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Weights of multipartitions and representations of Ariki–Koike algebras II: canonical bases

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Abstract

We consider the canonical basis for an integrable highest-weight module of the quantum algebra $U_v(\widehat{\mathfrak{sl}}_e)$; we show how the relationship between different canonical bases is controlled by the weight of a multipartition, generalising a theorem of Lascoux, Leclerc and Thibon for the level 1 case.

1 Introduction

Let $\mathbb{F}$ be a field, and n, r positive integers. The *Ariki–Koike algebra* (also called the *cyclotomic Hecke algebra*) is a finite-dimensional $\mathbb{F}$-algebra $\mathcal{H}_n$, whose definition depends on certain parameters $q, Q_1, \dots, Q_r$ in $\mathbb{F}$. This arises as a deformation of the group algebra of the wreath product $C_r \wr \mathfrak{S}_n$, i.e. the imprimitive complex reflection group of type $G(r, 1, n)$; the usual presentation of $\mathcal{H}_n$ is obtained by modifying the relations in the standard 'Coxeter-like' presentation of this group. The Iwahori–Hecke algebras of types A_{n-1} and B_n arise as the special cases $r = 1$ and $r = 2$.

The Ariki–Koike algebra was introduced by Ariki and Koike [6], and independently by Broué and Malle [8]. The representation theory of this algebra has seen a surge in interest in recent years, and (as long as the parameters $q, Q_1, \dots, Q_r$ are non-zero and $q \neq 1$) seems to generalise the much-studied representation theory of the Iwahori–Hecke algebra H_n of type A in many natural ways. In particular, the indexing of H_n-modules by partitions generalises to an indexing of $\mathcal{H}_n$-modules by multipartitions with r components, and a lot of attention centres on finding the appropriate generalisations of aspects of the combinatorics of partitions. As with the Iwahori–Hecke algebra, the main problem of interest in the representation theory of the Ariki–Koike algebra is the *decomposition number problem*, which asks for the composition multiplicities of the simple modules in the so-called Specht modules. By far the most significant theorem in this regard is Ariki's theorem, which says that, as long as $\mathbb{F}$ has

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infinite characteristic, the decomposition numbers equal the values at $v = 1$ of certain coefficients in the canonical basis for a given integrable highest-weight module for the quantum group $U_v(\widehat{\mathfrak{sl}_e})$, where e is the multiplicative order of the parameter q.

In an attempt to generalise the notion of the e-weight of a partition, which plays an important rôle in the representation theory of H_n, the author [11] introduced the notion of the weight of a multipartition. This yields a block invariant for $\mathcal{H}_n$, and seems to provide the same measure of the complexity of a block that the e-weight does for blocks of H_n. In fact, the blocks of $\mathcal{H}_n$ have only recently been classified by Lyle and Mathas [20], and the author's work [11, 12] on the weight of a multipartition plays an important part in the proof.

The purpose of this note is to connect the notion of weight to canonical bases, by generalising a theorem of Lascoux, Leclerc and Thibon which shows how to read off the e-weight of a partition from the corresponding canonical basis element. This manifestation of the weight of a multipartition was in fact the initial motivation for the definition of weight, and the main theorem here answers a question raised by several interested colleagues upon learning about the definition of weight.

The theorem of Lascoux, Leclerc and Thibon really reflects the symmetry of the decomposition matrix of H_n which arises from tensoring with the one-dimensional 'sign' representation. We address the analogue of this symmetry for $\mathcal{H}_n$, but the details here are more awkward, and involve comparing two different Ariki–Koike algebras. Much of this paper is devoted to working through details which many readers will find straightforward, but which do not seem to be written down.

For the remainder of this introduction, we summarise the background details we shall need, both from the representation theory of the Ariki–Koike algebra and the theory of canonical bases. In the interests of brevity, we unashamedly restrict attention to the special cases which interest us. In Section 2, we prove our main theorem on canonical bases. In Section 3, we see how this theorem relates to decomposition numbers for Ariki–Koike algebras.

1.1 Background on Ariki–Koike algebras

1.1.1 The Ariki–Koike algebra

From now on we fix a field $\mathbb{F}$, a positive integer r and non-zero elements $q, Q_1, \dots, Q_r$ of $\mathbb{F}$ with $q \neq 1$. For any $n \geqslant 0$, the *Ariki–Koike algebra* $\mathcal{H}_n$ is the (unital associative) $\mathbb{F}$-algebra with generators $T_0, \dots, T_{n-1}$ and relations

$$\begin{aligned}
(T_i + q)(T_i - 1) &= 0 && (1 \leqslant i \leqslant n-1)\\
(T_0 - Q_1)\dots(T_0 - Q_r) &= 0 &&\\
T_iT_j &= T_jT_i && (0 \leqslant i, j \leqslant n-1,\ |i-j| > 1)\\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && (1 \leqslant i \leqslant n-2)\\
T_0T_1T_0T_1 &= T_1T_0T_1T_0. &&
\end{aligned}$$

We refer to $Q_1, \dots, Q_r$ as the *cyclotomic parameters* of $\mathcal{H}_n$, and to $T_0, \dots, T_{n-1}$ as the *standard generators*.

1.1.2 Partitions, multipartitions and Specht modules

A *partition* is defined to be a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \dots$ and $|\lambda| = \sum_{i=1}^{\infty} \lambda_i$ is finite. When writing partitions, we may group equal parts and omit trailing zeroes, and we write the partition $(0, 0, \dots)$ as $\odot$.

A *multipartition* (with r components) is an r-tuple $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of partitions. We write $|\lambda| = |\lambda^{(1)}| + \dots + |\lambda^{(r)}|$, and say that λ is a multipartition of $|\lambda|$. We write $\mathcal{P}^r$ for the set of all multipartitions (with r components), and $\mathcal{P}^r(n)$ for the set of all $\lambda \in \mathcal{P}^r$ with $|\lambda| = n$. We use $\varnothing$ for the multipartition $(\odot, \dots, \odot)$.

If λ is a multipartition, the *Young diagram* of λ is a subset of $\mathbb{N}^2 \times \{1, \dots, r\}$; we refer to elements of the latter set as *nodes*, and write them in the form $(i, j)_k$, with $i, j \in \mathbb{N}$ and $1 \leqslant k \leqslant r$. The Young diagram of λ is the set

$$\left\{(i, j)_k \in \mathbb{N}^2 \times \{1, \dots, r\} \;\middle|\; j \leqslant \lambda_i^{(k)}\right\},$$

whose elements are called the nodes of λ. We may abuse notation by not distinguishing between λ and its Young diagram. A node $(i, j)_k$ of λ is *removable* if $\lambda \setminus \{(i, j)_k\}$ is the Young diagram of some multipartition, while a node $(i, j)_k$ not in λ is an *addable node of* λ if $\lambda \cup \{(i, j)_k\}$ is the Young diagram of a multipartition. We partially order the set $\mathbb{N}^2 \times \{1, \dots, r\}$ by saying that $(i, j)_k$ lies *above* or *higher than* $(i', j')_{k'}$ if either $k < k'$ or ($k = k'$ and $i - j < i' - j'$). Note that this order restricts to a total order on the set of all addable and removable nodes of a given multipartition.

With $q, Q_1, \dots, Q_r \in \mathbb{F}$ as above, we define the *residue* of a node $(i, j)_k$ to be the element $\operatorname{res}((i, j)_k) = q^{j-i} Q_k$ of $\mathbb{F}$. We may use the term $(q; Q_1, \dots, Q_r)$-residue if there is a danger of confusion.

The fundamentals of the representation theory of $\mathcal{H}_n$ arise from the fact that $\mathcal{H}_n$ is a *cellular algebra* (as defined in [14]). To each $\lambda \in \mathcal{P}^r(n)$ is associated a *Specht module* S^λ for $\mathcal{H}_n$. This has a naturally-defined submodule $S^{\lambda\perp}$, and the quotient $D^\lambda = S^\lambda / S^{\lambda\perp}$ is either zero or absolutely irreducible. The non-zero D^λ are pairwise non-isomorphic and give a complete set of irreducible $\mathcal{H}_n$-modules as λ ranges over $\mathcal{P}^r(n)$.

The main problem in the representation theory of $\mathcal{H}_n$ is to find the *decomposition numbers* $[S^\lambda : D^\mu]$ for pairs of multipartitions λ, μ with D^μ non-zero. A fundamental result concerning decomposition numbers is the following, which also comes from the fact that $\mathcal{H}_n$ is cellular. To state this, we need to define the dominance order $\trianglerighteqslant$ on multipartitions: given multipartitions λ and μ, we write $\lambda \trianglerighteqslant \mu$ if

$$|\lambda^{(1)}| + \dots + |\lambda^{(k-1)}| + \lambda_1^{(k)} + \dots + \lambda_i^{(k)} \geqslant |\mu^{(1)}| + \dots + |\mu^{(k-1)}| + \mu_1^{(k)} + \dots + \mu_i^{(k)}$$

for every $1 \leqslant k \leqslant r$ and $i \geqslant 1$.

Theorem 1.1. *Suppose λ and μ are multipartitions of n with D^μ non-zero.*

1. *If $\mu = \lambda$, then $[S^\lambda : D^\mu] = 1$.*
2. *If $[S^\lambda : D^\mu] > 0$, then $\lambda \trianglerighteqslant \mu$.*

1.1.3 Conjugate multipartitions

Define a bijection $'$ from $\mathbb{N}^2 \times \{1, \dots, r\}$ to itself by

$$(i, j)_k' = (j, i)_{r+1-k}.$$

Now given a multipartition λ, define the *conjugate multipartition*

$$\lambda' = \{\mathfrak{n}' \mid \mathfrak{n} \in \lambda\};$$

that is, if $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$, then $\lambda' = (\lambda^{(r)\prime}, \dots, \lambda^{(1)\prime})$, where $\lambda^{(i)\prime}$ is the usual conjugate partition to $\lambda^{(i)}$. We collect together the basic facts we need on conjugation in the following lemma, whose proof is obvious.

Lemma 1.2.

1. *If $\mathfrak{m}$ and $\mathfrak{n}$ are two nodes with $\mathfrak{m}$ above $\mathfrak{n}$, then $\mathfrak{m}'$ lies below $\mathfrak{n}'$.*
2. *If λ and μ are multipartitions, then $\lambda \trianglerighteq \mu$ if and only if $\lambda' \trianglelefteq \mu'$.*
3. *If $\mathfrak{n}$ is a node (or an addable node) of λ with $(q; Q_1, \dots, Q_r)$-residue f, then $\mathfrak{n}'$ is a node (or an addable node, respectively) of λ' with $(q^{-1}; Q_r, \dots, Q_1)$-residue f.*

1.1.4 The weight of a multipartition

The weight of a multipartition was introduced by the author in [11]; this is a generalisation of James's 'e-weight' for the case $r = 1$, and is a block invariant for $\mathcal{H}_n$ in the sense that if two Specht modules S^λ and S^μ lie in the same block of $\mathcal{H}_n$, then λ and μ have the same weight (note that each Specht module lies in one block of $\mathcal{H}_n$, and each block contains at least one Specht module, so in order to describe the blocks of $\mathcal{H}_n$ it suffices to give the appropriate partition of the set of Specht modules). The weight of a multipartition depends on the parameters $q; Q_1, \dots, Q_r$ for $\mathcal{H}_n$. Given $\lambda \in \mathcal{P}^r$, define

$$c_f(\lambda) = |\{(i, j)_k \in \lambda \mid \operatorname{res}((i, j)_k) = f\}|$$

for any $f \in \mathbb{F}$. Now define the weight of λ to be

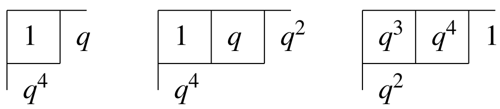
$$w(\lambda) = \sum_{k=1}^{r} c_{Q_k}(\lambda) - \frac{1}{2} \sum_{f \in \mathbb{F}} \left(c_f(\lambda) - c_{qf}(\lambda)\right)^2.$$

Several basic properties of this function are described in [11]. The main aim of this paper is to show how the weight of a multipartition appears in the setting of canonical bases. The only result we need from [11] is the following, which is a special case of [11, Lemma 3.6].

Lemma 1.3. *Suppose λ and μ are multipartitions, and that μ is obtained from λ by adding a node of residue $f \in \mathbb{F}$. Let $\delta_f(\mu)$ equal the number of removable nodes of μ of residue f minus the number of addable nodes of μ of residue f. Then*

$$w(\mu) - w(\lambda) = 1 - \delta_f(\mu).$$

Example. Suppose $r = 3$, that q is a primitive fifth root of unity in $\mathbb{F}$, and that $(Q_1, Q_2, Q_3) = (1, 1, q^3)$. Suppose $\lambda = (\varnothing, (2), (2))$ and $\mu = ((1), (2), (2))$. Then μ is obtained from λ by adding the node $(1, 1)_1$ of residue 1. The Young diagram of μ, with the residues of nodes and addable nodes marked, may be drawn as follows:



μ has one removable node of residue 1 and one addable node of residue 1 (namely, $(1,3)_3$). So $\delta_1(\mu) = 0$ and by Lemma 1.3 we should have $w(\mu) - w(\lambda) = 1$. And indeed one can easily calculate that $w(\mu) = 3$ and $w(\lambda) = 2$.

1.2 Background on canonical bases

Now we introduce the higher-level Fock spaces and canonical bases for the corresponding highest-weight representations. By a theorem of Ariki, these bases encode the decomposition numbers of Ariki–Koike algebras over fields of infinite characteristic. We assume the reader has a basic familiarity with integrable highest-weight modules for quantised enveloping algebras of symmetrisable Kac–Moody algebras; the book by Kashiwara [18] provides a sufficient introduction, tailored to our needs.

We let e denote the multiplicative order of q in $\mathbb{F}$; by our assumption that $q \neq 1$, we have $e \in \{2,3,\dots\} \cup \{\infty\}$. In some of what follows we make a tentative assumption that e is finite, though really this assumption is purely to simplify notation. All the results that follow hold when $e = \infty$ (and, as long as $r > 1$, are non-trivial), though a few of the definitions (chiefly, those concerning the algebra $U_v(\widehat{\mathfrak{sl}}_e)$) need to be modified. Therefore we will assume that $e < \infty$ without comment when it is convenient, leaving the reader to make the necessary modifications. We may abuse notation by not distinguishing between an integer and its residue modulo e.

We also make a more significant assumption for the rest of the paper, namely that each of the cyclotomic parameters $Q_1,\dots,Q_r$ is a power of q. Since these parameters may be simultaneously re-scaled without affecting the isomorphism type of $\mathcal{H}_n$, this assumption really just says that $Q_1,\dots,Q_r$ are *q-connected*, i.e. Q_i/Q_j is a power of q for each i, j. This assumption is justified by a theorem of Dipper and Mathas [10, Theorem 1.1], which says that if $Q_1,\dots,Q_r$ are not q-connected then $\mathcal{H}_n$ is Morita equivalent to a tensor product of smaller Ariki–Koike algebras. The implications of this assumption for combinatorial notions (such as Kleshchev multipartitions, or the weight of a multipartition) are easy to work out (and partly discussed in [11, §3.1]), and we leave the details to the reader. We now choose and fix integers $a_1,\dots,a_r$ such that $Q_i = q^{a_i}$ for each i. Note that with this assumption, the $(q;Q_1,\dots,Q_r)$-residue of any node $\mathfrak{n}$ is a power of q; we call a node of residue q^i an *i-node*.

1.2.1 The algebra $U_v(\widehat{\mathfrak{sl}}_e)$ and the Fock space

Let v be an indeterminate over $\mathbb{Q}$, and let $\mathcal{U}$ be the quantum algebra $U_v(\widehat{\mathfrak{sl}}_e)$. This has generators e_i, f_i, v^{h_i}, v^D for $i \in \mathbb{Z}/e\mathbb{Z}$. Defining relations are given in [19, §4.1] (where e and v are written as n and q, respectively).

The *Fock space* $\mathcal{F} = \mathcal{F}(q;Q_1,\dots,Q_r)$ is defined to be the $\mathbb{Q}(v)$-vector space with basis the set of multipartitions:

$$\mathcal{F} = \bigoplus_{\lambda\in\mathcal{P}^r} \mathbb{Q}(v)\lambda.$$

This has the structure of a $\mathcal{U}$-module; it suffices for our purposes to describe the action of the generators f_i ($i \in \mathbb{Z}/e\mathbb{Z}$) for the negative part of $\mathcal{U}$. Given two multipartitions λ and μ, we write $\lambda \xrightarrow{i} \mu$ if there is an addable i-node $\mathfrak{n}$ of λ such that $\mu = \lambda \cup \{\mathfrak{n}\}$. If this is the case, then we define the integer

$$N_i(\lambda,\mu) = (\text{number of addable } i\text{-nodes of } \mu \text{ below } \mathfrak{n}) - (\text{number of removable } i\text{-nodes of } \mu \text{ below } \mathfrak{n}).$$

Now for any multipartition λ and any $i \in \mathbb{Z}/e\mathbb{Z}$, the action of f_i is given by

$$f_i\lambda = \sum_{\lambda \xrightarrow{i} \mu} v^{N_i(\lambda,\mu)}\mu. \tag{1.2.1}$$

This is due to Ariki and Mathas [7, Proposition 2.5], generalising the statement for $r = 1$ given by Hayashi [17] (and transferred to the present combinatorial setting by Misra and Miwa [23]).

Let $\mathbf{M}$ denote the submodule of $\mathcal{F}$ generated by the empty multipartition $\varnothing$. $\mathbf{M}$ is isomorphic to the integrable highest-weight module $M(\Lambda_{a_1} + \cdots + \Lambda_{a_r})$ for $\mathcal{U}$, where $a_1, \dots, a_r$ are as above; we regard this highest-weight module as being embedded in $\mathcal{F}$ via (the inverse of) such an isomorphism.

1.2.2 The crystal basis at $v = 0$

Let $A \subset \mathbb{Q}(v)$ denote the set of rational functions which do not have a pole at $v = 0$. The *crystal basis* of $\mathcal{F}$ at $v = 0$ is the pair (L, B), where L is the lattice

$$L = \bigoplus_{\lambda \in \mathcal{P}^r} A\lambda$$

and B is the basis

$$B = \{\lambda + vL \mid \lambda \in \mathcal{P}^r\}$$

of the $\mathbb{Q}$-vector space L/vL. The important feature of the crystal basis is that the set $B \cup \{0\}$ is invariant under Kashiwara's 'crystal operators' $\tilde{e}_i, \tilde{f}_i$, for $i \in \mathbb{Z}/e\mathbb{Z}$. These operators are defined using the decomposition of $\mathcal{F}$ into irreducible $\mathcal{U}_i$-modules, where $\mathcal{U}_i$ is the subalgebra of $\mathcal{U}$ generated by e_i and f_i, isomorphic to $U_v(\mathfrak{sl}_2)$; see [18] for more details. The crystal operators endow B with the structure of a *crystal*, and their action has a simple combinatorial description. Given $\lambda \in \mathcal{P}^r$, define the *i-signature* of λ to be the sequence of signs obtained by examining the addable and removable i-nodes of λ in turn from higher to lower, writing a + for each addable i-node and a − for each removable i-node. Now construct the *reduced i-signature* by successively deleting adjacent pairs −+. If there are any − signs in the reduced i-signature, then the removable node corresponding to the leftmost of these is called the *good i*-node of λ, and if there are any + signs in the reduced i-signature, then the addable node corresponding to the rightmost of these is called the *cogood i*-node of λ. Now we have the following. This is due to Misra and Miwa [23] in the case $r = 1$; the generalisation is due to Ariki and Mathas [7, Theorem 2.9].

Theorem 1.4. *Suppose $\lambda \in \mathcal{P}^r$.*

1. *If λ does not have a good i-node, then $\tilde{e}_i(\lambda + vL) = 0$. Otherwise, $\tilde{e}_i(\lambda + vL) = \mu + vL$, where μ is the multipartition obtained by removing the good i-node from λ.*

2. *If λ does not have a cogood i-node, then $\tilde{f}_i(\lambda + vL) = 0$. Otherwise, $\tilde{f}_i(\lambda + vL) = \mu + vL$, where μ is the multipartition obtained by adding the cogood i-node to λ.*

The *crystal graph* of $\mathcal{F}$ is defined to be the directed labelled graph with vertex set $\mathcal{P}^r$, and an edge labelled with $i \in \mathbb{Z}/e\mathbb{Z}$ from λ to μ if and only if $\tilde{f}_i(\lambda+vL) = \mu+vL$ (or equivalently, if $\tilde{e}_i(\mu+vL) = \lambda+vL$). The crystal graph of the highest-weight module $\mathbf{M}$ is the connected component of this graph containing the empty multipartition $\varnothing$. A vertex of the crystal graph of $\mathbf{M}$ is called a *Kleshchev multipartition*. Of course, this definition depends on $q, Q_1, \dots, Q_r$, and we may use the term '$(q; Q_1, \dots, Q_r)$-Kleshchev' if there is a danger of ambiguity. We write $\mathcal{K}(q; Q_1, \dots, Q_r)$ for the set of $(q; Q_1, \dots, Q_r)$-Kleshchev multipartitions.

By definition, a multipartition λ lies in $\mathcal{K}(q; Q_1, \dots, Q_r)$ if and only if we can reach λ from the empty multipartition by successively adding good nodes. In the case $r = 1$, it is straightforward to show that the multipartition $(\lambda^{(1)})$ is Kleshchev if and only if $\lambda^{(1)}$ is an e-restricted partition, i.e. $\lambda^{(1)}_i - \lambda^{(1)}_{i+1} < e$ for all i. In the case $r = 2$, a non-recursive characterisation of Kleshchev multipartitions has recently been given by Ariki, Kreiman and Tsuchioka [5], but it remains an open problem to do this for $r \geqslant 3$.

An important property of Kleshchev multipartitions is the following.

Theorem 1.5. [3, Theorem 4.2] *Suppose $\lambda \in \mathcal{P}^r(n)$, and let D^λ be the corresponding $\mathcal{H}_n$-module. Then D^λ is non-zero if and only if λ is a Kleshchev multipartition.*

1.2.3 The bar involution and the canonical basis

$\mathcal{U}$ possesses an important $\mathbb{Q}(v + v^{-1})$-linear involution – the *bar involution* – given by

$$\overline{v} = v^{-1}, \quad \overline{e_i} = e_i, \quad \overline{f_i} = f_i, \quad \overline{v^{h_i}} = v^{-h_i}, \quad \overline{v^D} = v^{-D}$$

for $i \in \mathbb{Z}/e\mathbb{Z}$. This gives rise to a bar involution on $\mathbf{M}$, defined by setting $\overline{\varnothing} = \varnothing$ and $\overline{um} = \overline{u}\,\overline{m}$ for $u \in \mathcal{U}$ and $m \in \mathbf{M}$.

Now we can define the Lusztig–Kashiwara *canonical basis* of $\mathbf{M}$, which is the main object of study in this paper. For each Kleshchev multipartition μ, there is an element $G(\mu)$ of $\mathbf{M}$ uniquely determined by the properties

$$G(\mu) \equiv \mu \pmod{vL}$$

and

$$\overline{G(\mu)} = G(\mu).$$

The set $\{G(\mu) \mid \mu \in \mathcal{K}(q; Q_1, \dots, Q_r)\}$ is the canonical basis of $\mathbf{M}$. Expanding each $G(\mu)$ as

$$G(\mu) = \sum_\lambda d_{\lambda\mu}(v)\lambda$$

yields polynomials $d_{\lambda\mu}(v)$, which have become known as 'v-decomposition numbers', in view of the following deep theorem.

Theorem 1.6. [1, Theorem 4.4] *Suppose $\mathbb{F}$ has infinite characteristic. Suppose $\lambda, \mu \in \mathcal{P}^r(n)$ and that μ is $(q; Q_1, \dots, Q_r)$-Kleshchev. Then*

$$[S^\lambda : D^\mu] = d_{\lambda\mu}(1).$$

2 The main result

Our aim in this paper is to generalise [19, Theorem 7.2], which describes the symmetry of the canonical basis of a level 1 Fock space arising from conjugation of partitions; this corresponds to tensoring with the one-dimensional 'sign' representation of the Iwahori–Hecke algebra of type A. The e-weight of a partition is fundamental to this theorem, and in fact the theorem makes it very easy to read off the e-weight of an e-regular partition from the corresponding canonical basis element. We shall generalise this result using the weight of a multipartition.

This section closely follows [19, §7]. We take from there the definition of the involutory automorphism # of $\mathcal{U}$ given by

$$e_i^\# = e_{-i}, \qquad f_i^\# = f_{-i}, \qquad (v^{h_i})^\# = v^{h_{-i}}, \qquad (v^D)^\# = v^D.$$

This yields a linear bijection

$$\star : M(\Lambda_{a_1} + \cdots + \Lambda_{a_r}) \longrightarrow M(\Lambda_{-a_r} + \cdots + \Lambda_{-a_1})$$

which is uniquely specified by

$$(x_{\Lambda_{a_1}+\cdots+\Lambda_{a_r}})^\star = x_{\Lambda_{-a_r}+\cdots+\Lambda_{-a_1}}$$

and

$$(um)^\star = u^\# m^\star$$

for $u \in \mathcal{U}$ and $m \in \mathbf{M}$, where $x_{\Lambda_{a_1}+\cdots+\Lambda_{a_r}}$ and $x_{\Lambda_{-a_r}+\cdots+\Lambda_{-a_1}}$ denote highest-weight vectors. Our aim is to describe this map explicitly (i.e. non-recursively) in terms of the embeddings of these highest-weight modules in appropriate Fock spaces. The module $M(\Lambda_{a_1} + \cdots + \Lambda_{a_r})$ we identify with the submodule $\mathbf{M}$ of the Fock space $\mathcal{F}$ as above, while for the module $M(\Lambda_{-a_r} + \cdots + \Lambda_{-a_1})$, we use the submodule $\hat{\mathbf{M}}$ generated by the empty multipartition in the Fock space $\hat{\mathcal{F}} = \mathcal{F}(q^{-1}; Q_r, \dots, Q_1)$. The action of the generators f_i on $\hat{\mathcal{F}}$ is given by equation (1.2.1), but note that we must read 'i-node' to mean 'node whose $(q^{-1}; Q_r, \dots, Q_1)$-residue is $(q^{-1})^i$'; the same warning applies to the description of the crystal graph of $\hat{\mathbf{M}}$. Keeping in mind the embeddings of the highest-weight modules in these Fock spaces, we may replace both $x_{\Lambda_{a_1}+\cdots+\Lambda_{a_r}}$ and $x_{\Lambda_{-a_r}+\cdots+\Lambda_{-a_1}}$ with $\varnothing$ above.

It is easy to check that the map $\star$ respects the crystal operators, in the sense that

$$(\tilde{e}_i m)^\star = \tilde{e}_{-i}(m^\star), \qquad (\tilde{f}_i m)^\star = \tilde{f}_{-i}(m^\star)$$

for every i and every $m \in \mathbf{M}$. Since $\star$ is a #-twisted isomorphism, there is therefore a bijection between the crystal graphs of $\mathbf{M}$ and $\hat{\mathbf{M}}$, under which the labels of arrows are negated. That is, there is a bijection $^\diamond$ from $\mathcal{K}(q; Q_1, \dots, Q_r)$ to $\mathcal{K}(q^{-1}; Q_r, \dots, Q_1)$, with the properties that

- $\varnothing^\diamond = \varnothing$, and
- if λ is a multipartition with a good i-node $\mathfrak{n}$, then $\lambda^\diamond$ has a good $-i$-node $\mathfrak{m}$, and $(\lambda \setminus \{\mathfrak{n}\})^\diamond = \lambda^\diamond \setminus \{\mathfrak{m}\}$.

Example. As in the last example, suppose q is a primitive fifth root of unity and $(Q_1, Q_2, Q_3) = (1, 1, q^3)$. $\mathbf{M}$ is therefore isomorphic to the integrable highest-weight module $M(2\Lambda_0 + \Lambda_3)$ for $U_v(\widehat{\mathfrak{sl}}_5)$.

If we take $\mu = ((1), (2), (2))$ as in the last example, then the interval between $\varnothing$ and μ in the crystal graph for $\mathbf{M}$ is shown in Figure 1. The corresponding subgraph for $\hat{\mathbf{M}}$ is shown in Figure 2, and we have $\mu^\diamond = ((1), (1), (2, 1))$.

The bijection $^\diamond$ may be viewed as a generalisation of the Mullineux involution [24] for the case $r = 1$, though it remains a problem to generalise Mullineux's algorithm to calculate $\lambda^\diamond$ non-recursively.

Now we can state our main result. We use $d_{\lambda\mu}(v)$ to denote a v-decomposition number in $\mathcal{F}$, and $\hat{d}_{\lambda\mu}(v)$ for the corresponding v-decomposition number in $\hat{\mathcal{F}}$. $w(\mu)$ will always indicate the $(q; Q_1, \dots, Q_r)$-weight of a multipartition μ.

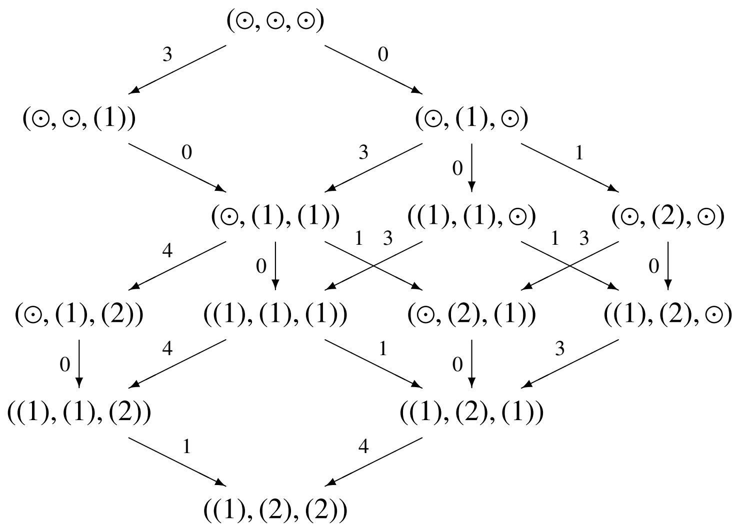


Figure 1

Theorem 2.1. *Suppose λ and μ are multipartitions of n, with μ $(q; Q_1, \dots, Q_r)$-Kleshchev. Then*

$$\hat{d}_{\lambda'\mu^\diamond}(v) = v^{w(\mu)} d_{\lambda\mu}(v^{-1}).$$

We continue to follow [19]. Writing $G(\mu)$ for a canonical basis element in $\mathcal{F}$ and $\hat{G}(\mu)$ for a canonical basis element in $\hat{\mathcal{F}}$, we have the following.

Lemma 2.2. *If $\mu \in \mathcal{K}(q; Q_1, \dots, Q_r)$, then*

$$G(\mu)^\star = \hat{G}(\mu^\diamond).$$

Proof. This is proved exactly as [19, Lemma 7.3]. □

Lemma 2.3. *Suppose m is an element of $\mathbf{M}$ with $\overline{m} = m$, and write*

$$m = \sum_\lambda p_\lambda(v)\lambda,$$

with each $p_\lambda \in \mathbb{Q}(v)$. Then

$$m^\star = \sum_\lambda v^{w(\lambda)} p_\lambda(v^{-1})\lambda'.$$

Proof. Since $\mathbf{M}$ is a highest-weight module, any element of $\mathbf{M}$ is a $\mathbb{Q}(v)$-linear combination of elements of the form

$$f_{i_1} \dots f_{i_l} \varnothing.$$

Hence from the definition of the bar involution any bar-invariant element is a $\mathbb{Q}(v + v^{-1})$-linear combination of such elements. Now $\varnothing^\star = \varnothing$, and the conclusion of the lemma is obviously preserved under

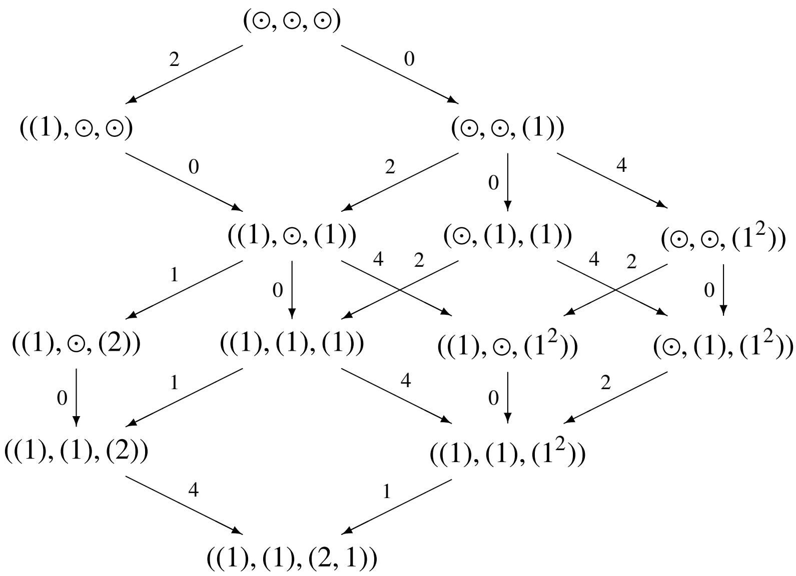


Figure 2

$\mathbb{Q}(v+v^{-1})$-linear extension, so it suffices to consider the action of each f_i and show that if the lemma holds for an element m, then it also holds for $f_i(m)$. That is, we assume that

$$\begin{aligned} m &= \sum_\lambda p_\lambda(v)\lambda, \\ m^\star &= \sum_\lambda v^{w(\lambda)} p_\lambda(v^{-1})\lambda', \\ f_i(m) &= \sum_\mu s_\mu(v)\mu \end{aligned}$$

for some polynomials $p_\lambda(v), s_\mu(v)$, and we must show that

$$f_{-i}(m^\star) = \sum_\mu v^{w(\mu)} s_\mu(v^{-1})\mu'.$$

In order to distinguish notation for the two Fock spaces $\mathcal{F}$ and $\hat{\mathcal{F}}$, we write $\lambda \xrightarrow{i} \mu$ and $N_i(\lambda,\mu)$ as above for $\mathcal{F}$, and we use the notation $\lambda \overset{i}{\dashrightarrow} \mu$ and $\hat{N}_i(\lambda,\mu)$ for $\hat{\mathcal{F}}$. $\lambda \overset{i}{\dashrightarrow} \mu$ therefore means that μ is obtained from λ by adding an addable node whose $(q^{-1};Q_r,\dots,Q_1)$-residue is q^{-i}. Now, using the description (1.2.1) of the action of f_i, it suffices to show that, given $\lambda,\mu \in \mathcal{P}^r$, we have

$$\lambda \xrightarrow{i} \mu \qquad \text{if and only if} \qquad \lambda' \overset{-i}{\dashrightarrow} \mu',$$

and that if these conditions hold then

$$v^{w(\mu)}v^{-N_i(\lambda,\mu)} = v^{w(\lambda)}v^{\hat{N}_{-i}(\lambda',\mu')}.$$

The first statement is simply Lemma 1.2(3). For the second statement, we need to show that if $\lambda \xrightarrow{i} \mu$ then

$$N_i(\lambda,\mu) + \hat{N}_{-i}(\lambda',\mu') = w(\mu) - w(\lambda).$$

If we write $\mathfrak{n}$ for the unique node of $\mu \setminus \lambda$, then by parts (1) and (3) of Lemma 1.2 we see that

$$\begin{aligned}\hat{N}_{-i}(\lambda', \mu') =&(\text{number of addable nodes of } \mu \text{ of } (q; Q_1, \ldots, Q_r)\text{-residue } q^i \text{ above } \mathfrak{n})\\ &- (\text{number of removable nodes of } \mu \text{ of } (q; Q_1, \ldots, Q_r)\text{-residue } q^i \text{ above } \mathfrak{n}).\end{aligned}$$

Hence

$$N_i(\lambda, \mu) + \hat{N}_{-i}(\lambda', \mu') = 1 - \delta_{q^i}(\mu),$$

and by Lemma 1.3 this equals $w(\mu) - w(\lambda)$. □

Combining Lemma 2.2 with Lemma 2.3 yields Theorem 2.1. We now observe that as a consequence of Theorem 2.1 we can read off the weight of a Kleshchev multipartition from the corresponding canonical basis vector.

Corollary 2.4. *If μ is a Kleshchev multipartition, then there is a multipartition λ such that the v-decomposition number $d_{\lambda\mu}(v)$ equals $v^{w(\mu)}$, while any other v-decomposition number $d_{\nu\mu}(v)$ has degree at most $w(\mu) - 1$.*

Proof. Put $\lambda = \mu^{\diamond\prime}$. Then we have $\hat{d}_{\lambda'\mu^\diamond}(v) = \hat{d}_{\mu^\diamond\mu^\diamond}(v) = 1$, while $\hat{d}_{\nu\mu^\diamond}(v)$ is divisible by v for any $\nu \neq \lambda'$. Now the result follows from Theorem 2.1. □

Example. We retain the set-up of the previous example, with $\mu = ((1), (2), (2))$. It is straightforward to calculate

$$\begin{aligned}G(\mu) =&\left(\tfrac{1}{v+v^{-1}}\right) f_1 f_4 f_0^2 f_3 \varnothing\\ =&((1), (2), (2)) + v((1), (2, 1), (1)) + v^2((1^2), (2), (1))\\ &+ v((2), (1), (2)) + v^2((2), (1^2), (1)) + v^3((2, 1), (1), (1)).\end{aligned}$$

We can immediately verify that $w(\mu) = 3$, and we have

$$\begin{aligned}\hat{G}(\mu^\diamond) =&((1), (1), (2, 1)) + v((1), (2), (1^2)) + v^2((1^2), (1), (1^2))\\ &+ v((1), (1^2), (2)) + v^2((1), (2, 1), (1)) + v^3((1^2), (1^2), (1)).\end{aligned}$$

3 Decomposition numbers for Ariki–Koike algebras

In this section we examine the connection between the results of the previous section and decomposition numbers. We wish to compare two different Ariki–Koike algebras, and we begin by introducing notation to prevent ambiguity. We let $\mathcal{H}_n'$ denote the Ariki–Koike algebra with parameters $q^{-1}, Q_r, \ldots, Q_1$, and write the standard generators of $\mathcal{H}_n'$ as $T_0', \ldots, T_{n-1}'$. If $\lambda \in \mathcal{P}^r(n)$, we write the corresponding Specht module for $\mathcal{H}_n'$ as $\hat{S}^\lambda$, and if $\lambda \in \mathcal{K}(q^{-1}; Q_r, \ldots, Q_1)$, we write the corresponding simple $\mathcal{H}_n'$-module as $\hat{D}^\lambda$.

It is easily checked that there is an isomorphism $\theta : \mathcal{H}_n \to \mathcal{H}_n'$ given by

$$T_0 \longmapsto T_0', \qquad T_i \longmapsto -qT_i' \quad (i = 1, \ldots, n-1).$$

This allows us to define a functor F_θ from the category of $\mathcal{H}'_n$-modules to the category of $\mathcal{H}_n$-modules in the usual way: if M is an $\mathcal{H}'_n$-module, then $F_\theta(M)$ has the same underlying vector space, with $\mathcal{H}_n$-action $h \cdot m = \theta(h)m$.

We write $S'(\lambda)$ for the image under F_θ of the Specht module $\hat{S}^\lambda$; this is referred to in [22, §4] as a *dual Specht module* for $\mathcal{H}_n$. The reason for this terminology relates to contragredient duality. There is an anti-automorphism ϕ of $\mathcal{H}_n$ given by

$$\phi : T_i \longmapsto T_i \qquad (i = 0, \dots, n-1),$$

and for any $\mathcal{H}_n$-module N this allows us to make the dual vector space N^* into an $\mathcal{H}_n$-module, which we denote $N^{\circledast}$; then we have an exact contravariant functor $N \mapsto N^{\circledast}$ from the category of $\mathcal{H}_n$-modules to itself. The effect of this functor on simple modules and Specht modules is as follows.

Proposition 3.1. *Suppose λ is a multipartition of n with r components.*

1. $(S^\lambda)^{\circledast} \cong S'(\lambda')$.
2. *If λ is Kleshchev, then $(D^\lambda)^{\circledast} \cong D^\lambda$.*

Proof.

1. This is Corollary 5.7 of [22]. Although that reference does not refer to the isomorphism θ, the definition of $S'(\lambda)$ is equivalent to ours. The module $S'(\lambda)$ is defined in [22] to be the cell module arising from a cellular basis $\{n_{\mathfrak{st}}\}$ of $\mathcal{H}_n$, while the Specht module $\hat{S}^\lambda$ arises from a cellular basis $\{m_{\mathfrak{st}}\}$ of $\mathcal{H}'_n$; it is easily checked that $\theta(n_{\mathfrak{st}})$ equals a scalar multiple of $m_{\mathfrak{st}}$, and since the definition of a cell module depends only on the space spanned by each vector in a cellular basis, the definitions yield the same module.

2. This follows from the cellularity of $\mathcal{H}_n$: the anti-automorphism ϕ coincides with the anti-automorphism given by $n_{\mathfrak{st}} \mapsto n_{\mathfrak{ts}}$, and so by [21, Exercise 2.7(iii)] each simple module is self-dual. □

Now we can give a module-theoretic counterpart to Theorem 2.1.

Corollary 3.2. *There is a bijection $^\square$ from $\mathcal{K}(q; Q_1, \dots, Q_r)$ to $\mathcal{K}(q^{-1}; Q_r, \dots, Q_1)$ such that for multipartitions λ and μ with μ $(q; Q_1, \dots, Q_r)$-Kleshchev, we have*

$$[S^\lambda : D^\mu] = [\hat{S}^{\lambda'} : \hat{D}^{\mu^\square}].$$

In particular,

$$[S^\lambda : D^\mu] = \begin{cases} 1 & (\lambda = (\mu^\square)') \\ 0 & (\lambda \ntrianglelefteq (\mu^\square)'). \end{cases}$$

Proof. Since θ is an isomorphism, the functor F_θ takes simple $\mathcal{H}'_n$-modules to simple $\mathcal{H}_n$-modules. Hence there is a bijection $^\square$ from $\mathcal{K}(q; Q_1, \dots, Q_r)$ to $\mathcal{K}(q^{-1}; Q_r, \dots, Q_1)$ such that $D^\mu \cong F_\theta(\hat{D}^{\mu^\square})$ for all $\mu \in \mathcal{K}(q; Q_1, \dots, Q_r)$. Now the first part follows from Proposition 3.1 and the fact that the functor $^{\circledast}$ is exact. The second part then follows by applying Theorem 1.1 to $\mathcal{H}'_n$, and using Lemma 1.2(2). □

In the case where $\mathbb{F}$ has infinite characteristic, we see from Theorems 1.6 and 2.1 that the two bijections $^\diamond$ and $^\square$ coincide. We conclude this paper by showing that this is true in general.

Proposition 3.3. *Suppose $\mu \in \mathcal{K}(q; Q_1, \dots, Q_r)$. Then $\mu^\diamond = \mu^\square$.*

One could prove this by combining the result in infinite characteristic with the theory of adjustment matrices [13], but we instead prove Proposition 3.3 by induction using the recently-proved modular branching rule. To this end we briefly address induction and restriction of simple modules. $\mathcal{H}_n$ is naturally embedded in $\mathcal{H}_{n+1}$, and accordingly there are induction and restriction functors $\uparrow^{\mathcal{H}_{n+1}}$ and $\downarrow_{\mathcal{H}_n}$ between the module categories of these two algebras. These functors may be refined by considering generalised eigenspaces for the *Jucys–Murphy elements*. These are the elements $L_0, \dots, L_{n-1}$ of $\mathcal{H}_n$ defined by

$$L_j = q^{-j} T_j T_{j-1} \dots T_1 T_0 T_1 \dots T_j$$

for $j = 0, \dots, n-1$. The symmetric polynomials in $L_0, \dots, L_{n-1}$ are central elements of $\mathcal{H}_n$, and so for any $\mathcal{H}_n$-module M a simultaneous generalised eigenspace for these symmetric polynomials will be a submodule of M. Following Ariki [2], we suppose $c = \{c_0, \dots, c_{n-1}\}$ is a multiset consisting of powers of q, and denote by $P_c(M)$ the subspace

$$\left\{ m \in M \;\middle|\; \begin{array}{c} (f(L_0, \dots, L_{n-1}) - f(c_0, \dots, c_{n-1}))^N m = 0 \\ \text{for all symmetric polynomials } f\text{, for } N \text{ sufficiently large} \end{array} \right\}.$$

Now Ariki's 'i-induction' functor is defined by

$$f_i(M) = \sum_c P_{c \cup \{q^i\}} \left(P_c(M) \uparrow^{\mathcal{H}_{n+1}} \right),$$

where $c \cup \{q^i\}$ denotes c with a copy of q^i added. There is an 'i-restriction' functor e_i defined in a similar way, but we do not need this.

The i-induction functor provides the following important link between the representation theory of $\mathcal{H}_n$ and the crystal graph of the corresponding highest-weight module. To state this, we use use the crystal operator $\tilde{f}_i$ (see Theorem 1.4 and the preceding discussion).

Theorem 3.4. *Suppose $\lambda \in \mathcal{K}(q; Q_1, \dots, Q_r)$ and $|\lambda| = n$. If $\tilde{f}_i(\lambda) = 0$, then $f_i(D^\lambda) = 0$. Otherwise, let μ be the multipartition such that $\tilde{f}_i(\lambda + vL) = \mu + vL$. Then $f_i(D^\lambda)$ is an indecomposable module with socle isomorphic to D^μ.*

This theorem is known as the modular branching rule; the special case at $r = 1$, due to Kleshchev and Brundan, was a major breakthrough in representation theory, and a survey of related results appears in [9]. The version for arbitrary r was proved much more recently, by Ariki; he used previous work of Grojnowski and Vazirani [16, 15], who had proved a modular branching rule for simple modules, controlled by an appropriate crystal, but with no connection to the labelling by multipartitions. Ariki [4, Theorem 6.1] completed the theorem by using a Fock space argument to show that the labellings match up appropriately.

The proof of Proposition 3.3 rests on a comparison of i-induction functors for $\mathcal{H}_n$ and $\mathcal{H}'_n$. If we let $L'_0, \dots, L'_{n-1}$ denote the Jucys–Murphy elements for $\mathcal{H}'_n$ and f'_i the i-induction functor from $\mathcal{H}'_n$-modules to $\mathcal{H}'_{n+1}$-modules, then it is easily checked that $\theta(L_j) = L'_j$ for each j, and hence that

$$f_i(F_\theta(M)) = F_\theta(f'_{-i}(M))$$

for any $\mathcal{H}'_n$-module M and any $i \in \mathbb{Z}/e\mathbb{Z}$ (note that $-i$ appears rather than i because we use q^{-1} in place of q in the definition of f'_i).

Proof. Proof of Proposition 3.3 We proceed by induction on $n = |\mu|$, with the case $n = 0$ being trivial. If $n > 0$, then μ has a good i-node for some i. Let ν be the multipartition obtained by removing this node. Then $\nu \in \mathcal{K}(q; Q_1, \dots, Q_r)$, and by induction $\nu^\diamond = \nu^\square$. By the definition of $^\diamond$, the multipartition $\nu^\diamond$ has a cogood $-i$-node, and the multipartition $\mu^\diamond$ is obtained by adding this node to $\nu^\diamond$.

Now using the modular branching rule we have

$$\begin{aligned} D^{\mu^\square} &\cong F_\theta(D^\mu) \\ &\cong F_\theta(\mathrm{soc}(f_i(D^\nu))) \\ &\cong \mathrm{soc}(F_\theta(f_i(D^\nu))) \\ &\cong \mathrm{soc}(f'_{-i}(F_\theta(D^\nu))) \\ &\cong \mathrm{soc}(f'_{-i}(D^{\nu^\square})) \\ &\cong \mathrm{soc}(f'_{-i}(D^{\nu^\diamond})) \\ &\cong D^{\mu^\diamond}, \end{aligned}$$

as required. Induction follows. □

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