rangle}_{B_1} \oplus D^{\langle 3,1\rangle}_{B_1}, & D^{\langle 3,1\rangle}_{B_1}{\uparrow}^{C_1} &\cong D^{\langle 2,1\rangle}_{C_1} \oplus D^{\langle 2,1\rangle}_{C_1}. \end{aligned}$$

Proof.

1. From the decomposition matrix for C_i, we see that $P(D^{\langle 3,3,3\rangle}_{C_i})$ has a filtration by the Specht modules $S^{\langle 3,3,3\rangle}_{C_i}$, $S^{\langle 2,3,3\rangle}_{C_i}$, $S^{\langle 2,3\rangle}_{C_i}$ and $S^{\langle 2,2,2\rangle}_{C_i}$. Since the partitions corresponding to these Specht modules are all 3-regular, $D^{\langle 3,3,3\rangle}_{C_i}$ cannot appear in the second Loewy layer of $P(D^{\langle 3,3,3\rangle}_{C_i})$.

2. The composition factors of the induced and restricted modules may be found using the Branching Rule, the decomposition matrices for B_i and C_i (which may be found in Appendix A) and the fact that induction and restriction are exact functors. Since none of the simple modules in B_1 or B_2 self-extends, the restricted modules in the cases $i = 1, 2$ must be semi-simple. Frobenius reciprocity then implies that the induced modules in these cases are semi-simple. By the Eckmann-Shapiro relations and the general fact that
$$\mathrm{Ext}^1(M, N \oplus N) \cong \mathrm{Ext}^1(M, N) \oplus \mathrm{Ext}^1(M, N),$$
we then find that none of the simple modules of C_2 self-extends. Since C_2 is Morita equivalent to B_3, none of the simple modules of the latter block self-extends, and the restricted modules in the case $i = 3$ are semi-simple, as are the induced modules.

□

By the Eckmann-Shapiro relations, then, we may determine most of the Ext-spaces in C_i; in the cases $i = 1, 2$, we know $\mathrm{Ext}^1(D^{\lambda}_{C_i}, D^{\mu}_{C_i})$ provided neither λ nor μ equals $\langle 3,3,3\rangle$, and in the case $i = 3$ we know $\mathrm{Ext}^1(D^{\lambda}_{C_i}, D^{\mu}_{C_i})$ provided neither λ nor μ equals $\langle 3,3,3\rangle$ or $\langle 2,2,2\rangle$ (although of course the latter case will follow once we know the extensions of $D^{\langle 3,3,3\rangle}_{C_2}$). We proceed, then, to find the extensions of $D^{\langle 3,3,3\rangle}_{C_i}$.

As noted in the proof of Proposition 3.1, the projective cover of $D^{\langle 3,3,3\rangle}_{C_i}$ is filtered by $S^{\langle 3,3,3\rangle}_{C_i}$, $S^{\langle 2,3,3\rangle}_{C_i}$, $S^{\langle 2,3\rangle}_{C_i}$ and $S^{\langle 2,2,2\rangle}_{C_i}$. From the decomposition matrix we see that the first of these Specht modules has structure

$$\begin{matrix} D^{\langle 3,3,3\rangle}_{C_i} \\ D^{\langle 3,3\rangle}_{C_i} \end{matrix},$$

and so $D^{\langle 3,3,3\rangle}_{C_i}$ can only possibly extend $D^{\langle 3,3\rangle}_{C_i}$, $D^{\langle 2,3,3\rangle}_{C_i}$, $D^{\langle 2,3\rangle}_{C_i}$ and $D^{\langle 2,2,2\rangle}_{C_i}$, and the corresponding Ext-spaces are at most one-dimensional.

Lemma 3.2. $S^{\langle 2\rangle}_{C_i}$ *is uniserial, with structure*

$$\begin{matrix} D^{\langle 2\rangle}_{C_i} \\ D^{\langle 2,3\rangle}_{C_i} \\ D^{\langle 2,3,3\rangle}_{C_i} \\ D^{\langle 3,3\rangle}_{C_i} \\ D^{\langle 3,3,3\rangle}_{C_i} \end{matrix},$$

and $\mathrm{Ext}^1(D^{\langle 3,3,3\rangle}_{C_i}, D^{\langle 2\rangle}_{C_i}) = 0$.

Proof. Since $\langle 2\rangle$ is column 3-regular, it has a simple socle isomorphic to $D^{\langle 2\rangle'}_{C_i} \otimes \mathrm{sgn}$, by [3, Theorem 8.15]. Using the Mullineux map we find that $D^{\langle 2\rangle'}_{C_i} \otimes \mathrm{sgn} \cong D^{\langle 3,3,3\rangle}_{C_i}$. The structure then follows from the part of the Ext-quiver of C_i we already know, in particular the fact that $\mathrm{Ext}^1(D^{\langle 2\rangle}_{C_i}, D^{\langle 2,3,3\rangle}_{C_i})$, $\mathrm{Ext}^1(D^{\langle 2\rangle}_{C_i}, D^{\langle 3,3\rangle}_{C_i})$ and $\mathrm{Ext}^1(D^{\langle 2,3\rangle}_{C_i}, D^{\langle 3,3\rangle}_{C_i})$ are all zero.

From the decomposition matrix of C_i, we see that the head of $S^{\langle 2\rangle}_{C_i}$ is the only copy of $D^{\langle 2\rangle}_{C_i}$ lying in $P(D^{\langle 3,3,3\rangle}_{C_i})$; from the above Specht structure, it does not lie in the second socle layer, and so does not extend $D^{\langle 3,3,3\rangle}_{C_i}$. □

Corollary 3.3.

$$\mathrm{Ext}^1(D^{\langle 3,3,3\rangle}_{C_i}, D^{\langle 2,3,3\rangle}_{C_i}) = \mathrm{Ext}^1(D^{\langle 3,3,3\rangle}_{C_i}, D^{\langle 2,3\rangle}_{C_i}) = 0.$$

Proof. From the structure of $S^{\langle 2\rangle}_{C_i}$, we see that $P(D^{\langle 3,3,3\rangle}_{C_i})$ has a uniserial submodule

$$\begin{matrix} D^{\langle 2,3\rangle}_{C_i} \\ D^{\langle 2,3,3\rangle}_{C_i} \\ D^{\langle 3,3\rangle}_{C_i} \\ D^{\langle 3,3,3\rangle}_{C_i} \end{matrix}.$$

Since this projective cover is self-dual, it has a uniserial quotient

$$\begin{matrix} D^{\langle 3,3,3\rangle}_{C_i} \\ D^{\langle 3,3\rangle}_{C_i} \\ D^{\langle 2,3,3\rangle}_{C_i} \\ D^{\langle 2,3\rangle}_{C_i} \end{matrix};$$

hence the heads of $S^{\langle 2,3,3\rangle}_{C_i}$ and $S^{\langle 2,3\rangle}_{C_i}$ lie in the third and fourth Loewy layers of $P(D^{\langle 3,3,3\rangle}_{C_i})$, and so do not extend $D^{\langle 3,3,3\rangle}_{C_i}$. □

Since $D^{\langle 3,3,3\rangle}_{C_i}$ evidently extends $D^{\langle 3,3\rangle}_{C_i}$, we have determined all the Ext-spaces. Thus we may draw the quivers of all symmetric group blocks of weight three in characteristic three. We put these together in Figure 4, with the equivalence classes arranged as in Figure 2. We change notation, and represent the simple modules in each class by using an abacus with fifteen beads for the 'smallest' block in each class. Next to each quiver we note which notation we are using.

4 The projective cover of $D^{\langle 3,3,3\rangle}$

For this section, we shall restrict attention to the block C_i, for a fixed i; Specht modules and simple modules will be assumed to be in this block unless otherwise indicated. Let P denote the projective cover of $D^{\langle 3,3,3\rangle}$. Since we have already discovered some of the structure of P, it seems worth while to determine this structure completely. Recall that P is filtered by the Specht modules $S^{\langle 3,3,3\rangle}$, $S^{\langle 2,3,3\rangle}$, $S^{\langle 2,3\rangle}$ and $S^{\langle 2,2,2\rangle}$, and that we have determined the structures of the first and last of these. For the others we use Schaper's formula, together with a consideration of homomorphisms between Specht modules. First of all we determine the structure of $S^{\langle 2,3,3\rangle}$.

Lemma 4.1. *The Specht module $S^{\langle 2,3,3\rangle}$ has structure*

$$\begin{matrix} D^{\langle 2,3,3\rangle} \\ D^{\langle 3,3\rangle} \\ D^{\langle 3\rangle}D^{\langle 3,3,3\rangle} \\ D^{\langle 3,3\rangle} \end{matrix}.$$

Proof. Using Schaper's formula, we find that the bounds for the multiplicities for the factors of $S^{\langle 2,3,3\rangle}$ other than $D^{\langle 2,3,3\rangle}$ actually equal their multiplicities. Hence $S^{\langle 2,3,3\rangle}_{(2)} = 0$, and $\mathrm{rad}(S^{\langle 2,3,3\rangle}) = S^{\langle 2,3,3\rangle}_{(1)}$ is self-dual. Since $D^{\langle 2,3,3\rangle}$ constitutes the cosocle of $S^{\langle 2,3,3\rangle}$, it must extend each composition factor of $\mathrm{cosoc}(S^{\langle 2,3,3\rangle}_{(1)})$; from the Ext-quiver of C_i, we find that $D^{\langle 2,3,3\rangle}$ extends neither $D^{\langle 3\rangle}$ nor $D^{\langle 3,3,3\rangle}$, and so $S^{\langle 2,3,3\rangle}_{(1)}$ must be isomorphic to the indecomposable module

$$\begin{matrix} D^{\langle 3,3\rangle} \\ D^{\langle 3\rangle}D^{\langle 3,3,3\rangle} \\ D^{\langle 3,3\rangle} \end{matrix}.$$

□

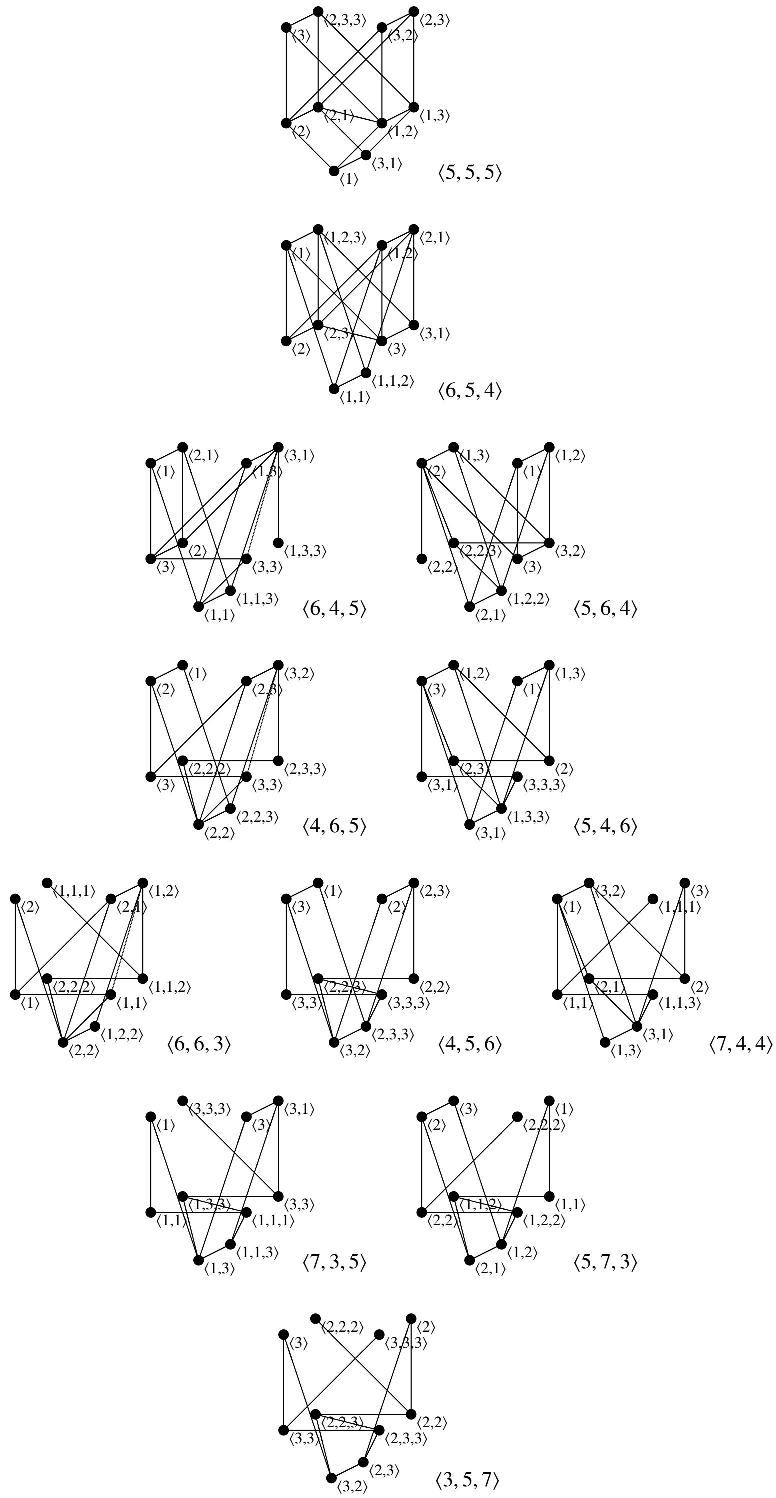

Figure 4: Ext-quivers of weight three blocks in characteristic three

We can also determine the structure of $S^{\langle 2,3\rangle}$.

Lemma 4.2. *For the Specht module $S^{\langle 2,3\rangle}$, we have*

$$\frac{S^{\langle 2,3\rangle}_{(0)}}{S^{\langle 2,3\rangle}_{(1)}} \cong D^{\langle 2,3\rangle},$$

$$\frac{S^{\langle 2,3\rangle}_{(1)}}{S^{\langle 2,3\rangle}_{(2)}} \cong D^{\langle 2,3,3\rangle} \oplus D^{\langle 3,2\rangle},$$

$$\frac{S^{\langle 2,3\rangle}_{(2)}}{S^{\langle 2,3\rangle}_{(3)}} \sim D^{\langle 3\rangle} + D^{\langle 3,3,3\rangle} + D^{\langle 3,3\rangle} \times 2,$$

$$S^{\langle 2,3\rangle}_{(3)} = 0.$$

Proof. Schaper's formula gives

$$\sum_{i=1}^{\infty} S^{\langle 2,3\rangle}_{(i)} \sim D^{\langle 2,3,3\rangle} + D^{\langle 3,2\rangle} + 2 \times D^{\langle 3\rangle} + 2 \times D^{\langle 3,3,3\rangle} + 4 \times D^{\langle 3,3\rangle}.$$

For those composition factors which only appear once, this determines which layer $S^{\langle 2,3\rangle}_{(i)}/S^{\langle 2,3\rangle}_{(i+1)}$ they lie in. $D^{\langle 3,3\rangle}$ appears twice in $S^{\langle 2,3\rangle}$, so it could lie in $S^{\langle 2,3\rangle}_{(1)}/S^{\langle 2,3\rangle}_{(2)}$ and $S^{\langle 2,3\rangle}_{(3)}/S^{\langle 2,3\rangle}_{(4)}$, or twice in $S^{\langle 2,3\rangle}_{(2)}/S^{\langle 2,3\rangle}_{(3)}$. But in any case $S^{\langle 2,3\rangle}_{(1)}/S^{\langle 2,3\rangle}_{(2)}$ is semi-simple, and so $D^{\langle 2,3\rangle}$ extends each factor of $S^{\langle 2,3\rangle}_{(1)}/S^{\langle 2,3\rangle}_{(2)}$. But $\mathrm{Ext}^1(D^{\langle 2,3\rangle}, D^{\langle 3,3\rangle}) = 0$. □

Lemma 4.3.

$$\mathrm{Hom}(D^{\langle 3\rangle}, S^{\langle 2,3\rangle}) = 0.$$

Proof. We have

$$\begin{aligned}
\mathrm{Hom}(D^{\langle 3\rangle}, S^{\langle 2,3\rangle}) &\leqslant \mathrm{Hom}(D^{\langle 3\rangle} \oplus D^{\langle 3\rangle}, S^{\langle 2,3\rangle}) \\
&\cong \mathrm{Hom}(D^{\langle 2\rangle}_{B_i}, S^{\langle 2,3\rangle}\downarrow_{B_i}) \\
&\leqslant \mathrm{Hom}(D^{\langle 2\rangle}_{B_i}, (S^{\langle 3\rangle}_{B_i} \oplus S^{\langle 2,2,3\rangle}_{B_i} \oplus S^{\langle 2,2,2\rangle}_{B_i})^{\oplus 2})
\end{aligned}$$

by Frobenius reciprocity and the Branching Rule. Now $[S^{\langle 2,2,3\rangle}_{B_i} : D^{\langle 2\rangle}_{B_i}] = 0$, and $\mathrm{soc}(S^{\langle 2,2,2\rangle}_{B_i}) \cong D^{\langle 3\rangle}_{B_i}$ by [3, Theorem 8.15]. By examining the Ext-quiver of B_i we see that $S^{\langle 3\rangle}_{B_i}$ is uniserial with socle $D^{\langle 2,2\rangle}_{B_i}$. Hence $D^{\langle 2\rangle}_{B_i}$ does not inject into any of these three Specht modules. □

Lemma 4.4. *$S^{\langle 2,3\rangle}_{(2)}$ has the structure*

$$\begin{array}{c} D^{\langle 3,3\rangle} \\ D^{\langle 3\rangle} D^{\langle 3,3,3\rangle} \\ D^{\langle 3,3\rangle} \end{array}.$$

Hence $S^{\langle 2,3\rangle}$ has a distributive submodule lattice, and its structure is given by

$$\begin{tikzcd}
 & D^{\langle 2,3\rangle} \arrow[dl, dash] \arrow[dr, dash] & & \\
D^{\langle 3,2\rangle} \arrow[ddr, dash] & & D^{\langle 2,3,3\rangle} \arrow[d, dash] & \\
 & & D^{\langle 3,3\rangle} \arrow[dl, dash] \arrow[dr, dash] & \\
 & D^{\langle 3\rangle} \arrow[dr, dash] & & D^{\langle 3,3,3\rangle} \arrow[dl, dash] \\
 & & D^{\langle 3,3\rangle} &
\end{tikzcd}.$$

Proof. The composition factor $D^{\langle 3,3,3\rangle}$ only extends $D^{\langle 3,3\rangle}$, and so a copy of the latter must lie above it in $S^{\langle 2,3\rangle}$; the other copy of $D^{\langle 3,3\rangle}$ must lie below $D^{\langle 3,3,3\rangle}$, since $S^{\langle 2,3\rangle}_{(2)}$ is self-dual. $D^{\langle 3\rangle}$ does not lie in the socle of $S^{\langle 2,3\rangle}$, so it also must lie in between the two copies of $D^{\langle 3,3\rangle}$. Thus the structure of $S^{\langle 2,3\rangle}_{(2)}$ is as stated.

Now the only factor of $S^{\langle 2,3\rangle}_{(0)}/S^{\langle 2,3\rangle}_{(2)}$ which extends $D^{\langle 3,3\rangle}$ is $D^{\langle 2,3,3\rangle}$, and so this extension must occur in $S^{\langle 2,3\rangle}$. Of the factors of $S^{\langle 2,3\rangle}_{(2)}$, $D^{\langle 2,3\rangle}$ extends none and $D^{\langle 3,2\rangle}$ can only possibly extend $D^{\langle 3\rangle}$. This gives the structure of $S^{\langle 2,3\rangle}$, except that it is not clear that there is an edge joining $D^{\langle 3,2\rangle}$ to $D^{\langle 3\rangle}$.

In order to show that this edge exists, we need to consider homomorphisms between Specht modules. From the module structures we have so far, it is clear that

$$\mathrm{Hom}(S^{\langle 3,3\rangle}, S^{\langle 3,2\rangle}) \cong \mathrm{Hom}(S^{\langle 3,2\rangle}, S^{\langle 2,3\rangle}) \cong \mathrm{Hom}(S^{\langle 3,3\rangle}, S^{\langle 2,3\rangle}) \cong k.$$

We need to show that the image of the homomorphism from $S^{\langle 3,2\rangle}$ to $S^{\langle 2,3\rangle}$ contains the image of the homomorphism from $S^{\langle 3,3\rangle}$ to $S^{\langle 2,3\rangle}$, i.e. that the latter factors through $S^{\langle 3,2\rangle}$. Since the corresponding Hom-spaces are all one-dimensional, it is sufficient to show that, for some $0 \neq f \in \mathrm{Hom}(S^{\langle 3,3\rangle}, S^{\langle 3,2\rangle})$ and $0 \neq g \in \mathrm{Hom}(S^{\langle 3,2\rangle}, S^{\langle 2,3\rangle})$, the composition gf is non-zero.

We concentrate on the case $i = 3$; the other cases are essentially identical. Here we have $\langle 3,3\rangle = (12,7,2^2,1^2)$, $\langle 3,2\rangle = (12,4^2,3,1^2)$, $\langle 2,3\rangle = (9,6,5,3,1^2)$. Now, if λ and μ are partitions of n, [3, Theorem 13.13] tells us that a basis for $\mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, M^\mu)$ is given by

$$\{\hat{\Theta}_T \mid T \in \mathcal{T}_0(\lambda,\mu)\},$$

where $\mathcal{T}_0(\lambda,\mu)$ is the set of semistandard λ-tableaux of type μ, and $\hat{\Theta}_T$ is the semistandard homomorphism corresponding to T. Looking at our case, we find that there is just one semistandard $(12,7,2^2,1^2)$-tableau of type $(12,4^2,3,1^2)$, namely

$$T_1 = \begin{array}{llllllllllll} 1&1&1&1&1&1&1&1&1&1&1&1\\ 2&2&2&2&3&3&4&&&&&\\ 3&3&&&&&&&&&&\\ 4&4&&&&&&&&&&\\ 5&&&&&&&&&&&\\ 6&&&&&&&&&&& \end{array},$$

and one $(12,4^2,3,1^2)$-tableau of type $(9,6,5,3,1^2)$, namely

$$T_2 = \begin{array}{llllllllllll} 1&1&1&1&1&1&1&1&1&2&2&3\\ 2&2&2&2&&&&&&&&\\ 3&3&3&3&&&&&&&&\\ 4&4&4&&&&&&&&&\\ 5&&&&&&&&&&&\\ 6&&&&&&&&&&& \end{array}.$$

So we may take $f = \hat{\Theta}_{T_1}$ and $g = \hat{\Theta}_{T_2}$. Given a λ-tableau T of type μ, $\hat{\Theta}_T$ is constructed as the restriction of the homomorphism $\Theta_T : M^\lambda \to M^\mu$, which depends on a specific choice of a λ-tableau t (of type λ). Choosing the $(12, 7, 2^2, 1^2)$-tableau

$$t_1 = \begin{array}{llllllllllll} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \\ c_1 & c_2 \\ d_1 & d_2 \\ e_1 \\ f_1 \end{array}$$

and the $(12, 4^2, 3, 1^2)$-tableau

$$t_2 = \begin{array}{llllllllllll} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & b_5 & b_6 \\ d_1 & d_2 & b_7 \\ e_1 \\ f_1 \end{array},$$

we get

$$\Theta_{T_1}(\{t_1\}) = \sum_{\substack{1 \leqslant l < m \leqslant 7 \\ 1 \leqslant n \leqslant 7 \\ l \neq n \neq m}} \left\{ \begin{array}{l} a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8\ a_9\ a_{10}\ a_{11}\ a_{12} \\ b_1\ \dots\ \hat{b_l}\ \dots\ \hat{b_m}\ \dots\ \hat{b_n}\ \dots\ b_7 \\ c_1\ c_2\ b_l\ b_m \\ d_1\ d_2\ b_n \\ e_1 \\ f_1 \end{array} \right\}$$

and

$$\Theta_{T_2}(\{t_2\}) = \sum_{\substack{1 \leqslant i < j \leqslant 12 \\ 1 \leqslant k \leqslant 12 \\ i \neq k \neq j}} \left\{ \begin{array}{l} a_1\ \dots\ \dots\ \hat{a_i}\ \dots\ \dots\ \hat{a_j}\ \dots\ \dots\ \hat{a_k}\ \dots\ \dots\ a_{12} \\ b_1\ b_2\ b_3\ b_4\ b_5\ b_6\ b_7\ a_i\ a_j \\ c_1\ c_2\ a_k \\ d_1\ d_2 \\ e_1 \\ f_1 \end{array} \right\},$$

where, as usual, a circumflex accent indicates omission, and $\{t\}$ denotes the tabloid containing the tableau t. Hence we have

$$\Theta_{T_2}(\Theta_{T_1}(\{t_1\})) = \sum_{\substack{1 \leqslant i < j \leqslant 12 \\ 1 \leqslant k \leqslant 12 \\ i \neq k \neq j}} \sum_{\substack{1 \leqslant l < m \leqslant 7 \\ 1 \leqslant n \leqslant 7 \\ l \neq n \neq m}} \left\{ \begin{array}{l} a_1\ \dots\ \dots\ \hat{a_i}\ \dots\ \dots\ \hat{a_j}\ \dots\ \dots\ \hat{a_k}\ \dots\ \dots\ a_{12} \\ b_1\ \dots\ \hat{b_l}\ \dots\ \hat{b_m}\ \dots\ \hat{b_n}\ \dots\ b_7\ a_i\ a_j \\ c_1\ c_2\ a_k\ b_l\ b_m \\ d_1\ d_2\ b_n \\ e_1 \\ f_1 \end{array} \right\}.$$

To show that the composite $\hat{\Theta}_{T_2}\hat{\Theta}_{T_1}$ is non-zero, we need only show that the image under $\Theta_{T_2}\Theta_{T_1}$ of some polytabloid is non-zero; we shall take the polytabloid e_{t_1} corresponding to t_1, that is the sum $\sum_\pi (-1)^\pi \{\pi t\}$, where the sum is over all column permutations of t_1. Consider the coefficient of the tabloid

$$U = \left\{ \begin{array}{l} a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8\ a_9 \\ b_1\ b_2\ b_3\ b_4\ a_{10}\ a_{11} \\ c_1\ c_2\ b_5\ b_6\ a_{12} \\ d_1\ d_2\ b_7 \\ e_1 \\ f_1 \end{array} \right\}$$

in $\Theta_{T_2}(\Theta_{T_1}(e_{t_1}))$. Since a column permutation π of t_1 only permutes entries of t_1 with the same index, we must have $i = 10, j = 11, k = 12, l = 5, m = 6, n = 7$ in order to get

$$\pi \left\{ \begin{array}{l} a_1\ \dots\ \dots\ \hat{a_i}\ \dots\ \dots\ \hat{a_j}\ \dots\ \dots\ \hat{a_k}\ \dots\ \dots\ a_{12} \\ b_1\ \dots\ \hat{b_l}\ \dots\ \hat{b_m}\ \dots\ \hat{b_n}\ \dots\ b_7\ a_i\ a_j \\ c_1\ c_2\ a_k\ b_l\ b_m \\ d_1\ d_2\ b_n \\ e_1 \\ f_1 \end{array} \right\} = U;$$

it is then clear that we must have $\pi = 1$, so that the coefficient of U in $\Theta_{T_2}(\Theta_{T_1}(e_{t_1}))$ is 1. $\square$

Combining the structures of the Specht modules in P, we are able to find the structure of the latter. Using the structures of the Specht modules and their duals, and using the Ext-quiver of C_i to rule out any further extensions, we obtain the following.

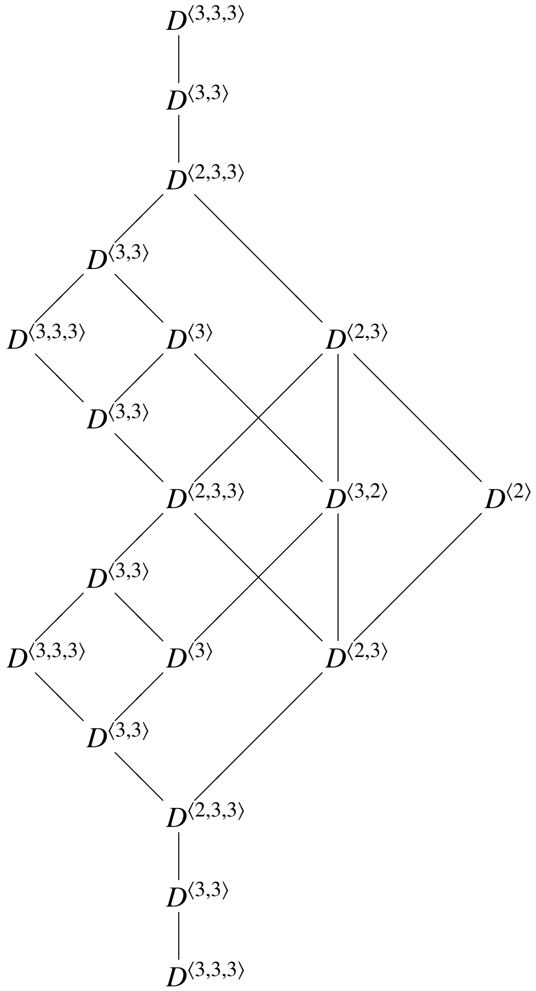

Figure 5: The submodule structure of $P(D^{\langle 3,3,3\rangle})$

Proposition 4.5. *The projective module* $P = P(D^{\langle 3,3,3\rangle})$ *has a distributive submodule lattice, and its structure is as in Figure 5.*

It is extremely unusual for a projective module in a block of such large defect to have a distributive submodule lattice. The other indecomposable projectives in C_i have many more composition factors, and can easily be shown not to have distributive submodule lattices.

Notice that the Loewy length of this module is thirteen, and that it is not stable, i.e. its Loewy series does not coincide with its socle series. However, if we ignore all the copies of the simple module $D^{\langle 3,3\rangle}$, then the resulting structure is stable, with seven layers. Martin conjectures that for a symmetric group block of weight $\omega < p$, the projective modules have stable structures and Loewy length exactly $2\omega+1$; from the above evidence, it seems as though, with some modified definition of the Loewy series, this conjecture might hold for $\omega \geqslant p$ as well.

A Decomposition matrices

The matrices found here were calculated using Schaper's formula and other means. The (λ, μ)th entry of each matrix records the composition multiplicity $[S^\lambda : D^\mu]$, with a blank space indicating zero. The partitions are given in the abacus notation specified for each matrix.

A.1 The block of $k\mathfrak{S}_{13}$ with 3-core $(3,1)$ ($\langle 3,5,3\rangle$-notation)

	$\langle 2\rangle$	$\langle 2,2\rangle$	$\langle 2,3\rangle$	$\langle 3\rangle$	$\langle 3,2\rangle$	$\langle 2,2,3\rangle$	$\langle 2,2,2\rangle$	$\langle 1,2\rangle$	$\langle 1\rangle$	$\langle 3,1\rangle$
$\langle 2\rangle$	1									
$\langle 2,2\rangle$	1	1								
$\langle 2,3\rangle$	2	1	1							
$\langle 2,1\rangle$			1							
$\langle 3\rangle$	1	1	1	1						
$\langle 3,2\rangle$				1	1					
$\langle 2,2,3\rangle$			1	1	1	1				
$\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$\langle 1,2,2\rangle$							1			
$\langle 1,2\rangle$	2	1	1			1	1	1		
$\langle 1,2,3\rangle$	2	1						1		
$\langle 2,3,3\rangle$						1		1		
$\langle 1,1,2\rangle$						1				
$\langle 1\rangle$	1	1			1	1	1	1	1	
$\langle 3,1\rangle$	2	1	1			1		1	1	1
$\langle 3,3\rangle$	1				1	1		1	1	1
$\langle 1,3\rangle$	1		1			1				1
$\langle 1,1\rangle$					1	1			1	1
$\langle 1,3,3\rangle$	1									1
$\langle 1,1,3\rangle$									1	1
$\langle 3,3,3\rangle$					1				1	
$\langle 1,1,1\rangle$					1					

A.2 The block of $k\mathfrak{S}_{15}$ with 3-core $(4,2)$ ($\langle 3,3,5\rangle$-notation)

	$\langle 3\rangle$	$\langle 3,3\rangle$	$\langle 3,2\rangle$	$\langle 3,3,3\rangle$	$\langle 3,3,2\rangle$	$\langle 2,3\rangle$	$\langle 1,3\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle 2,2\rangle$
$\langle 3\rangle$	1									
$\langle 3,3\rangle$	1	1								
$\langle 3,2\rangle$	2	1	1							
$\langle 3,1\rangle$			1							
$\langle 3,3,3\rangle$		1		1						
$\langle 3,3,2\rangle$	1	2		1	1					
$\langle 3,3,1\rangle$					1					
$\langle 2,3\rangle$	1	2	1	1	1	1				
$\langle 1,3\rangle$	2	1	1		1	1	1			
$\langle 1,2,3\rangle$	2	1					1			
$\langle 3,2,2\rangle$						1	1			
$\langle 3,1,1\rangle$						1				
$\langle 2\rangle$		1		1	1	1		1		
$\langle 1\rangle$	1	1			1	1	1	1	1	
$\langle 2,1\rangle$	2	1	1			1	1		1	1
$\langle 2,2\rangle$	1					1	1	1	1	1
$\langle 1,2\rangle$	1		1			1				1
$\langle 1,1\rangle$						1		1	1	1
$\langle 1,2,2\rangle$	1									1
$\langle 1,1,2\rangle$									1	1
$\langle 2,2,2\rangle$								1	1	
$\langle 1,1,1\rangle$								1		

A.3 The block of $k\mathfrak{S}_{17}$ with 3-core $(4,2,1^2)$ ($\langle 3,6,4\rangle$-notation)

	$\langle 2\rangle$	$\langle 2,2\rangle$	$\langle 2,3\rangle$	$\langle 3\rangle$	$\langle 3,2\rangle$	$\langle 2,2,3\rangle$	$\langle 2,2,2\rangle$	$\langle 2,3,3\rangle$	$\langle 3,3\rangle$	$\langle 1\rangle$
$\langle 2\rangle$	1									
$\langle 2,2\rangle$	1	1								
$\langle 2,3\rangle$	2	1	1							
$\langle 2,1\rangle$			1							
$\langle 3\rangle$	1	1	1	1						
$\langle 3,2\rangle$				1	1					
$\langle 2,2,3\rangle$			1	1	1	1				
$\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$\langle 1,2,2\rangle$							1			
$\langle 2,3,3\rangle$	2	1	1				1	1		
$\langle 1,2,3\rangle$			1			1	1	1		
$\langle 1,2\rangle$								1		
$\langle 1,1,2\rangle$						1				
$\langle 3,3\rangle$	1	1	1		1	1	1	1	1	
$\langle 1\rangle$	1		1		1	1		1	1	1
$\langle 3,1\rangle$					2	1	1	1	1	
$\langle 1,3\rangle$	1		1			1				1
$\langle 1,3,3\rangle$	1									1
$\langle 3,3,3\rangle$					1	1			1	1
$\langle 1,1,3\rangle$					2	1			1	
$\langle 1,1\rangle$					1				1	
$\langle 1,1,1\rangle$					1					

A.4 The block of $k\mathfrak{S}_{19}$ with 3-core $(5,3,1^2)$ ($\langle 3,4,6\rangle$-notation)

	$\langle 3\rangle$	$\langle 3,3\rangle$	$\langle 3,2\rangle$	$\langle 3,3,3\rangle$	$\langle 2,3,3\rangle$	$\langle 2,3\rangle$	$\langle 2,2,3\rangle$	$\langle 2\rangle$	$\langle 2,2\rangle$	$\langle 2,1\rangle$
$\langle 3\rangle$	1									
$\langle 3,3\rangle$	1	1								
$\langle 3,2\rangle$	2	1	1							
$\langle 3,1\rangle$			1							
$\langle 3,3,3\rangle$		1		1						
$\langle 2,3,3\rangle$	1	2		1	1					
$\langle 1,3,3\rangle$					1					
$\langle 2,3\rangle$	1	2	1	1	1	1				
$\langle 2,2,3\rangle$	2	1	1		1		1			
$\langle 1,2,3\rangle$			1		1	1	1			
$\langle 1,3\rangle$							1			
$\langle 1,1,3\rangle$						1				
$\langle 2\rangle$		1		1	1	1		1		
$\langle 2,2\rangle$	1	1	1		1	1	1	1	1	
$\langle 2,1\rangle$					1	1	1	2	1	
$\langle 1\rangle$	1		1			1	1	1	1	1
$\langle 1,2\rangle$	1		1			1				1
$\langle 1,2,2\rangle$	1									1
$\langle 2,2,2\rangle$						1		1	1	1
$\langle 1,1,2\rangle$						1		2	1	
$\langle 1,1\rangle$								1	1	
$\langle 1,1,1\rangle$								1		

A.5 The block of $k\mathfrak{S}_{23}$ with 3-core $(5,3,2^2,1^2)$ ($\langle 3,7,5\rangle$-notation)

	$\langle 2\rangle$	$\langle 2,2\rangle$	$\langle 2,3\rangle$	$\langle 3\rangle$	$\langle 3,2\rangle$	$\langle 2,2,3\rangle$	$\langle 2,2,2\rangle$	$\langle 2,3,3\rangle$	$\langle 3,3\rangle$	$\langle 3,3,3\rangle$
$\langle 2\rangle$	1									
$\langle 2,2\rangle$	1	1								
$\langle 2,3\rangle$	2	1	1							
$\langle 2,1\rangle$			1							
$\langle 3\rangle$	1	1	1	1						
$\langle 3,2\rangle$				1	1					
$\langle 2,2,3\rangle$			1	1	1	1				
$\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$\langle 1,2,2\rangle$							1			
$\langle 2,3,3\rangle$	2	1	1				1	1		
$\langle 1,2,3\rangle$			1			1	1	1		
$\langle 1,2\rangle$								1		
$\langle 1,1,2\rangle$						1				
$\langle 3,3\rangle$	1	1	1		1	1	1	1	1	
$\langle 3,1\rangle$					2	1	1	1	1	
$\langle 3,3,3\rangle$	1		1					1	1	1
$\langle 1,3,3\rangle$			1		1	1		1	2	1
$\langle 1,3\rangle$					1			1	2	1
$\langle 1,1,3\rangle$					2	1			1	
$\langle 1\rangle$									1	1
$\langle 1,1\rangle$					1				1	
$\langle 1,1,1\rangle$					1					

A.6 The block of $k\mathfrak{S}_{25}$ with 3-core $(6,4,2^2,1^2)$ ($\langle 3,5,7\rangle$-notation)

	$\langle 3\rangle$	$\langle 3,3\rangle$	$\langle 3,2\rangle$	$\langle 3,3,3\rangle$	$\langle 2,3,3\rangle$	$\langle 2,3\rangle$	$\langle 2,2,3\rangle$	$\langle 2\rangle$	$\langle 2,2\rangle$	$\langle 2,2,2\rangle$
$\langle 3\rangle$	1									
$\langle 3,3\rangle$	1	1								
$\langle 3,2\rangle$	2	1	1							
$\langle 3,1\rangle$			1							
$\langle 3,3,3\rangle$		1		1						
$\langle 2,3,3\rangle$	1	2		1	1					
$\langle 1,3,3\rangle$					1					
$\langle 2,3\rangle$	1	2	1	1	1	1				
$\langle 2,2,3\rangle$	2	1	1		1		1			
$\langle 1,2,3\rangle$			1		1	1	1			
$\langle 1,3\rangle$							1			
$\langle 1,1,3\rangle$						1				
$\langle 2\rangle$		1		1	1	1		1		
$\langle 2,2\rangle$	1	1	1		1	1	1	1	1	
$\langle 2,1\rangle$					1	1	1	2	1	
$\langle 2,2,2\rangle$	1		1				1		1	1
$\langle 1,2,2\rangle$			1			1	1	1	2	1
$\langle 1,2\rangle$							1	1	2	1
$\langle 1,1,2\rangle$						1		2	1	
$\langle 1\rangle$									1	1
$\langle 1,1\rangle$								1	1	
$\langle 1,1,1\rangle$								1		

References

[1] D. Benson, 'Some remarks on the decomposition numbers for the symmetric groups', *Proceedings of Symposia in Pure Mathematics* **47** (1987), 381–94.

[2] M. Fayers, 'On the blocks of $\mathfrak{S}_{13}$ over a field of characteristic three', *Math. Proc. Cambridge Philos. Soc.* **133** (2002), 1–29.

[3] G. James, *The representation theory of the symmetric groups*, Lecture Notes in Mathematics **682**, Springer-Verlag, New York/Berlin, 1978.

[4] G. James & A. Kerber, *The representation theory of the symmetric group*, Encyclopædia of Mathematics and its Applications **16**, Addison–Wesley, 1981.

[5] J. Scopes, 'Cartan matrices and Morita equivalence for blocks of the symmetric groups', *J. Algebra* **142** (1991), 441–55.

[6] K. M. Tan, 'On the principal blocks of $\mathfrak{S}_9$ and $\mathfrak{S}_{10}$ over a field of characteristic three', *J. Pure Appl. Algebra* **153** (2000), 79–106.

[7] K. M. Tan, 'On the principal blocks of $\mathfrak{S}_{11}$ and $\mathfrak{S}_{12}$ over a field of characteristic three', *Math. Proc. Cambridge Philos. Soc.* **128** (2000), 395–423.