Data for weight 2 blocks of Hecke algebras of type A

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Contents

1 Introduction

In this note we present some data (decomposition numbers and Cartan matrix entries) for weight two blocks of Iwahori–Hecke algebras of type A. This has been studied in detail before, and the material here is not new, though our presentation is. In particular, we give a classification of Specht modules according to composition length, and give the possible composition lengths of a projective indecomposable module and each of its Loewy layers. (This answers a question of Erdmann and Danz: the third Loewy layer of a projective indecomposable cannot be simple.)

We attempt to present results efficiently by judicious choice of notation. In particular, we employ the 'pyramids' described by Richards. We also use a new labelling of the simple modules: for any block in 'quantum characteristic' e, the simple modules are labelled by symbols $\lceil i \rfloor$ and $\lceil i,j \rfloor$, where $1 \leqslant i \leqslant j < e$. Our labelling has the property that two simple modules which correspond under the Scopes equivalence have the same label. This will simplify formulæ considerably. In particular, our Proposition 3.1 describes the decomposition numbers of all blocks of weight 2 rather more concisely than the tables in Richards's paper. However, some of our later results are nonetheless rather complicated. But we hope that some of these results will be useful. This note is expected to evolve over time; readers are invited to suggest new items for inclusion.

Correctness of the results

No proofs are given in this paper, because the results really are already known in a different form, thanks to the work of Scopes, Richards, Chuang–Tan and others. So the method of proof here is just careful book-keeping. The results were obtained theoretically, and then tested by computer on all (Scopes classes of) blocks of weight 2 in quantum characteristic at most 7. So we are very confident in their correctness.

2 Notation

2.1 Partitions, Specht modules and the abacus

We summarise the background and notation we use. The reader should consult the references for a less brief account.

Throughout, we work over a field $\mathbb{F}$ of arbitrary characteristic, and fix a root of unity $q \in \mathbb{F}$. We define the *quantum characteristic* e to be the least positive integer such that $1 + q + \cdots + q^{e-1} = 0$ in $\mathbb{F}$. Given a positive integer n, let $\mathcal{H}_n$ denote the Iwahori–Hecke algebra of type A_{n-1} over $\mathbb{F}$, with quantum parameter q. *We assume throughout that $e > 2$.* Many of our results still apply when $e = 2$, but modifications are needed in certain cases, and the results we use on extensions between simple modules fail badly when $e = 2$.

$\mathcal{H}_n$ has *Specht modules* indexed by the partitions of n; we use the Specht modules defined by Dipper and James. Given a partition λ of n, one defines the *beta-numbers*

$$\beta_i = \lambda_i - i \qquad \text{for } i \geqslant 1.$$

The *e-runner abacus* is an abacus with e infinite vertical runners, with positions marked on the runners. These positions are labelled with integers, in such a way that the positions on the ith runner from the left are labelled with the integers congruent to $i - 1$ modulo e, increasing down the runner. For example, the 5-runner abacus is marked as follows.

$$\begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \\ | & | & | & | & | \\ -5 & -4 & -3 & -2 & -1 \\ | & | & | & | & | \\ 0 & 1 & 2 & 3 & 4 \\ | & | & | & | & | \\ 5 & 6 & 7 & 8 & 9 \\ | & | & | & | & | \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

The *abacus display* for λ is obtained by placing a bead at position β_i for each i. By sliding beads up their runners into empty spaces, we obtain the abacus display for the *e-core* of λ; this is a partition of $n - ew$, where w is the *e-weight* of λ. Two Specht modules S^λ and S^μ for $\mathcal{H}_n$ lie in the same block if and only if λ and μ have the same e-core. This automatically implies that they have the same e-weight, and so we may speak of the (e-)weight and (e-)core of a block. In this note we are entirely concerned with blocks of weight 2.

If λ is an e-regular partition, then S^λ has an irreducible cosocle D^λ, and the D^λ realise all irreducible representations of $\mathcal{H}_n$. In this note we are mainly concerned with the decomposition numbers $\left[S^\lambda : D^\mu\right]$, and the Cartan invariants $\left[P^\lambda : D^\mu\right]$, where P^λ denotes the projective cover of D^λ for λ an e-regular partition.

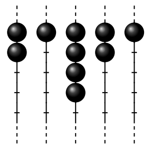
2.2 Pyramids

Suppose B is a block of $\mathcal{H}_n$ of weight 2. The *pyramid* of B is a triangular array of integers which encodes some of the combinatorics of B. Take the abacus display for the e-core of B, and consider the position of the lowest bead on each runner. This yields a set of e integers, which we write as $p_0 < \cdots < p_{e-1}$. Now for $0 \leqslant i \leqslant j \leqslant e-1$, we define

$$ {}_iB_j = \begin{cases} 1 & (\text{if } p_j - p_i < e) \\ 0 & (\text{if } p_j - p_i > e). \end{cases} $$

The array $({}_iB_j)$ is the pyramid of B. Note that we adhere to Richards's original convention for the pyramid values, rather than that employed by the author and Tan.

Example. Suppose $e = 5$, and that B is the block of $\mathcal{H}_{24}$ with core $(8,4,1,1)$. The abacus display for this partition is



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and we see that $(p_0, p_1, p_2, p_3, p_4) = (-9, -6, -5, -2, 7)$. Hence the pyramid is given by the following diagram:

$$ \begin{array}{ccccccccc} &&&&0&&&& \\ &&&0&&0&&& \\ &&1&&1&&0&& \\ &1&&1&&1&&0& \\ 1&&1&&1&&1&&1 \\ \hline 0&&1&&2&&3&&4 \end{array} $$

We employ shorthand notation, for example writing ${}_i0_j$ to mean that ${}_iB_j = 0$. For convenience, we extend the definition of the pyramid to allow i and j to be any integers: if $i > j$, then we define ${}_iB_j = 1$. Otherwise, if $i < 0$ or $j \geqslant e$, then we define ${}_iB_j = 0$.

2.3 Labelling Specht modules and simple modules

Now we describe our labelling of partitions and e-regular partitions in weight 2 blocks, which is slightly unusual.

Suppose B is the weight 2 block with core ν, and consider the abacus display for ν. Let $p_0 < \cdots < p_{e-1}$ be the integers defined above, and label the runners of the abacus $0, \dots e-1$ so that integer p_i lies on runner i for each i. For example, taking $e = 5$ and $\nu = (8,4,1,1)$ as in the last example, the runners are labelled $2,0,4,3,1$ from left to right.

Suppose λ is a partition in B. Then the abacus display for λ is obtained from the abacus display for ν by twice moving a bead into an empty space below it. If these moves occur on runners i and j for $i < j$, then we denote λ as $\langle i, j\rangle$. On the other hand, if the lowest bead on runner i is moved down two spaces, we denote λ as $\langle i\rangle$, while if the lowest two beads on runner i are moved down one space each, we denote λ as $\langle i^2\rangle$.

Thus B contains the $\binom{e+2}{2} - 1$ partitions, with labels $\langle i, j\rangle$ for $0 \leqslant i < j < e$ or $\langle i\rangle$ or $\langle i^2\rangle$ for $0 \leqslant i < e$. Which of these partitions are e-regular depends on B, so we introduce a different notation for e-regular partitions, which will have the advantage that the set of labels of e-regular partitions depends only on e, not on B.

Given $1 \leqslant i < e$, we define

$$ \lceil i \rfloor = \begin{cases} \langle i \rangle & (\text{if } {}_i0_{i+1}) \\ \langle i, i{+}1 \rangle & (\text{if } {}_i1_{i+1}). \end{cases} $$

Given $1 \leqslant i \leqslant j < e$, we define

$$\lceil i,j \rfloor = \begin{cases} \langle i,j \rangle & (\text{if } {}_i0_j) \\ \langle j^2 \rangle & (\text{if } {}_i1_j \text{ and } {}_{i-1}0_j) \\ \langle i{-}1 \rangle & (\text{if } {}_{i-1}1_j \text{ and } {}_{i-1}0_{j+1}) \\ \langle i{-}1, j{+}1 \rangle & (\text{if } {}_{i-1}1_{j+1}). \end{cases}$$

It is easy to check that the symbols $\lceil i \rfloor$ for $1 \leqslant i < e$ and $\lceil i,j \rfloor$ for $1 \leqslant i \leqslant j < e$ all denote different partitions in B, and in fact denote precisely the e-regular partitions in B.

Our notation has the property that if A and B are weight 2 blocks of m and n forming a $[2:k]$-*pair* in the sense of Scopes [S], then a simple module in A has the same label as the corresponding simple module in B. (The reader who is unfamiliar with the Scopes correspondence should ignore this remark, since we do not use it explicitly in what follows.)

3 Decomposition numbers

Here we present the decomposition numbers for a block of weight 2. The information here is essentially that in Tables 1 and 2 of [R], but our notation for simple modules makes the statement a little shorter.

3.1 Columns of the decomposition matrix

First we present the decomposition numbers 'by columns'; that is, for a given simple module, we say which Specht modules contains it as a composition factor.

Proposition 3.1. *Suppose B is a block of $\mathcal{H}_n$ of weight 2, with pyramid $({}_iB_j)$.*

1. *Suppose $1 \leqslant i < e$. Then the decomposition number $\left[S^\lambda : D^{\lceil i \rfloor}\right]$ equals 1 if one of the following occurs, and 0 otherwise.*

$\lambda = \langle i \rangle$,	${}_i0_{i+1}$;	$\lambda = \langle i{-}1 \rangle$,	${}_{i-1}1_i$, ${}_{i-1}0_{i+1}$;
$\lambda = \langle i{-}2 \rangle$,	${}_{i-2}1_{i-1}$, ${}_{i-2}0_i$;	$\lambda = \langle i{+}1^2 \rangle$,	${}_{i-1}0_{i+1}$, ${}_i1_{i+1}$;
$\lambda = \langle i^2 \rangle$,	${}_{i-2}0_i$, ${}_{i-1}1_i$;	$\lambda = \langle i{-}1^2 \rangle$,	${}_{i-2}0_{i-1}$;
$\lambda = \langle i,i{+}1 \rangle$,	${}_i1_{i+1}$;	$\lambda = \langle i{-}1,i \rangle$,	${}_{i-1}0_i$;
$\lambda = \langle i{-}2,i{-}1 \rangle$,	${}_{i-2}1_{i-1}$;	$\lambda = \langle i{-}1,i{+}1 \rangle$,	${}_{i-1}1_{i+1}$;
$\lambda = \langle i{-}2,i \rangle$,	${}_{i-2}1_i$.		

2. *Suppose $1 \leqslant i \leqslant j < e$. Then the decomposition number $\left[S^\lambda : D^{\lceil i,j \rfloor}\right]$ equals 1 if one of the following occurs, and 0 otherwise.*

$\lambda = \langle i \rangle$,	${}_i1_{j-1}$, ${}_i0_j$;	$\lambda = \langle i{-}1 \rangle$,	${}_{i-1}1_{j-1}$, ${}_{i-1}0_{j+1}$;
$\lambda = \langle i{-}2 \rangle$,	${}_{i-2}1_j$, ${}_{i-2}0_{j+1}$;	$\lambda = \langle j{+}1^2 \rangle$,	${}_{i-2}0_{j+1}$, ${}_{i-1}1_{j+1}$;
$\lambda = \langle j^2 \rangle$,	${}_{i-2}0_j$, ${}_i1_j$;	$\lambda = \langle j{-}1^2 \rangle$,	${}_{i-1}0_{j-1}$, ${}_i1_{j-1}$;
$\lambda = \langle i,j{-}1 \rangle$,	${}_i0_{j-1}$;	$\lambda = \langle i,j \rangle$,	${}_i0_j$;
$\lambda = \langle i{-}1,j{-}1 \rangle$,	${}_{i-1}0_{j-1}$;	$\lambda = \langle i{-}1,j \rangle$;	
$\lambda = \langle i{-}1,j{+}1 \rangle$,	${}_{i-1}1_{j+1}$;	$\lambda = \langle i{-}2,j \rangle$,	${}_{i-2}1_j$;
$\lambda = \langle i{-}2,j{+}1 \rangle$,	${}_{i-2}1_{j+1}$.		

This leads to a surprisingly simple formula for the diagonal entries of the Cartan matrix.

Corollary 3.2. *Suppose B is a block of $\mathcal{H}_n$ of weight 2, with pyramid $({}_iB_j)$.*

1. *Suppose $1 \leqslant i < e$. Then the diagonal Cartan number $\left[P^{\lceil i \rfloor} : D^{\lceil i \rfloor}\right]$ equals*

$$3 + {}_{i-2}B_{i-1} + {}_{i-1}B_i + {}_iB_{i+1} - {}_{i-2}B_i - {}_{i-1}B_{i+1}.$$

2. *Suppose $1 \leqslant i \leqslant j < e$. Then the diagonal Cartan number $\left[P^{\lceil i,j \rfloor} : D^{\lceil i,j \rfloor}\right]$ equals*

$$4 + {}_iB_{j-1} - {}_{i-1}B_{j-1} - {}_iB_j + {}_{i-2}B_j + {}_{i-1}B_{j+1} - {}_{i-2}B_{j+1}.$$

3.2 Rows of the decomposition matrix

Next, we present the same information 'by rows', giving a list of the composition factors of a given Specht module. In the following result, we use $[M]$ to denote the image of a module M in the Grothendieck group of $\mathcal{H}_n$. The term $\left[D^{\lceil i,j \rfloor}\right]$ should be interpreted as zero if the conditions $1 \leqslant i \leqslant j < e$ fail, and similarly for $\left[D^{\lceil i \rfloor}\right]$.

Proposition 3.3. *Suppose B is a block of $\mathcal{H}_n$ of weight 2, with pyramid $({}_iB_j)$.*

1. *Suppose $0 \leqslant k < e$, and let $l \in \{k, \dots, e-1\}$ be maximal such that ${}_k1_l$. Then*

$$\left[S^{\langle k \rangle}\right] = \delta_{kl}\left[D^{\lceil k \rfloor}\right] + \delta_{(k+1)l}\left(\left[D^{\lceil k+1 \rfloor}\right] + \left[D^{\lceil k+2 \rfloor}\right]\right) + \left[D^{\lceil k,l+1 \rfloor}\right] + \left[D^{\lceil k+1,l+1 \rfloor}\right] + \left[D^{\lceil k+1,l \rfloor}\right] + \left[D^{\lceil k+2,l \rfloor}\right].$$

2. *Suppose $0 \leqslant l < e$, and let $k \in \{0, \dots, l\}$ be minimal such that ${}_k1_l$. Then*

$$\left[S^{\langle l^2 \rangle}\right] = \delta_{kl}\left[D^{\lceil l+1 \rfloor}\right] + \delta_{k(l-1)}\left(\left[D^{\lceil l-1 \rfloor}\right] + \left[D^{\lceil l \rfloor}\right]\right) + \left[D^{\lceil k,l \rfloor}\right] + \left[D^{\lceil k+1,l \rfloor}\right] + \left[D^{\lceil k,l+1 \rfloor}\right] + \left[D^{\lceil k+1,l-1 \rfloor}\right].$$

3. *Suppose $1 \leqslant l < e$, and ${}_{l-1}1_l$. Then*

$$\left[S^{\langle l-1,l \rangle}\right] = \left[D^{\lceil l-1 \rfloor}\right] + \left[D^{\lceil l+1 \rfloor}\right] + \left[D^{\lceil l,l \rfloor}\right].$$

4. *Suppose $1 \leqslant l < e$, and ${}_{l-1}0_l$. Then*

$$\left[S^{\langle l-1,l \rangle}\right] = \left[D^{\lceil l \rfloor}\right] + \left[D^{\lceil l,l \rfloor}\right] + \left[D^{\lceil l-1,l \rfloor}\right] + \left[D^{\lceil l,l+1 \rfloor}\right] + \left[D^{\lceil l-1,l+1 \rfloor}\right].$$

5. *Suppose $2 \leqslant l < e$, and ${}_{l-2}1_l$. Then*

$$\left[S^{\langle l-2,l \rangle}\right] = \left[D^{\lceil l-1 \rfloor}\right] + \left[D^{\lceil l \rfloor}\right] + \left[D^{\lceil l-1,l-1 \rfloor}\right] + \left[D^{\lceil l,l \rfloor}\right] + \left[D^{\lceil l-1,l \rfloor}\right].$$

6. *Suppose $0 \leqslant k \leqslant l-2 < e-2$, and ${}_k0_l$. Then*

$$\left[S^{\langle k,l \rangle}\right] = \left[D^{\lceil k,l \rfloor}\right] + \left[D^{\lceil k,l+1 \rfloor}\right] + \left[D^{\lceil k+1,l \rfloor}\right] + \left[D^{\lceil k+1,l+1 \rfloor}\right].$$

7. *Suppose $0 \leqslant k \leqslant l-3 < e-3$, and ${}_k1_l$. Then*

$$\left[S^{\langle k,l \rangle}\right] = \left[D^{\lceil k+1,l-1 \rfloor}\right] + \left[D^{\lceil k+1,l \rfloor}\right] + \left[D^{\lceil k+2,l-1 \rfloor}\right] + \left[D^{\lceil k+2,l \rfloor}\right].$$

Corollary 3.4. *Suppose B is a block of $\mathcal{H}_n$ of weight 2, with pyramid $({}_iB_j)$, and λ is a partition in B. Then the composition length of S^λ is 1, 2, 3, 4 or 5.*

1. *S^λ is simple if and only if:*

- ◇ $\lambda = \langle e-1\rangle$;
- ◇ $\lambda = \langle 0^2\rangle$;
- ◇ $\lambda = \langle 0\rangle$ and ${}_00_1$;
- ◇ $\lambda = \langle e-1^2\rangle$ and ${}_{e-2}0_{e-1}$; or
- ◇ $\lambda = \langle 0, e-1\rangle$ and ${}_00_{e-1}$.

2. *S^λ has composition length 2 if and only if:*

- ◇ $\lambda = \langle k\rangle$, ${}_k1_{e-1}$ and $k \leqslant e-2$;
- ◇ $\lambda = \langle l^2\rangle$, ${}_01_l$ and $1 \leqslant l$;
- ◇ $\lambda = \langle 0,1\rangle$ and ${}_01_1$;
- ◇ $\lambda = \langle e-2, e-1\rangle$ and ${}_{e-2}1_{e-1}$;
- ◇ $\lambda = \langle 0,l\rangle$, ${}_00_l$ and $2 \leqslant l \leqslant e-2$; or
- ◇ $\lambda = \langle k, e-1\rangle$, ${}_k0_{e-1}$ and $1 \leqslant k \leqslant e-3$.

3. *S^λ has composition length 3 if and only if:*

- ◇ $\lambda = \langle 0\rangle$, ${}_01_2$ and ${}_00_{e-1}$;
- ◇ $\lambda = \langle e-1^2\rangle$, ${}_{e-3}1_{e-1}$ and ${}_00_{e-1}$;
- ◇ $\lambda = \langle k\rangle$, ${}_k0_{k+1}$ and $1 \leqslant k \leqslant e-2$;
- ◇ $\lambda = \langle l^2\rangle$, ${}_{l-1}0_l$ and $1 \leqslant l \leqslant e-2$;
- ◇ $\lambda = \langle l-1, l\rangle$, ${}_{l-1}1_l$ and $2 \leqslant l \leqslant e-2$;
- ◇ $\lambda = \langle 0,1\rangle$ and ${}_00_1$; or
- ◇ $\lambda = \langle e-2, e-1\rangle$ and ${}_{e-2}0_{e-1}$.

4. *S^λ has composition length 4 if and only if:*

- ◇ $\lambda = \langle k\rangle$, ${}_k1_{k+2}$, ${}_k0_{e-1}$ and $k \geqslant 1$;
- ◇ $\lambda = \langle l^2\rangle$, ${}_{l-2}1_l$, ${}_00_l$ and $l \leqslant e-2$;
- ◇ $\lambda = \langle 0\rangle$, ${}_01_1$ and ${}_00_2$;
- ◇ $\lambda = \langle e-1^2\rangle$, ${}_{e-2}1_{e-1}$ and ${}_{e-3}0_{e-1}$;
- ◇ $\lambda = \langle k,l\rangle$, ${}_k0_l$ and $1 \leqslant k \leqslant l-2 \leqslant e-4$; or
- ◇ $\lambda = \langle k,l\rangle$, ${}_k1_l$ and $0 \leqslant k \leqslant l-3 \leqslant e-4$.

5. *S^λ has composition length 5 if and only if*

- ◇ $\lambda = \langle k\rangle$, ${}_k1_{k+1}$, ${}_k0_{k+2}$ and $1 \leqslant k \leqslant e-2$;
- ◇ $\lambda = \langle l^2\rangle$, ${}_{l-2}0_l$, ${}_{l-1}1_l$ and $2 \leqslant l \leqslant e-2$;
- ◇ $\lambda = \langle l-1, l\rangle$, ${}_{l-1}0_l$ and $2 \leqslant l \leqslant e-2$; or
- ◇ $\lambda = \langle l-2, l\rangle$ and ${}_{l-2}1_l$.

4 Richards's ∂-function and a bipartition of the simple modules

Since $e \geqslant 3$, the quiver of a weight 2 block is bipartite. The corresponding bipartition of the set of simple modules can be derived from Richards's ∂-function, which assigns to each partition λ in a weight 2 block (and hence to the corresponding simple module, if λ is e-regular) an integer $\partial\lambda$ in the range $0, \dots, e-1$. It is shown by Chuang and Tan [CT1, CT2] that when $e \geqslant 3$, $\mathrm{Ext}^1_{\mathcal{H}_n}(D^\lambda, D^\mu)$ can be non-zero only if $\partial\lambda - \partial\mu = \pm 1$. Furthermore, Richards defines a colour (either black or white) for each λ with $\partial\lambda = 0$, which plays a role in his formula for decomposition numbers. Here we describe in our notation the ∂ function and the colour function on simple modules.

Proposition 4.1. *Suppose B is a block of $\mathcal{H}_n$ of weight 2, with pyramid $({}_iB_j)$.*

1. *If $1 \leqslant i < e$, then $\partial\lceil i\rfloor = 0$. $\lceil i\rfloor$ is white if $i \equiv e \pmod 2$, and black otherwise.*

2. *If $1 \leqslant i \leqslant j < e$, then $\partial\lceil i,j\rfloor = j - i + {}_{i-1}B_j$. If ${}_{i-1}0_i$, then $\lceil i,i\rfloor$ is black if $i \equiv e \pmod 2$, and white otherwise.*

We define the *parity* of an e-regular partition μ to be the parity of the integer $\partial\mu$.

5 The Cartan matrix

Now we use the decomposition numbers to obtain the entries of the Cartan matrix. Since $e \geqslant 3$, we can then easily obtain the Loewy layers of the indecomposable projectives, since it is known that the quiver of a weight 2 block is bipartite and that the projectives have Loewy length 5 and are rigid.

Proposition 5.1. *Suppose B is a weight 2 block of $\mathcal{H}_n$ with pyramid $({}_iB_j)$, and $\lambda \neq \mu$ are e-regular partitions in B of the same parity. Then the Cartan matrix entry $c_{\lambda\mu} = [P^\lambda : D^\mu] = [P^\mu : D^\lambda]$ equals $0, 1$ or 2.*

1. *$c_{\lambda\mu} = 1$ if and only if the pair $\{\lambda, \mu\}$ corresponds to one of the entries in the following table.*

λ	μ	*conditions*	$\lvert\partial\lambda - \partial\mu\rvert$
$\lceil i \rfloor$	$\lceil i{+}1 \rfloor$	${}_{i-1}B_i + {}_iB_{i+1} + {}_{i-1}B_{i+1}$ *is odd*	0
$\lceil i \rfloor$	$\lceil i{+}2 \rfloor$	${}_i1_{i+1}$	0
$\lceil i \rfloor$	$\lceil i{-}1, i{-}1 \rfloor$	${}_{i-2}0_{i-1}$	0
$\lceil i \rfloor$	$\lceil i, i \rfloor$	${}_{i-1}0_i$	0
$\lceil i \rfloor$	$\lceil i{+}1, i{+}1 \rfloor$	${}_i0_{i+1}$	0
$\lceil i, j \rfloor$	$\lceil i, j{+}1 \rfloor$	${}_{i-1}1_j$ *and* ${}_{i-1}0_{j+1}$	0
$\lceil i, j \rfloor$	$\lceil i{+}1, j \rfloor$	${}_{i-1}0_j$ *and* ${}_i1_j$	0
$\lceil i, j \rfloor$	$\lceil i{+}1, j{+}1 \rfloor$	${}_i0_j$ *or* ${}_{i-1}1_{j+1}$	0
$\lceil i \rfloor$	$\lceil i{-}1, i \rfloor$	${}_{i-2}1_i$	2
$\lceil i \rfloor$	$\lceil i, i{+}1 \rfloor$	${}_{i-1}1_{i+1}$	2
$\lceil i \rfloor$	$\lceil i{-}2, i \rfloor$	${}_{i-2}1_{i-1}$ *and* ${}_{i-2}0_i$	2
$\lceil i \rfloor$	$\lceil i{-}1, i{+}1 \rfloor$	${}_{i-1}0_i$ *or* ${}_{i-2}B_i + {}_{i-1}B_{i+1} = 1$	2
$\lceil i \rfloor$	$\lceil i, i{+}2 \rfloor$	${}_{i-1}0_{i+1}$ *and* ${}_i1_{i+1}$	2
$\lceil i, j \rfloor$	$\lceil i{-}1, j{+}1 \rfloor$	${}_{i-1}0_j$ *or* ${}_{i-2}1_{j+1}$	2
$\lceil i, j \rfloor$	$\lceil i{-}1, j{+}2 \rfloor$	${}_{i-2}0_{j+1}$ *and* ${}_{i-1}1_{j+1}$	2
$\lceil i, j \rfloor$	$\lceil i{+}2, j{-}1 \rfloor$	${}_i1_{j-1}$ *and* ${}_i0_j$	2

2. *$c_{\lambda\mu} = 2$ if and only if the pair $\{\lambda, \mu\}$ corresponds to one of the entries in the following table.*

λ	μ	*conditions*	$\lvert\partial\lambda - \partial\mu\rvert$
$\lceil i \rfloor$	$\lceil i{+}1 \rfloor$	${}_{i-1}1_i1_{i+1}$ *and* ${}_{i-1}0_{i+1}$	0
$\lceil i \rfloor$	$\lceil i{-}1, i{+}1 \rfloor$	${}_{i-2}0_i$, ${}_{i-1}1_i$ *and* ${}_{i-1}0_{i+1}$	2

Proposition 5.2. *Suppose B is a weight 2 block of $\mathcal{H}_n$ with pyramid $({}_iB_j)$, and λ, μ are e-regular partitions in B. Then $\dim_{\mathbb{F}} \mathrm{Ext}^1_{\mathcal{H}_n}(D^\lambda, D^\mu)$ equals 0 or 1, and is 1 if and only if the pair $\{\lambda, \mu\}$ equals:*

- ◇ $\{\lceil i \rfloor, \lceil i{-}1, i{-}1 \rfloor\}$, *where* ${}_{i-2}1_{i-1}$;
- ◇ $\{\lceil i \rfloor, \lceil i, i \rfloor\}$, *where* ${}_{i-1}1_i$;
- ◇ $\{\lceil i \rfloor, \lceil i{+}1, i{+}1 \rfloor\}$, *where* ${}_i1_{i+1}$;
- ◇ $\{\lceil i \rfloor, \lceil i{-}1, i \rfloor\}$, *where* ${}_{i-2}0_i$;
- ◇ $\{\lceil i \rfloor, \lceil i, i{+}1 \rfloor\}$, *where* ${}_{i-1}0_{i+1}$;
- ◇ $\{\lceil i, j \rfloor, \lceil i{-}1, j{+}1 \rfloor\}$, *where* ${}_{i-2}0_{j+1}$ *and* ${}_{i-1}1_j$;
- ◇ $\{\lceil i, j \rfloor, \lceil i, j{+}1 \rfloor\}$, *where* ${}_{i-1}0_j$ *or* ${}_{i-1}1_{j+1}$; *or*
- ◇ $\{\lceil i, j \rfloor, \lceil i{+}1, j \rfloor\}$, *where* ${}_i0_j$ *or* ${}_{i-1}1_j$.

Proposition 5.3. *Suppose B is a weight 2 block of $\mathcal{H}_n$ with pyramid $({}_iB_j)$, and λ is an e-regular partition in B. Let n_λ be the number of e-regular partitions μ such that $\mathrm{Ext}^1_{\mathcal{H}_n}(D^\lambda, D^\mu) \neq 0$, i.e. the composition length of the second (or fourth) Loewy layer of P^λ.*

1. ◇ *If $\lambda = \lceil i \rfloor$ for $1 \leqslant i < e$, then*

$$n_\lambda = 2 - \delta_{i1} - \delta_{i(e-1)} + {}_{i-2}B_{i-1} + {}_{i-1}B_i + {}_iB_{i+1} - {}_{i-2}B_i - {}_{i-1}B_{i+1}.$$

◇ *If $\lambda = \lceil i,i \rfloor$ for $1 \leqslant i < e$, then*

$$n_\lambda = (2 - \delta_{i1} - \delta_{i(e-1)})\,(1 + {}_{i-1}B_i) + {}_{i-2}B_i + {}_{i-1}B_{i+1} - {}_{i-2}B_{i+1}.$$

◇ *If $\lambda = \lceil i,i{+}1 \rfloor$ for $1 \leqslant i \leqslant e{-}2$, then*

$$n_\lambda = 6 - \delta_{i1} - \delta_{i(e-2)} - {}_{i-1}B_i - {}_iB_{i+1} + (-1 + \delta_{i1}\delta_{i(e-2)})_{i-1}B_{i+1} + {}_{i-2}B_{i+1} + {}_{i-1}B_{i+2} - {}_{i-2}B_{i+2}.$$

◇ *If $\lambda = \lceil i,j \rfloor$ for $1 \leqslant i \leqslant j{-}2 \leqslant e{-}3$, then*

$$n_\lambda = 4 - \delta_{i1} - \delta_{j(e-1)} + {}_iB_{j-1} - {}_{i-1}B_{j-1} - {}_iB_j + \delta_{i1}\delta_{j(e-1)}({}_{i-1}B_j) + {}_{i-2}B_j + {}_{i-1}B_{j+1} - {}_{i-2}B_{j+1}.$$

2. ◇ *If $e = 3$, then n_λ can take any of the values $1, 2, 3, 4$.*

◇ *If $e \geqslant 4$ then n_λ can take any of the values $1, 2, 3, 4, 5, 6$.*

Combining Corollary 3.2 and Proposition 5.1, we can obtain the composition length of the third Loewy layer of a projective indecomposable module.

Proposition 5.4. *Suppose B is a weight 2 block of $\mathcal{H}_n$ with pyramid $({}_iB_j)$, and λ is an e-regular partition in B. Let m_λ be the composition length of the third Loewy layer of $[P^\lambda : D^\mu]$.*

1. ◇ *If $\lambda = \lceil i \rfloor$ for $1 \leqslant i < e$, then*

$$\begin{aligned} m_\lambda = 5 &- 2\delta_{i1} - 2\delta_{i(e-1)} + (3 - 2\delta_{i2})({}_{i-2}B_{i-1}) + (3 - 2\delta_{i1} - 2\delta_{i(e-1)})({}_{i-1}B_i) \\ &+ (3 - 2\delta_{i(e-2)})({}_iB_{i+1}) + (-3 + \delta_{i2} + \delta_{i(e-1)})({}_{i-2}B_i) + (-3 + \delta_{i1} + \delta_{i(e-2)})({}_{i-1}B_{i+1}). \end{aligned}$$

◇ *If $\lambda = \lceil i,i \rfloor$ for $1 \leqslant i < e$, then*

$$\begin{aligned} m_\lambda = 5 &- 2\delta_{i1} - 2\delta_{i(e-1)} + (-2 + \delta_{i1} + \delta_{i(e-1)})({}_{i-1}B_i) + (2 - \delta_{i2} - \delta_{i(e-1)} + \delta_{i2}\delta_{i(e-1)})({}_{i-2}B_i) \\ &+ (2 - \delta_{i1} - \delta_{i(e-2)} + \delta_{i1}\delta_{i(e-2)})({}_{i-1}B_{i+1}) + (-2 + \delta_{i2} + \delta_{i(e-2)})({}_{i-2}B_{i+1}). \end{aligned}$$

◇ *If $\lambda = \lceil i,i{+}1 \rfloor$ for $1 \leqslant i \leqslant e{-}2$, then*

$$\begin{aligned} m_\lambda = 6 &- 2\delta_{i1} - 2\delta_{i(e-2)} + \delta_{i1}\delta_{i(e-2)} + (-1 + \delta_{i1})({}_{i-1}B_i) + (-1 + \delta_{i(e-2)})({}_iB_{i+1}) \\ &+ (1 - \delta_{i1}\delta_{i(e-2)})({}_{i-1}B_{i+1}) + (2 - \delta_{i2} - \delta_{i(e-2)} + \delta_{i2}\delta_{i(e-2)})({}_{i-2}B_{i+1}) \\ &+ (2 - \delta_{i1} - \delta_{i(e-3)} + \delta_{i1}\delta_{i(e-3)})({}_{i-1}B_{i+2}) + (-2 + \delta_{i2} + \delta_{i(e-3)})({}_{i-2}B_{i+2}). \end{aligned}$$

◇ *If $\lambda = \lceil i,i{+}2 \rfloor$ for $1 \leqslant i \leqslant e{-}3$, then*

$$\begin{aligned} m_\lambda = 7 &- 2\delta_{i1} - 2\delta_{i(e-3)} + \delta_{i1}\delta_{i(e-3)} + 3({}_iB_{i+1}) + (-3 + \delta_{i1})({}_{i-1}B_{i+1}) + (-3 + \delta_{i(e-3)})({}_iB_{i+2}) \\ &- \delta_{i1}\delta_{i(e-3)}({}_{i-1}B_{i+2}) + (2 - \delta_{i2} - \delta_{i(e-3)} + \delta_{i2}\delta_{i(e-3)})({}_{i-2}B_{i+2}) \\ &+ (2 - \delta_{i1} - \delta_{i(e-4)} + \delta_{i1}\delta_{i(e-4)})({}_{i-1}B_{i+3}) + (-2 + \delta_{i2} + \delta_{i(e-4)})({}_{i-2}B_{i+3}). \end{aligned}$$

- *If $\lambda = \lceil i,j \rfloor$ for $1 \leqslant i \leqslant j-3 \leqslant e-4$, then*

$$\begin{aligned} m_\lambda = 6 &- 2\delta_{i1} - 2\delta_{j(e-1)} + \delta_{i1}\delta_{j(e-1)} + 2({}_iB_{j-1}) + (-2+\delta_{i1})({}_{i-1}B_{j-1}) + (-2+\delta_{j(e-1)})({}_iB_j) \\ &- \delta_{i1}\delta_{j(e-1)}({}_{i-1}B_j) + (2-\delta_{i2}-\delta_{j(e-1)}+\delta_{i2}\delta_{j(e-1)})({}_{i-2}B_j) \\ &+ (2-\delta_{i1}-\delta_{j(e-2)}+\delta_{i1}\delta_{j(e-2)})({}_{i-1}B_{j+1}) + (-2+\delta_{i2}+\delta_{j(e-2)})({}_{i-2}B_{j+1}). \end{aligned}$$

2. - *If $e = 3$, then m_λ can take any of the values 2–5.*
   - *If $e = 4$, then m_λ can take any of the values 2–10.*
   - *If $e = 5$, then m_λ can take any of the values 2–12.*
   - *If $e \geqslant 6$, then m_λ can take any of the values 2–12 or 14.*

Now we consider the composition length of an indecomposable projective module. Combining Propositions 5.3 and 5.4, one can easily write down an explicit formula for the length of a projective module. We save space by omitting this, but instead give the possible values such a formula can take.

Proposition 5.5. *Suppose B is a weight 2 block of $\mathcal{H}_n$, and λ is an e-regular partition in B, and let c_λ denote the composition length of P^λ.*

- *If $e = 3$, then c_λ can take any of the values 7–11 or 13.*
- *If $e = 4$, then c_λ can take any of the values 7, 8, 10–19 or 22.*
- *If $e = 5$, then c_λ can take any of the values 7–21 or 24.*
- *If $e \geqslant 6$, then c_λ can take any of the values 7–22, 24 or 26.*

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