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Weight two blocks of Iwahori–Hecke algebras in characteristic two

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Abstract

We study blocks of the Iwahori–Hecke algebra $\mathcal{H}_q(\mathfrak{S}_n)$ of weight two over a field of characteristic two. Using techniques and notation developed by Scopes, Richards, Chuang and Tan for the case of odd characteristic, we find the decomposition numbers and classify extensions between simple modules for these blocks.

Contents

1 Introduction

Let $\mathbb{F}$ be a field, q an element of $\mathbb{F}$ and n a non-negative integer. We consider the Iwahori–Hecke algebra $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$, which we shall henceforth denote $\mathcal{H}_n$. Of course, if $q = 1$ then $\mathcal{H}_n \cong \mathbb{F}\mathfrak{S}_n$. We let e denote the smallest integer such that $1 + q + \cdots + q^{e-1} = 0$ in $\mathbb{F}$ if such an integer exists, and put $e = \infty$ otherwise. Each block of $\mathcal{H}_n$ then has an *e-weight*, which is a non-negative integer. In this paper, we consider blocks of e-weight two in the case where $\mathbb{F}$ has characteristic two. Blocks of weight zero and one are very well understood, and blocks of weight two are understood in the case when char($\mathbb{F}$) > 2; in the symmetric group case, Scopes [14] initiated the study of these blocks; Richards [12] described their decomposition matrices combinatorially and then Chuang and Tan [2, 3] described the Ext^1 space between two simple modules. The q-analogue of the Jantzen–Schaper formula due to James and Mathas [8] ensures that these results also hold for the Iwahori–Hecke algebras. We use Richards's combinatorial techniques together with the Jantzen–Schaper formula to determine the decomposition matrices in the case char($\mathbb{F}$) $= 2$ (that is, we find the 'adjustment matrices' for these blocks, in the sense of James [7]); by analysing $[2:1]$-pairs in the cases where the adjustment matrix is non-trivial, we are also able to calculate the Ext^1 space for two simple modules (when $e > 2$).

An excellent introduction to the representation theory of $\mathcal{H}_n$ can be found in Mathas's book [10]; we shall summarise the relevant points here.

For each partition λ of n one defines a *Specht module* S^λ for $\mathcal{H}_n$. (There are conflicting definitions of the Specht module; we actually use that defined by Dipper and James [4] rather than that defined by Mathas.) If λ is e-regular (i.e. if λ does not have e equal non-zero parts) then S^λ has an irreducible self-dual cosocle D^λ, and the set of modules D^λ gives a complete set of irreducible $\mathcal{H}_n$-modules as λ ranges over the set of e-regular partitions of n. Furthermore, each composition factor of S^λ is of the form D^μ for some partition μ with $\mu \trianglerighteq \lambda$, where $\trianglerighteq$ is the usual dominance order on partitions; if λ is e-regular, then D^λ occurs exactly once as a composition factor of S^λ.

Let $T_1, \dots, T_{n-1}$ be the standard generators of $\mathcal{H}_n$. Let $\sharp : \mathcal{H}_n \to \mathcal{H}_n$ be the involutory automorphism sending T_i to $q-1-T_i$, and let $* : \mathcal{H}_n \to \mathcal{H}_n$ be the anti-automorphism sending T_i to T_i. Given a module M for $\mathcal{H}_n$, define $M^\sharp$ to be the module with the same underlying vector space and with action

$$h \cdot m = h^\sharp m,$$

and define M^* to be the module with underlying vector space dual to M and with $\mathcal{H}_n$-action

$$h \cdot f(m) = f(h^* m).$$

Let λ' denote the partition conjugate to λ. Then we have the following.

Lemma 1.1. [10, Exercise 3.14(iii)] *For any partition λ,*

$$S^{\lambda'} \cong (S^{\lambda\sharp})^*.$$

Given an e-regular partition λ, let $\lambda^\diamond$ denote the e-regular partition such that $D^{\lambda\sharp} \cong D^{\lambda^\diamond}$; the bijection $\lambda \mapsto \lambda^\diamond$ is described by the Mullineux algorithm [11]. Since conjugation of partitions reverses the dominance order, we find that if D^μ is a composition factor of S^λ, then $\lambda \trianglerighteq \mu^{\diamond\prime}$.

1.1 Blocks and the abacus

If e is finite, then partitions of n are conveniently represented on an abacus. If λ is a partition, choose an integer r greater than the number of parts of λ, and define

$$\beta_i = \lambda_i + r - i$$

for $i = 1, \dots, r$. The set $\{\beta_1, \dots, \beta_r\}$ is then said to be a set of *beta-numbers* for λ. Now take an abacus with e vertical runners numbered $0, \dots, e-1$ from left to right, and number the positions on runner i from the top as $i, i+e, i+2e, \dots$. Then place a bead on the abacus at position β_i for each i. The resulting configuration is said to be an abacus display for λ. The partition whose abacus display is obtained from this by moving all the beads as far up their runners as they will go is called the *e-core* of λ; it is a partition of $n - we$ for some w, which is called the *weight* of λ. Moving a bead up s spaces on its runner corresponds to removing a rim hook of length es from the Young diagram. 'Nakayama's Conjecture' says that two Specht modules S^λ and S^μ lie in the same block of $\mathcal{H}_n$ (we shall abuse notation by saying that λ and μ lie in this block) if and only if they have the same e-core; this means that they also have the same weight, and this is called the weight of the block.

1.2 The Janzten–Schaper formula

One of the most important tools in finding the decomposition numbers of $\mathcal{H}_n$ is the (q-analogue of the) Jantzen–Schaper formula. We describe this very briefly.

Given partitions λ and μ of n, define $H(\lambda, \mu)$ to be the set of ordered pairs (g, h), where

- g is a rim hook of the Young diagram $[\lambda]$ of λ;
- h is a rim hook of the Young diagram $[\mu]$ of μ;
- $[\lambda] \setminus g = [\mu] \setminus h$.

Now define

$$c_{\lambda,\mu} = \sum_{(g,h)\in H(\lambda,\mu)} (-1)^{l(g)+l(h)+1} \nu_{e,p}(|g|);$$

here $|g|$ is the number of nodes of g and $l(g)$ its leg length, and $\nu_{e,p}(x)$ is defined for a positive integer x to be $a + 1$ if x may be written in the form $x = ep^a b$ with b an integer not divisible by p, and 0 otherwise.

Each Specht module S^λ has a filtration

$$S^\lambda = S^{\lambda(0)} \geqslant S^{\lambda(1)} \geqslant \dots$$

in which $S^{\lambda(0)}/S^{\lambda(1)}$ is isomorphic to D^λ if λ is e-regular and 0 otherwise. The Jantzen–Schaper formula may be stated as follows.

Theorem 1.2. [8, Theorem 4.7] *For partitions λ, μ of n with μ e-regular,*

$$\sum_i i\left[\left(S^{\lambda(i)}/S^{\lambda(i+1)}\right) : D^\mu\right] = \sum_{\nu \rhd \lambda} c_{\lambda,\nu}[S^\nu : D^\mu].$$

In particular, if $\lambda \neq \mu$ then

$$[S^\lambda : D^\mu] \leqslant \sum_{\nu \rhd \lambda} c_{\lambda,\nu}[S^\nu : D^\mu],$$

and the left-hand side of the above inequality is non-zero if and only if the right-hand side is.

This theorem greatly restricts the possible decomposition numbers for $\mathcal{H}_n$. To use it, one studies one e-regular partition μ at a time, and tries to find possible sets of values $\left[\left(S^{\lambda(i)}/S^{\lambda(i+1)}\right) : D^\mu\right]$ satisfying the Jantzen–Schaper formula, bearing in mind that $[S^\lambda : D^\mu] = 0$ unless $\mu \unrhd \lambda \unrhd \mu^{\diamond\prime}$, and that

$$\left[\left(S^{\mu(0)}/S^{\mu(1)}\right) : D^\mu\right] = [S^\mu : D^\mu] = [S^{\mu^{\diamond\prime}} : D^\mu] = 1.$$

Remark. One might ask in which layer of the Janzten–Schaper filtration of $S^{\mu^{\diamond\prime}}$ the composition factor D^μ lies, or more generally how the Jantzen–Schaper filtrations of S^λ and $S^{\lambda'}$ correspond under the isomorphism in Lemma 1.1. The author has answered this question in the symmetric group case [5, Theorem 4.8], and from the calculations used in the present work an analogous result would seem to hold for the Iwahori–Hecke algebras. But we shall not need this here.

1.3 Blocks of weight two

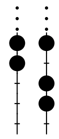
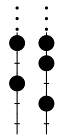
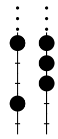
In this paper we shall be entirely concerned with blocks of weight two. Scopes [14] began the study of these blocks in the symmetric group case, and developed the notion of a $[2:k]$-*pair*. If A is a block of $\mathcal{H}_{n-k}$ and B a block of $\mathcal{H}_n$, then we say that A and B form a $[2:k]$-pair if there is an abacus for A and some integer $0 \leqslant i \leqslant e-1$ such that interchanging runners i and $i-1$ gives an abacus for B.

If $k \geqslant 2$, then $[2:k]$-pairs are particularly useful.

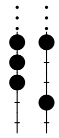
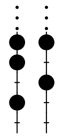
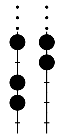
Theorem 1.3. [10, p. 127] *Suppose that A and B form a $[2:k]$-pair as above, with $k \geqslant 2$. For each partition λ in A, define the partition $\Phi(\lambda)$ in B by interchanging runners i and $i-1$ of the abacus display for λ. Then:*

- *Φ gives a bijection between the set of partitions in A and the set of partitions in B;*
- *$\Phi(\lambda)$ is e-regular if and only if λ is;*
- *for partitions λ and μ in A with μ e-regular, we have $[S^\lambda : D^\mu] = [S^{\Phi(\lambda)} : D^{\Phi(\mu)}]$;*
- *the correspondence $D^\lambda \leftrightarrow D^{\Phi(\lambda)}$ gives a Morita equivalence between A and B.*

Thus the difficult part of the study of blocks of weight two is to understand [2 : 1]-pairs. Suppose that A and B form a [2 : 1]-pair, and denote by α,β,γ the following partitions in B (we indicate only runners $i-1$ and i; on all other runners the beads must be as far up as possible):

$\alpha =$  $\beta =$  $\gamma =$ .

We also denote by $\bar\alpha,\bar\beta,\bar\gamma$ the following partitions in A:

$\bar\alpha =$  $\bar\beta =$  $\bar\gamma =$ .

Now if λ is a partition in A other than $\bar\alpha,\bar\beta$ or $\bar\gamma$, then define the partition $\Phi(\lambda)$ in B by interchanging runners i and $i-1$ of the abacus display for λ. Then the following result follows from [14] and [1].

Theorem 1.4. *Let $A, B, \alpha, \beta, \gamma, \bar\alpha, \bar\beta, \bar\gamma, \Phi$ be as above. Then*

- *Φ gives a bijection between the set of partitions in A other than $\bar\alpha,\bar\beta,\bar\gamma$ and the set of partitions in B other than α,β,γ;*
- *$\Phi(\lambda)$ is e-regular if and only if λ is;*
- *if $\lambda \neq \bar\alpha,\bar\beta,\bar\gamma$, then*
$$S^\lambda{\uparrow}_A^B \cong S^{\Phi(\lambda)}, \qquad S^{\Phi(\lambda)}{\downarrow}_A^B \cong S^\lambda$$
and (if λ is e-regular)
$$D^\lambda{\uparrow}_A^B \cong D^{\Phi(\lambda)}, \qquad D^{\Phi(\lambda)}{\downarrow}_A^B \cong D^\lambda;$$
- *$\bar\beta$ is e-regular if and only if γ is, and in this case*
$$D^{\bar\beta}{\uparrow}_A^B \cong D^\gamma, \qquad D^\gamma{\downarrow}_A^B \cong D^{\bar\beta};$$
- *$\bar\gamma$ is e-regular if and only if β is, and in this case*
$$D^{\bar\gamma}{\uparrow}_A^B \cong D^\beta, \qquad D^\beta{\downarrow}_A^B \cong D^{\bar\gamma};$$
- *$\bar\alpha$ and α are always e-regular, $D^{\bar\alpha}{\uparrow}_A^B$ is self-dual with socle isomorphic to D^α, and $D^\alpha{\downarrow}_A^B$ is self-dual with socle isomorphic to $D^{\bar\alpha}$;*
- *if λ,μ are partitions in A other than $\bar\alpha,\bar\beta,\bar\gamma$ with μ e-regular, then $[S^\lambda : D^\mu] = [S^{\Phi(\lambda)} : D^{\Phi(\mu)}]$;*
- *if λ is a partition in A other than $\bar\alpha,\bar\beta,\bar\gamma$ and $\bar\beta$ is e-regular, then $[S^\lambda : D^{\bar\beta}] = [S^{\Phi(\lambda)} : D^\gamma]$;*
- *if λ is a partition in A other than $\bar\alpha,\bar\beta,\bar\gamma$ and $\bar\gamma$ is e-regular, then $[S^\lambda : D^{\bar\gamma}] = [S^{\Phi(\lambda)} : D^\beta]$;*
- *if λ is a partition in A other than $\bar\alpha,\bar\beta,\bar\gamma$, then $[S^\lambda : D^{\bar\alpha}] = [S^{\Phi(\lambda)} : D^\alpha] = 0$.*

1.4 Pyramids

Richards was able to understand blocks of weight two (and, in particular, to classify all possible cases of the Jantzen–Schaper formula for these blocks) using the combinatorics of pyramids.

Given a block B of $\mathcal{H}_n$ of weight two, take an abacus for (the e-core of) B and let $p_0 < \cdots < p_{e-1}$ be those integers such that there is a bead at position p_i but no bead at position $p_i + e$, for each i. Thus exactly one of $p_0, \ldots, p_{e-1}$ will lie in each (mod e) congruence class. Now, for $0 \leqslant i \leqslant j \leqslant e-1$, define

$$ {}_iB_j = \begin{cases} 1 & (p_j < p_i + e) \\ 0 & (p_j > p_i + e). \end{cases} $$

We also define ${}_{-1}B_i = {}_iB_e = 0$ for all i. The set of integers ${}_iB_j$ then called the pyramid for B. We shall use shorthand notation such as ${}_i0_j1_k$ to indicate ${}_iB_j = 0$, ${}_jB_k = 1$, when it is clear which block B is.

Now, given a partition λ in B, we encode it with a pair of integers:

- if λ is obtained by moving beads down from positions p_i and p_j one space each, then we write λ as $\{2i+1, 2j+1\}$;
- if λ is obtained by moving beads down from positions p_i and $p_i - e$, then we define k to be the integer such that
$$ {}_{k-1}0_i, \qquad {}_k1_i, $$
and write λ as $\{2k, 2i+1\}$;
- if λ is obtained by a moving a bead down two spaces from position p_i, let k be the integer such that
$$ {}_i1_k, \qquad {}_i0_{k+1}, $$
and write λ as $\{2i+1, 2k+2\}$.

Lemma 1.5. [12, Lemma 4.5] *Suppose $\{a, b\}$ and $\{c, d\}$ are two partitions written as above. Then (assuming $a < b$ and $c < d$) $\{a, b\}$ dominates $\{c, d\}$ if and only if $a \geqslant c$ and $b \geqslant d$.*

Given a partition λ of weight two, we may remove two rim hooks of length e from the Young diagram of λ to obtain the Young diagram for the e-core of λ. The difference between the leg lengths of these rim hooks is independent of the way in which they are chosen, and we call this value $\partial\lambda$.

Using these combinatorial data, Richards was able to describe the partition $\mu^{\diamond\prime}$ for each e-regular partition μ of weight two, and to find all partitions λ with $\mu \trianglerighteq \lambda \trianglerighteq \mu^{\diamond\prime}$. The Mullineux algorithm depends only on e, and not on the underlying field, and so Richards's results are of use to us. These are summarised in Table 1, which is essentially Tables 1 and 2 of [12]. Note that we have made some changes to these tables in order to aid clarity. In labelling the rows, we identify 'conjugate pairs': thus if λ satisfies the conditions given in row A, then $\lambda^\diamond$ satisfies the conditions given in row A$^\diamond$. Those rows for which no conjugate is given are 'self-conjugate'. For partitions whose existence imposes certain conditions on the pyramid, we have given these conditions (between parentheses), in order to make it clear that the table exhausts all possibilities; for example, the partition $\{2i+1, 2i+2\}$ only exists if ${}_i0_{i+1}$. Finally, note that row P of Table 1 corresponds to two almost identical rows of Richards's Table 2.

By analysing each of these cases, Richards showed that, in the case where char($\mathbb{F}$) > 2, the Jantzen–Schaper formula completely determines the decomposition numbers. If λ and μ are partitions with μ e-regular, say that λ and μ are *adjacent* if $\mu \trianglerighteq \lambda \trianglerighteq \mu^{\diamond\prime}$ and $|\partial\lambda - \partial\mu| = 1$. Then we have the following.

	μ	Conditions	$\mu^{\diamond\prime}$	$\partial\lambda-\partial\mu=-2$	$\partial\lambda-\partial\mu=-1$	$\partial\lambda-\partial\mu=0$	$\partial\lambda-\partial\mu=1$	$\partial\lambda-\partial\mu=2$	$\partial\lambda-\partial\mu=3$
A	$\{2i+1,2j+1\}$ $0\leqslant i\leqslant j-2\leqslant e-3$	${}_{i}1_{j}$ ${}_{i-1}0_{j-1}$	$\{2i,2j-1\}$	—	$\{2i+1,2j-1\}$	—	$\{2i,2j+1\}$	—	—
B	$\{2i+1,2j+1\}$ $1\leqslant i\leqslant j-2\leqslant e-3$	${}_{i}1_{j}$ ${}_{i-1}0_{j}$ ${}_{i-1}1_{j-1}$	$\{2i-1,2j-1\}$	—	$\{2i+1,2j-1\}$	—	$\{2i,2j+1\}$ $\{2i-1,2j\}$	$\{2i-1,2j+1\}$	—
C	$\{2i+1,2j+1\}$ $1\leqslant i\leqslant j-2\leqslant e-3$	${}_{i-1}1_{j}$	$\{2i-1,2j-1\}$	—	$\{2i+1,2j-1\}$	—	$\{2i-1,2j+1\}$	—	—
D	$\{2i+1,2j+1\}$ $1\leqslant i\leqslant j-1\leqslant e-2$	${}_{i}0_{j}$ ${}_{i-1}1_{j-1}$	$\{2i-1,2j\}$	—	$\{2i+1,2j\}$	—	$\{2i-1,2j+1\}$	—	—
E	$\{2i+1,2j+1\}$ $1\leqslant i\leqslant j-2\leqslant e-3$	${}_{i}0_{j}$ ${}_{i}1_{j-1}$ ${}_{i-1}0_{j-1}$	$\{2i-1,2j-1\}$	$\{2i+1,2j-1\}$	$\{2i+1,2j\}$ $\{2i,2j-1\}$	—	$\{2i-1,2j+1\}$	—	—
F	$\{2i+1,2i+3\}$ $1\leqslant i\leqslant e-2$	${}_{i}0_{i+1}$ ${}_{i-1}0_{i}$	$\{2i-1,2i+1\}$	—	$\{2i+1,2i+2\}$ $\{2i,2i+1\}$	—	$\{2i-1,2i+3\}$	—	—
G	$\{2i+1,2j+1\}$ $1\leqslant i\leqslant j-2\leqslant e-3$	${}_{i}0_{j-1}$	$\{2i-1,2j-1\}$	—	$\{2i+1,2j-1\}$	—	$\{2i-1,2j+1\}$	—	—
H	$\{2i+1,2i+3\}$ $1\leqslant i\leqslant e-2$	${}_{i}1_{i+1}$ ${}_{i-1}0_{i}$ ${}_{i-2}0_{i-1}$	$\{2i-2,2i-1\}$	—	—	$\{2i,2i+1\}$ $\{2i-1,2i\}$	$\{2i,2i+3\}$ $\{2i-1,2i+1\}$	$\{2i-1,2i+3\}$	—
I	$\{2i+1,2i+3\}$ $2\leqslant i\leqslant e-2$	${}_{i}1_{i+1}$ ${}_{i-1}0_{i}$ ${}_{i-2}1_{i-1}$	$\{2i-3,2i-1\}$	—	—	$\{2i,2i+1\}$ $\{2i-1,2i\}$	$\{2i,2i+3\}$ $\{2i-1,2i+1\}$ $\{2i-3,2i\}$	$\{2i-1,2i+3\}$ $\{2i-3,2i+1\}$	$\{2i-3,2i+3\}$
J	$\{2i+1,2i+3\}$ $1\leqslant i\leqslant e-2$	${}_{i}1_{i+1}$ ${}_{i-1}0_{i+1}$ ${}_{i-1}1_{i}$ ${}_{i-2}0_{i-1}$	$\{2i-2,2i-1\}$	—	—	$\{2i-1,2i+1\}$	$\{2i,2i+3\}$ $\{2i-1,2i+2\}$ $\{2i-2,2i+1\}$	$\{2i-1,2i+3\}$	—
K	$\{2i+1,2i+3\}$ $2\leqslant i\leqslant e-2$	${}_{i}1_{i+1}$ ${}_{i-1}0_{i+1}$ ${}_{i-1}1_{i}$ ${}_{i-2}0_{i}$ ${}_{i-2}1_{i-1}$	$\{2i-3,2i-1\}$	—	—	$\{2i-1,2i+1\}$	$\{2i,2i+3\}$ $\{2i-1,2i+2\}$ $\{2i-2,2i+1\}$ $\{2i-3,2i\}$	$\{2i-1,2i+3\}$ $\{2i-3,2i+1\}$	$\{2i-3,2i+3\}$
L	$\{2i+1,2i+3\}$ $2\leqslant i\leqslant e-2$	${}_{i}1_{i+1}$ ${}_{i-1}0_{i+1}$ ${}_{i-2}1_{i}$	$\{2i-3,2i-1\}$	—	—	$\{2i-1,2i+1\}$	$\{2i,2i+3\}$ $\{2i-1,2i+2\}$ $\{2i-3,2i+1\}$	$\{2i-1,2i+3\}$ $\{2i-3,2i+2\}$	$\{2i-3,2i+3\}$
M	$\{2i+1,2i+3\}$ $1\leqslant i\leqslant e-2$	${}_{i-1}1_{i+1}$ ${}_{i-2}0_{i-1}$	$\{2i-2,2i-1\}$	—	—	$\{2i-1,2i+1\}$	$\{2i-1,2i+3\}$ $\{2i-2,2i+1\}$	$\{2i-2,2i+3\}$	—
$\mathrm{L}^{\diamond}$	$\{2i+1,2i+3\}$ $2\leqslant i\leqslant e-2$	${}_{i-1}1_{i+1}$ ${}_{i-2}0_{i}$ ${}_{i-2}1_{i-1}$	$\{2i-3,2i-1\}$	—	—	$\{2i-1,2i+1\}$	$\{2i-1,2i+3\}$ $\{2i-2,2i+1\}$ $\{2i-3,2i\}$	$\{2i-2,2i+3\}$ $\{2i-3,2i+1\}$	$\{2i-3,2i+3\}$
N	$\{2i+1,2i+3\}$ $2\leqslant i\leqslant e-2$	${}_{i-1}1_{i+1}$ ${}_{i-2}0_{i+1}$ ${}_{i-2}1_{i}$	$\{2i-3,2i-1\}$	—	—	$\{2i-1,2i+1\}$	$\{2i-1,2i+3\}$ $\{2i-3,2i+1\}$	$\{2i-2,2i+3\}$ $\{2i-3,2i+2\}$	$\{2i-3,2i+3\}$
O	$\{2i+1,2i+3\}$ $2\leqslant i\leqslant e-2$	${}_{i-2}1_{i+1}$	$\{2i-3,2i-1\}$	—	—	$\{2i-1,2i+1\}$	$\{2i-1,2i+3\}$ $\{2i-3,2i+1\}$	$\{2i-3,2i+3\}$	
P	$\{2k,2i+1\}$ $1\leqslant k\leqslant i\leqslant e-1$	$({}_{k}1_{i})$ ${}_{k-1}1_{i-1}$ $({}_{k-1}0_{i})$	$\{2k-1,2i\}$	—	—	—	$\{2k-1,2i+1\}$	—	—
$\mathrm{D}^{\diamond}$	$\{2k,2i+1\}$ $1\leqslant k\leqslant i-1\leqslant e-2$	$({}_{k}1_{i})$ ${}_{k-1}0_{i-1}$	$\{2k-1,2i-1\}$	—	$\{2k,2i-1\}$	—	$\{2k-1,2i+1\}$	—	—
R	$\{2i+1,2k+2\}$ $0\leqslant i\leqslant k-1\leqslant e-2$	$({}_{i}0_{k+1})$ $({}_{i}1_{k})$ ${}_{i-1}0_{k}$	$\{2i,2k+1\}$	—	$\{2i+1,2k+1\}$	—	—	—	—
$\mathrm{A}^{\diamond}$	$\{2i+1,2k+2\}$ $1\leqslant i\leqslant k-1\leqslant e-2$	$({}_{i}0_{k+1})$ ${}_{i-1}1_{k}$	$\{2i-1,2k+1\}$	—	$\{2i+1,2k+1\}$	—	$\{2i-1,2k+2\}$	—	—
S	$\{2i+1,2i+2\}$ $1\leqslant i\leqslant e-1$	$({}_{i}0_{i+1})$ ${}_{i-1}0_{i}$ ${}_{i-2}0_{i-1}$	$\{2i-2,2i-1\}$	—	—	$\{2i,2i+1\}$ $\{2i-1,2i\}$	$\{2i-1,2i+1\}$	—	—
$\mathrm{H}^{\diamond}$	$\{2i+1,2i+2\}$ $2\leqslant i\leqslant e-1$	$({}_{i}0_{i+1})$ ${}_{i-1}0_{i}$ ${}_{i-2}1_{i-1}$	$\{2i-3,2i-1\}$	—	—	$\{2i,2i+1\}$ $\{2i-1,2i\}$	$\{2i-1,2i+1\}$ $\{2i-3,2i\}$	$\{2i-3,2i+1\}$	—
T	$\{2i+1,2i+2\}$ $1\leqslant i\leqslant e-1$	$({}_{i}0_{i+1})$ ${}_{i-1}1_{i}$ ${}_{i-2}0_{i-1}$	$\{2i-2,2i-1\}$	—	—	$\{2i-1,2i+1\}$	$\{2i-1,2i+2\}$ $\{2i-2,2i+1\}$	—	—
$\mathrm{J}^{\diamond}$	$\{2i+1,2i+2\}$ $2\leqslant i\leqslant e-1$	$({}_{i}0_{i+1})$ ${}_{i-1}1_{i}$ ${}_{i-2}0_{i}$ ${}_{i-2}1_{i-1}$	$\{2i-3,2i-1\}$	—	—	$\{2i-1,2i+1\}$	$\{2i-1,2i+2\}$ $\{2i-2,2i+1\}$ $\{2i-3,2i\}$	$\{2i-3,2i+1\}$	—
$\mathrm{M}^{\diamond}$	$\{2i+1,2i+2\}$ $2\leqslant i\leqslant e-1$	$({}_{i}0_{i+1})$ ${}_{i-2}1_{i}$	$\{2i-3,2i-1\}$	—	—	$\{2i-1,2i+1\}$	$\{2i-1,2i+2\}$ $\{2i-3,2i+1\}$	$\{2i-3,2i+2\}$	—

Table 1

Theorem 1.6. [12, Theorem 4.4] *Let B be a block of weight two, and suppose* $\mathrm{char}(\mathbb{F}) > 2$*. If λ and μ are partitions in B with μ e-regular, then $[S^\lambda : D^\mu]$ equals 1 if $\lambda = \mu$, $\lambda = \mu^{\diamond\prime}$ or λ and μ are adjacent, and 0 otherwise.*

2 Decomposition numbers

In this section, we find the decomposition numbers of weight two blocks in characteristic two. Here we find that the decomposition numbers are not completely determined by the Jantzen–Schaper formula, and we have to analyse some $[2 : 1]$-pairs. To begin with, we find what we can from the Jantzen–Schaper formula.

Proposition 2.1. *Suppose that* $\mathrm{char}(\mathbb{F}) = 2$*, B is a block of $\mathcal{H}_n$ of weight two, and μ is an e-regular partition in B.*

1. *Suppose μ is of the form $\{2i + 1, 2i + 2 + {}_iB_{i+1}\}$ and that ${}_{i-1}0_i$. Then:*
   - (a) *if $\lambda = \{2i - 1, 2i\}$ or $\{2i, 2i + 1\}$ then $[S^\lambda : D^\mu] = 1$;*
   - (b) *if $\lambda = \{2i - 1, 2i + 1\}$ then $[S^\lambda : D^\mu] = 2$;*
   - (c) *for any other partition λ in B, $[S^\lambda : D^\mu]$ equals 1 if $\lambda = \mu$, $\lambda = \mu^{\diamond\prime}$ or λ and μ are adjacent, and 0 otherwise.*
2. *Suppose μ is of the form $\{2i + 1, 2i + 2 + {}_iB_{i+1}\}$ and that ${}_{i-1}1_i$ and ${}_{i-2}0_i$. Then there exists an integer $a_\mu \in \{0, 1\}$ such that:*
   - (a) *if $\lambda = \{2i - 1, 2i\}$ or $\{2i, 2i + 1\}$ then $[S^\lambda : D^\mu] = 1 + a_\mu$;*
   - (b) *if $\lambda = \{2i - 1, 2i + 1\}$ then $[S^\lambda : D^\mu] = a_\mu$;*
   - (c) *for any other partition λ in B, $[S^\lambda : D^\mu]$ equals 1 if $\lambda = \mu$, $\lambda = \mu^{\diamond\prime}$ or λ and μ are adjacent, and 0 otherwise.*
3. *Suppose μ is not of one of the forms listed in (1) or (2). Then for any partition λ in B, $[S^\lambda : D^\mu]$ equals 1 if $\lambda = \mu$, $\lambda = \mu^{\diamond\prime}$ or λ and μ are adjacent, and 0 otherwise.*

Proof. We simply check through the cases in Table 1, trying to find possible decomposition numbers which satisfy the Jantzen–Schaper formula. (1) corresponds to cases H, I, S and H$^\diamond$, (2) corresponds to cases J, K, T and J$^\diamond$, and (3) covers all the remaining cases. We consider 'conjugate pairs' of cases simultaneously, bearing in mind that $[S^\lambda : D^\mu] = [S^{\lambda'} : D^{\mu^\diamond}]$. The Jantzen–Schaper formula is then sufficient to prove the proposition, except in case S. We illustrate this case explicitly. Putting $\mu = \{2i+1, 2i+2\}$ and supposing that ${}_{i-2}0_{i-1}0_i0_{i+1}$ as in row S of Table 1, the coefficients $c_{\lambda,\nu}$ which appear in the Theorem 1.2 are as follows, for $\mu \unrhd \nu \rhd \lambda \unrhd \mu^{\diamond\prime}$:

		ν: $\{2i+1, 2i+2\}$	$\{2i, 2i+1\}$	$\{2i-1, 2i+1\}$	$\{2i-1, 2i\}$
λ	$\{2i, 2i+1\}$	$+1$			
	$\{2i-1, 2i+1\}$	$+1$	$+1$		
	$\{2i-1, 2i\}$	-2	$+2$	$+1$	
	$\{2i-2, 2i-1\}$	$+2$	-2	$+1$	$+1$

.

Applying Theorem 1.2 simultaneously to μ and $\mu^\diamond = \{2(e-i)+1, 2(e-i)+2\}$ (whch lies in a block with ${}_{e-i-2}0_{e-i-1}0_{e-i}0_{e-i+1}$), we find that the column of the decomposition matrix corresponding to D^μ is

$$\begin{bmatrix} [S^{\{2i+1,2i+2\}} : D^\mu] \\ [S^{\{2i,2i+1\}} : D^\mu] \\ [S^{\{2i-1,2i+1\}} : D^\mu] \\ [S^{\{2i-1,2i\}} : D^\mu] \\ [S^{\{2i-2,2i-1\}} : D^\mu] \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1+a \\ 1 \\ 1 \end{bmatrix},$$

where a equals 0 or 1. If $i > 1$, then we may resolve this ambiguity by applying Theorem 1.2 to the Specht module $S^{\{2i-3,2i+1\}}$ – we find that $[S^{\{2i-3,2i+1\}} : D^\mu] \leqslant a-1$, which implies that $a = 1$. If $i < e-1$, then we may replace μ with $\mu^\diamond$ and apply the same argument. If $e-1 \leqslant i \leqslant 1$, then $e = 2$ and we may simply look up the decomposition number in the appendix of [6] (applying Theorem 1.3). □

To determine the decomposition numbers completely, it suffices to determine the integers a_μ which appear in Proposition 2.1. In fact, these integers are all zero; this follows from the next result.

Proposition 2.2. *Suppose B is a block of $\mathcal{H}_n$ of weight two, and that λ, μ are partitions in B with μ e-regular and $[S^\lambda : D^\mu] = 2$. Then λ is of the form $\{2i-1, 2i+1\}$.*

Proof. We proceed by induction on n; the initial case is the block of $\mathcal{H}_{2e}$ whose core is the partition of zero and for which ${}_i1_j$ whenever $0 \leqslant i \leqslant j \leqslant e-1$. From Proposition 2.1 we can see that the decomposition numbers in this block are all 0 or 1, and so the present proposition certainly holds.

For the inductive step, suppose blocks A and B form a $[2:k]$-pair and that we have proved the proposition for A. For a contradiction, we require $a_\mu = 1$ for some μ in B. This means that $[S^{\{2i-1,2i\}} : D^\mu] = [S^{\{2i,2i+1\}} : D^\mu] = 2$ for this particular μ.

If the partition $\{2i-1, 2i\}$ is of the form $\Phi(\nu)$ for some ν in A (i.e. if either $k > 1$ or $\{2i-1, 2i\}$ is not one of α,β,γ), then ν is of the form $\{2h+1, 2j\}$ for some h, j, and $[S^\nu : D^\xi] = 2$ by Theorem 1.3 or 1.4, contradicting the inductive hypothesis. Similarly if $\{2i, 2i+1\}$ is of the form $\Phi(\nu)$ we get a contradiction. So we must have $k = 1$, with $\{2i-1, 2i\}$ and $\{2i, 2i+1\}$ being two of α,β,γ. In fact we must have $\{2i-1, 2i\} = \alpha$ and $\{2i, 2i+1\} = \gamma$, and the runners interchanged to get from A to B must be those on which positions p_{i-1} and p_i lie. But this implies that ${}_{i-1}0_{i+1}$, contradicting our assumption. □

We may therefore state our main result, giving the decomposition numbers for blocks of weight two in characteristic two.

Corollary 2.3. *Suppose B is a block of $\mathcal{H}_n$ of weight two, and that* $\mathrm{char}(\mathbb{F}) = 2$. *Suppose μ is an e-regular partition in B.*

1. *Suppose μ is of the form $\{2i+1, 2i+2+{}_iB_{i+1}\}$ with ${}_{i-1}0_i$. Then:*
   - *(a) if $\lambda = \{2i-1, 2i\}$ or $\{2i, 2i+1\}$ then $[S^\lambda : D^\mu] = 1$;*
   - *(b) if $\lambda = \{2i-1, 2i+1\}$ then $[S^\lambda : D^\mu] = 2$;*
   - *(c) for any other partition λ in B, $[S^\lambda : D^\mu]$ equals 1 if $\lambda = \mu$, $\lambda = \mu^{\diamond\prime}$ or λ and μ are adjacent, and 0 otherwise.*
2. *Suppose μ is not of the type described in (1). Then for any partition λ in B, $[S^\lambda : D^\mu]$ equals 1 if $\lambda = \mu$, $\lambda = \mu^{\diamond\prime}$ or λ and μ are adjacent, and 0 otherwise.*

Corollary 2.3 may alternatively be stated in terms of *adjustment matrices*. Let B be as above, and let B^0 be the weight two block of $\mathcal{H}_{\mathbb{C},q}(\mathfrak{S}_n)$ with the same e-core as B, where q is a primitive eth root of unity in $\mathbb{C}$. Let D and D^0 be the decomposition matrices of B and B^0 respectively. Then, by [10, Theorem 6.35], there is a unitriangular square matrix A with non-negative integer entries such that $D = D^0A$. A is called the adjustment matrix for B. Corollary 2.3 now implies the following.

Corollary 2.4. *Suppose B is a block of $\mathcal{H}_n$ of weight two, and that* $\mathrm{char}(\mathbb{F}) = 2$. *Let A be the adjustment matrix for B. Then for e-regular partitions λ, μ in B we have*

$$A_{\lambda\mu} = \begin{cases} 1 & (\lambda = \mu) \\ 1 & (\lambda = \{2i, 2i+1\}, \mu = \{2i+1, 2i+2+{}_iB_{i+1}\} \text{ and } {}_{i-1}0_i) \\ 0 & (\text{otherwise}). \end{cases}$$

3 Extensions between simple modules

In this section, we calculate the space $\mathrm{Ext}^1_B(D^\lambda, D^\mu)$ for two simple modules D^λ and D^μ in a weight two block B in characteristic two. From now on we must assume that $e > 2$. The corresponding results in characteristic greater than two were found by Chuang and Tan in [2] and (by different methods) [3]; although their results are stated for symmetric groups, they hold in full generality.

Theorem 3.1. [2, Theorem 2] *Suppose* $\mathrm{char}(\mathbb{F}) > 2$, *that B is a block of $\mathcal{H}_n$ of weight two, and that λ and μ are e-regular partitions in B. Then* $\dim_{\mathbb{F}} \mathrm{Ext}^1_B(D^\lambda, D^\mu)$ *equals* 1 *if either λ and μ are adjacent or μ and λ are adjacent, and* 0 *otherwise.*

The corresponding result in characteristic two is slightly different.

Theorem 3.2. *Suppose* $\mathrm{char}(\mathbb{F}) = 2$, *that B is a block of $\mathcal{H}_n$ of weight two and that λ and μ are e-regular partitions in B.*

- *If neither λ nor μ is of the form $\{2i, 2i+1\}$ with ${}_{i-1}0_i$, then* $\dim_{\mathbb{F}} \mathrm{Ext}^1_B(D^\lambda, D^\mu)$ *equals* 1 *if either λ and μ are adjacent or μ and λ are adjacent, and* 0 *otherwise.*
- *Suppose $\lambda = \{2i, 2i+1\}$ for some i with ${}_{i-1}0_i$. Then* $\dim_{\mathbb{F}} \mathrm{Ext}^1_B(D^\lambda, D^\mu)$ *equals* 1 *if $\mu = \{2i+1, 2i+2+{}_iB_{i+1}\}$, and* 0 *otherwise.*

Say that a block of weight two is *good* if ${}_{i-1}1_i$ for all $1 \leqslant i \leqslant e-1$, and *bad* otherwise. Corollary 2.3 and Theorem 3.2 say that for good blocks of weight two, the decomposition matrices and Ext-quivers are the same as for the corresponding blocks in larger characteristic. We prove Theorem 3.2 separately for good and bad blocks.

Lemma 3.3. *Suppose* $\mathrm{char}(\mathbb{F}) = 2$ *and that B is a good weight two block. Then there exists a sequence $A_0, A_1, \dots, A_r = B$ of good weight two blocks such that A_0 is the block of $\mathcal{H}_{2n}$ with e-core the partition of zero and such that A_{i-1} and A_i form a $[2:k]$-pair for all i.*

Proof. Certainly there exists such a chain of (not necessarily good) weight two blocks. But in fact it is easily checked that if A and B form a $[2:k]$-pair and B is good, then A is good. □

Proposition 3.4. *Theorem 3.2 holds for good blocks.*

Proof. We may copy the proof of [2, Theorem 2]. This proceeds by induction via $[2:k]$-pairs, with the initial case being the weight two block of $\mathcal{H}_{2n}$; the proof depends upon the quiver of this block, the fact that projective modules for (good) weight two blocks are stable with Loewy length five, and the decomposition numbers for (good) weight two blocks. The quiver of the weight two block of $\mathcal{H}_{2e}$ was found by Martin in the symmetric group case [9]; his result uses only the decomposition numbers for this block and blocks of weight one, and so is valid in the present context, even in characteristic two. The fact that the projective modules are stable with Loewy length five was proved by Scopes [13]; again, her results use only the decomposition numbers, and so are valid here, in view of Corollary 2.3. □

Now we turn to the bad weight two blocks in characteristic two.

Lemma 3.5. *Suppose B is a bad weight two block of $\mathcal{H}_n$ in characteristic two, that A and B form a $[2:k]$-pair with $k \geqslant 2$ and that Theorem 3.2 holds for A. Then Theorem 3.2 holds for B.*

Proof. Let Φ be as defined in 1.3. In view of Theorem 1.3, we need only check that Φ preserves the appropriate relationships between e-regular partitions. But in fact A and B have the same pyramid [12, Lemma 3.1], and for each partition $\{a,b\}$, we have $\Phi(\{a,b\}) = \{a,b\}$. Moreover, $\partial\{a,b\}$ depends only on a and b and the pyramid, and so we are done. □

So in order to complete a proof of Theorem 3.2 by induction, we need only consider bad blocks B for which there is no block A forming a $[2:k]$-pair with $k \geqslant 2$.

Lemma 3.6. *Suppose B is a bad weight two block in characteristic two, and that there is no block A forming a $[2:k]$-pair with B for $k \geqslant 2$. Then there is an integer i such that $p_{i-1} = p_i - e - 1$.*

Proof. Take any i such that ${}_{i-1}B_i = 0$. Suppose p_{i-1} lies on runner 0 (adjusting the number of beads on the abacus if necessary) and p_i on runner l, and that $l > 1$. Suppose moreover that there are s beads on runner 0 and t beads on runner l; we have $t > s$, since ${}_{i-1}0_i$. The number of beads on any runner between 0 and l must be either strictly less than s or strictly greater than t, by the definition of p_{i-1} and p_i. If there is some $1 < j < l$ such that there are more than t beads on runner j and fewer than s on runner $j-1$, then we may construct a $[2:k]$-pair for $k \geqslant 2$ by interchanging runners $j-1$ and j. But if this doesn't happen, then there are either more than t beads on runner 1, or fewer than s beads on runner $l-1$. In either case, we get a $[2:k]$-pair with $k \geqslant 2$ and a contradiction. So we must have $l = 1$, whence $p_{i-1} = p_i - e(t-s) - 1$; we then get a $[2:t-s]$-pair by swapping runners 0 and 1, and we must have $t - s = 1$. □

We shall use the $[2:1]$-pair constructed in the last proof to complete the proof of Theorem 3.2.

Proposition 3.7. *Suppose that A and B are weight two blocks of $\mathcal{H}_n$ in characteristic two, forming a $[2:1]$-pair. Suppose that the abacus for A is obtained from that for B by interchanging runners $j-1$ and j, and that in the abacus for B, positions p_{l-1} and p_l lie on runners $j-1$ and j respectively. If Theorem 3.2 holds for A, then it holds for B.*

Proof. We may find many of the Ext^1 spaces in B by the Eckmann–Shapiro relations, using Theorem 1.4. Suppose λ and μ are e-regular partitions in B. If neither λ nor μ is the partition α defined in 1.3, then

$$D^{\lambda} \cong D^{\bar{\lambda}}{\uparrow}_A^B, \qquad D^{\mu}{\downarrow}_A^B \cong D^{\bar{\mu}}$$

for some e-regular partitions $\bar\lambda,\bar\mu$ in A, and we have $\dim_{\mathbb{F}}\operatorname{Ext}^1_B(D^\lambda,D^\mu)=\dim_{\mathbb{F}}\operatorname{Ext}^1_A(D^{\bar\lambda},D^{\bar\mu})$. Since Theorem 3.1 is true, it must be the case that λ and μ are adjacent if and only if $\bar\lambda$ and $\bar\mu$ are.

If neither λ nor μ is of the form $\{2i,2i+1\}$ with ${}_{i-1}B_i=0$, then neither $\bar\lambda$ nor $\bar\mu$ is of the form $\{2i,2i+1\}$ with ${}_{i-1}A_i=0$. So Theorem 3.2 holds in this case.

Now suppose λ is of the form $\{2i,2i+1\}$ with ${}_{i-1}B_i=0$ and $i\neq l$. Then $\bar\lambda$ equals $\{2i,2i+1\}$ with ${}_{i-1}A_i=0$ (the pyramid for A is identical to that for B except that ${}_{l-1}A_l=1$). So, by Theorem 3.2 applied to A, $\operatorname{Ext}^1_A(D^{\bar\lambda},D^{\bar\mu})$ is one-dimensional if $\bar\mu=\{2i+1,2i+2+{}_iA_{i+1}\}$ and zero otherwise. If $i\neq l-1$, then ${}_iA_{i+1}={}_iB_{i+1}$, and $\bar\mu$ equals $\{2i+1,2i+2+{}_iA_{i+1}\}$ if and only if μ equals $\{2i+1,2i+2+{}_iB_{i+1}\}$; so the theorem holds in this case. If $i=l-1$, then we have ${}_iA_{i+1}=1$, ${}_iB_{i+1}=0$. So $\operatorname{Ext}^1_A(D^{\bar\lambda},D^{\bar\mu})$ is non-zero if and only if $\bar\mu=\{2i+1,2i+3\}=\bar\beta$; if $\bar\mu=\bar\beta$, then $\mu=\gamma=\{2i+1,2i+2\}=\{2i+1,2i+2+{}_iB_{i+1}\}$, and the theorem is proved in this case.

It remains to determine $\dim_{\mathbb{F}}\operatorname{Ext}^1_B(D^\lambda,D^\mu)$ when $\lambda=\{2l,2l+1\}=\alpha$. We do this by explicitly constructing the projective cover P of D^λ. As in [14], P has a filtration with factors S^α, S^β and S^γ from the top down. Using Corollary 2.3, we write down the composition factors of these Specht modules. We assume that $2\leqslant l\leqslant e-2$; in other cases there are fewer composition factors, but similar arguments still apply (as long as we continue to assume $e>2$). In this case, S^α, S^β and S^γ have irreducible cosocles and (by Lemma 1.1) socles, and we may tabulate their composition factors as follows.

	cosocle	heart	socle
S^α	$D^\alpha=D^{\{2l,2l+1\}}$	$D^{\{2l+1,2l+2+{}_lB_{l+1}\}}$, $D^{\{2l+1-{}_lB_{l+1},2l+3\}}$	$D^{\{2l+3,2l+4+{}_{l+1}B_{l+2}\}}$
S^β	$D^\beta=D^{\{2l-1,2l+1\}}$	$D^{\{2l+1,2l+2+{}_lB_{l+1}\}}$ (twice), $D^{\{2l-1,2l+3\}}$, $D^{\{2l,2l+1\}}$	$D^{\{2l+1-{}_lB_{l+1},2l+3\}}$
S^γ	$D^\gamma=D^{\{2l-1,2l\}}$	$D^{\{2l+1,2l+2+{}_lB_{l+1}\}}$, $D^{\{2l-1,2l+1\}}$	$D^\alpha=D^{\{2l,2l+1\}}$

We may easily calculate

$$\partial\{2l+1,2l+2+{}_lB_{l+1}\}=\partial\{2l+3,2l+4+{}_{l+1}B_{l+2}\}=0,$$

so, from the Ext^1 spaces we have already calculated, we have

$$\operatorname{Ext}^1_B\left(D^{\{2l+1,2l+2+{}_lB_{l+1}\}},D^{\{2l+3,2l+4+{}_{l+1}B_{l+2}\}}\right)=0.$$

Hence S^α must be uniserial with composition factors

$$D^{\{2l,2l+1\}},\quad D^{\{2l+1,2l+2+{}_lB_{l+1}\}},\quad D^{\{2l+1-{}_lB_{l+1},2l+3\}},\quad D^{\{2l+3,2l+4+{}_{l+1}B_{l+2}\}}$$

from top to bottom. Similarly, S^γ must be uniserial, with factors

$$D^{\{2l-1,2l\}},\quad D^{\{2l-1,2l+1\}},\quad D^{\{2l+1,2l+2+{}_lB_{l+1}\}},\quad D^{\{2l,2l+1\}}$$

from top to bottom.

Now the only simple modules which can lie in the second Loewy layer of P are those in the second Loewy layer of S^α (namely $D^{\{2l+1,2l+2+{}_lB_{l+1}\}}$) and those in the cosocles of S^β and S^γ, namely $D^{\{2l-1,2l+1\}}$ and $D^{\{2l-1,2l\}}$. On the other hand, the only simple modules which can lie in the second socle layer of P are those in the second socle layer of S^γ (namely $D^{\{2l+1,2l+2+{}_lB_{l+1}\}}$) and those in the socles of S^α and S^γ, namely $D^{\{2l+3,2l+4+{}_{l+1}B_{l+2}\}}$ and $D^{\{2l+1-{}_lB_{l+1},2l+3\}}$. We conclude that $\operatorname{Ext}^1_B(D^\lambda,D^\mu)=0$ except when $\mu=\{2l+1,2l+2+{}_lB_{l+1}\}$, and that in this case $\operatorname{Ext}^1_B(D^\lambda,D^\mu)$ is one-dimensional. □

Proof of Theorem 3.2. If B is good, then Theorem 3.2 holds by Proposition 3.4. For bad blocks, we proceed by induction on n; if there is a block A such that A and B form a $[2:k]$-pair with $k\geqslant 2$, then we may apply Lemma 3.5. If there is no such, then we apply Lemma 3.6 and Proposition 3.7. □

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