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Weights of multipartitions and representations of Ariki–Koike algebras

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Abstract

We consider representations of the Ariki–Koike algebra, a q-deformation of the group algebra of the complex reflection group $C_r \wr \mathfrak{S}_n$. The representations of this algebra are naturally indexed by *multipartitions* of n, and for each multipartition λ we define a non-negative integer called the *weight* of λ. We prove some basic properties of this weight function, and examine blocks of small weight.

Contents

1 Introduction

Let $\mathfrak{S}_n$ denote the symmetric group on n letters. This has the famous Coxeter presentation with generators $T_1, \dots, T_{n-1}$ and relations

$$\begin{aligned} T_i^2 &= 1 && (1 \leqslant i \leqslant n-1) \\ T_iT_j &= T_jT_i && (1 \leqslant i, j \leqslant n-1,\ |i-j| > 1) \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && (1 \leqslant i \leqslant n-2). \end{aligned}$$

If we view this as a presentation for a (unital associative) algebra over a field $\mathbb{F}$, then of course the algebra we get is the group algebra $\mathbb{F}\mathfrak{S}_n$. Now we can introduce a 'deformation', by replacing the relation $T_i^2 = 1$ with

$$(T_i + q)(T_i - 1) = 0$$

for each i. The resulting algebra $H_n = H_{n,q}(\mathfrak{S}_n)$ is the *Iwahori–Hecke algebra* (of $\mathfrak{S}_n$). This algebra arises naturally in the study of groups with BN-pairs, and its representation theory has been extensively studied. An excellent introduction to this theory is provided by Mathas's book [17]. As long as q is non-zero, the representation theory of H_n bears a remarkable resemblance to the representation theory of $\mathfrak{S}_n$.

Now let G be the complex reflection group $C_r \wr \mathfrak{S}_n$. This has a 'Coxeter-like' presentation with generators $T_0, \dots, T_{n-1}$ and relations

$$\begin{aligned} T_i^2 &= 1 && (1 \leqslant i \leqslant n-1) \\ T_0^r &= 1 && \\ T_iT_j &= T_jT_i && (0 \leqslant i, j \leqslant n-1,\ |i-j| > 1) \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && (1 \leqslant i \leqslant n-2) \\ T_0T_1T_0T_1 &= T_1T_0T_1T_0. && \end{aligned}$$

Let $\mathbb{F}$ be a field, suppose $q, Q_1, \dots, Q_r$ are elements of $\mathbb{F}$, with q non-zero. The *Ariki–Koike algebra* $\mathcal{H}_n$ is defined to be the unital associative $\mathbb{F}$-algebra with presentation

$$\begin{aligned} (T_i + q)(T_i - 1) &= 0 && (1 \leqslant i \leqslant n-1) \\ (T_0 - Q_1)\dots(T_0 - Q_r) &= 0 && \\ T_iT_j &= T_jT_i && (0 \leqslant i, j \leqslant n-1,\ |i-j| > 1) \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && (1 \leqslant i \leqslant n-2) \\ T_0T_1T_0T_1 &= T_1T_0T_1T_0. && \end{aligned}$$

This algebra was introduced by Ariki and Koike [3], and independently by Broué and Malle [5]. It is less well-studied than the Iwahori–Hecke algebra, but many facts are known about it. Ariki gave a necessary and sufficient criterion in terms of the parameters $q, Q_1, \dots, Q_r$ for $\mathcal{H}_n$ to be semi-simple, and described the simple modules in this case. These are indexed by *multipartitions* of n with r components, and in general the combinatorics underpinning the representation theory of $\mathcal{H}_n$ seems to be analogous to that of the Iwahori–Hecke algebra, but extended from partitions to multipartitions. It has been shown that $\mathcal{H}_n$ is a cellular algebra (in the sense of Graham and Lehrer [12]), and this provides a great deal of information about the representation theory of $\mathcal{H}_n$. In particular, it gives us a classification of the simple modules of $\mathcal{H}_n$, in terms of 'Kleshchev multipartitions', although at present we only have a recursive definition of these. An alternative non-recursive parameterisation of the simple modules of $\mathcal{H}_n$ has been given by Foda et al. [11].

One of the deepest results concerning the Ariki–Koike algebras is Ariki's theorem [1] which describes the decomposition matrix of $\mathcal{H}_n$ in the case where $\mathbb{F}$ has infinite characteristic. This is done via a 'Fock space' representation of the quantum algebra $U_v(\widehat{\mathfrak{sl}_e})$.

The purpose of this paper is to provide further generalisation of the combinatorics of H_n to that of $\mathcal{H}_n$ by introducing a notion of 'weight' for multipartitions. If we let e be the least positive integer such that $1 + q + \cdots + q^{e-1} = 0$ in $\mathbb{F}$, then the 'e-weight' (or simply the weight) of a partition is a non-negative integer defined in a simple combinatorial way. It was introduced by James in the context of representation theory of symmetric groups (though it generalises immediately to the Iwahori–Hecke algebras). The weight of a partition turns out to be a block invariant (throughout this paper, we shall speak of a partition (or multipartition) λ as lying in a block of H_n (or $\mathcal{H}_n$, respectively) by which we shall mean that the corresponding Specht module S^λ lies in that block), and so we may define the weight of a block of H_n, and this turns out to be an excellent measure of how 'complicated' the representation theory of that block is. Indeed, for the representation theory of the symmetric groups over a field of characteristic p (where $e = p$), the weight of a block equals the defect of that block provided the defect group is abelian, and in general the defect group of a block of weight w is isomorphic to a Sylow subgroup of $\mathfrak{S}_{wp}$.

Much more is true: for fixed choices of $\mathbb{F}$, e and w (but allowing n to vary), all blocks of weight w contain the same numbers of Specht modules and simple modules, and the 'Scopes isometries' provide Morita equivalences between many of these blocks, so that there are in fact only finitely many blocks of weight w up to Morita equivalence.

Much of the study of the modular representation theory of the symmetric groups has adopted a 'bottom up' approach, restricting attention to blocks of small weight, and these are now understood to greater or lesser extents. Blocks of weight 0 are simple (and indeed, every simple block has weight zero). Blocks of weight 1 have been very well understood for some time; the key facts are summarised in Theorem 4.2 below. Blocks of weight 2 were first studied by (Erdmann and) Martin [16, 8] and then by Richards [18], who gave a combinatorial description of the decomposition numbers of these blocks (in the cases where the characteristic of $\mathbb{F}$ is not 2), showing that these decomposition numbers are bounded above by 1. Blocks of weight 3 have been studied by several authors, and the present author finally showed [9] that, as long as the characteristic of $\mathbb{F}$ is not 2 or 3, the decomposition numbers for weight three blocks are all at most 1.

Given the richness of this weight function for partitions, it seems odd that, as far as the author can tell, no definition of weight has been given for multipartitions until now. We do this in this paper, and examine some of the properties of our weight function. We begin by showing in Section 2 that our definition really is a generalisation of the weight of a partition. Then in Section 3 we introduce the abacus for a multipartition. This is a generalisation of James's abacus for partitions, which is a very useful combinatorial device for studying the partitions in a given block. We show how to calculate the weight of a multipartition from its abacus display, and as a consequence we find that the weight is non-negative (which is not at all obvious from our definition). Finally, in Section 4, we begin to study blocks of small weight, by looking at blocks of weight at most 1. In a subsequent paper [10] we examine blocks of weight 2 in the case $r = 2$.

Remarks.

1. The version of this paper published in *Advances in Mathematics* contains a reference to a (now discredited) preprint by Grojnowski containing a purported proof of the classification of the blocks of the Ariki–Koike algebra. This classification has now been proved by Lyle and Mathas [15, Theorem 2.11], but their proof uses the results of the present paper. Accordingly, we cannot assume this classification in this paper. This is not critical, but it necessitates changing some of the statements of results in this paper, so that they refer to 'combinatorial blocks' rather than (algebraic) blocks. The part of the block classification proved by Graham and Lehrer tells us that a combinatorial block is a union of blocks, but we cannot assume here that a combinatorial block is a single block. Of course, when re-reading this paper in the light of the work of Lyle and Mathas, the word 'combinatorial' may be ignored.

2. The results in this paper apply equally well to the 'cyclotomic q-Schur algebra' $\mathcal{S}_n$ of Dipper,

James and Mathas [6]. This plays the same rôle for $\mathcal{H}_n$ as the q-Schur algebra does for H_n, and much of the combinatorics inherent in the representation theory is the same. In particular, representations of $\mathcal{S}_n$ are indexed by multipartitions of n, and we may define a combinatorial block of $\mathcal{S}_n$ in the same way as for $\mathcal{H}_n$. So all the combinatorial results in this paper (Sections 2 and 3) apply to $\mathcal{S}_n$. The representation-theoretic results of Section 4 are slightly different (because there are simple $\mathcal{S}_n$-modules for all multipartitions of n, not just Kleshchev multipartitions), and we shall indicate the differences when we state the results.

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1.1 Background and notation

From now on, we fix a field $\mathbb{F}$ and elements $q, Q_1, \dots Q_r$ of $\mathbb{F}$, with q non-zero, and we let $\mathcal{H}_n$ denote the Ariki–Koike algebra defined above. We refer to $Q_1, \dots, Q_r$ as the 'cyclotomic parameters' of $\mathcal{H}_n$. We assume in this paper that q is not equal to 1, and that none of the Q_i equals 0. Much of the representation theory of $\mathcal{H}_n$ is unchanged without these restrictions, but certain technical differences make them essential for this paper. We let e denote the multiplicative order of q in $\mathbb{F}$; our assumptions on q mean that $e \in \{2, 3, \dots, \infty\}$.

Given a non-negative integer m, a *partition* of m is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of (weakly) decreasing non-negative integers summing to m. We often write partitions grouping together equal parts and omitting zeroes, so that $(5, 4, 3, 3, 3, 0, 0, 0, 0, \dots)$ is written as $(5, 4, 3^3)$. The unique partition of 0 is usually written $\varnothing$. If λ is a partition of m, we write $|\lambda| = m$.

A *multipartition of n with r components* is an r-tuple $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of partitions such that $|\lambda^{(1)}| + \dots + |\lambda^{(r)}| = n$. If r is understood, we shall just call this a multipartition of n. As with partitions, we write the unique multipartition of 0 as $\varnothing$, and if $\boldsymbol{\lambda}$ is a multipartition of n then we write $|\boldsymbol{\lambda}| = n$.

1.1.1 Residues and Kleshchev multipartitions

Given a multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$, we define its Young diagram $[\boldsymbol{\lambda}]$ to be the subset

$$[\boldsymbol{\lambda}] = \{(i, j, k) \in \mathbb{N}^3 \mid j \leqslant \lambda_i^{(k)}\}$$

of $\mathbb{N} \times \mathbb{N} \times \{1, \dots, r\}$, whose elements we call *nodes*. We may abuse notation by not distinguishing a multipartition from its Young diagram. We say that a node (i, j, k) of $[\boldsymbol{\lambda}]$ is *removable* if $[\boldsymbol{\lambda}] \setminus \{(i, j, k)\}$ is the Young diagram of a multipartition. We say that a triple (i, j, k) *not* in $[\boldsymbol{\lambda}]$ is an *addable node of* $[\boldsymbol{\lambda}]$ if $[\boldsymbol{\lambda}] \cup \{(i, j, k)\}$ is the Young diagram of a multipartition. We emphasise the potentially confusing point that an addable node of $[\boldsymbol{\lambda}]$ is not a node of $[\boldsymbol{\lambda}]$. We define the *residue* of the node (i, j, k) to be the element $q^{j-i}Q_k$ of $\mathbb{F}$.

Now we describe a certain subset $\mathcal{K}$ of the set of all multipartitions, which will index the simple modules for $\mathcal{H}_n$. Given a multipartition $\boldsymbol{\lambda}$ and an element f of $\mathbb{F}$, we totally order the set of all addable and removable nodes of $[\boldsymbol{\lambda}]$ of residue f by stipulating that $(i, j, k) < (i', j', k')$ if and only if $k < k'$ or ($k = k'$ and $i < i'$). We define the *f-signature* of $\boldsymbol{\lambda}$ to be the sequence of + and − signs obtained by examining the addable and removable nodes of residue f in increasing order and writing a + for each addable node and a − for each removable node. From this, we form the *reduced f-signature* by successively deleting all adjacent pairs $-+$. If there are any − signs in the reduced f-signature, then the removable node of $[\boldsymbol{\lambda}]$ corresponding to the first of these is referred to as a *good* node of $[\boldsymbol{\lambda}]$.

Now define the set $\mathcal{K}$ of *Kleshchev multipartitions* by saying that a multipartition $\boldsymbol{\lambda}$ of n lies in $\mathcal{K}$ if and only if there is a sequence $\boldsymbol{\lambda} = \boldsymbol{\lambda}(n), \boldsymbol{\lambda}(n-1), \dots, \boldsymbol{\lambda}(0) = \varnothing$ such that each $\boldsymbol{\lambda}(i)$ has a good node $\mathfrak{n}$ with $[\boldsymbol{\lambda}(i)] \setminus \{\mathfrak{n}\} = [\boldsymbol{\lambda}(i-1)]$. That is, $\boldsymbol{\lambda}$ is Kleshchev if and only if we can get from the Young diagram for $\boldsymbol{\lambda}$ to the empty Young diagram by successively removing good nodes. Obviously, the definition of a Kleshchev multipartition depends on the parameters $q, Q_1, \dots, Q_r$ (and indeed on the order of $Q_1, \dots, Q_r$), but we shall simply say 'Kleshchev' without fear of confusion.

We shall need a slightly stronger statement about which multipartitions are Kleshchev.

Proposition 1.1. *Suppose $\boldsymbol{\lambda}$ is a multipartition with a good node $\mathfrak{n}$, and let $\boldsymbol{\mu}$ be the multipartition with $[\boldsymbol{\mu}] = [\boldsymbol{\lambda}] \setminus \{\mathfrak{n}\}$. Then $\boldsymbol{\lambda}$ is Kleshchev if and only if $\boldsymbol{\mu}$ is.*

Proof. This follows from [4, Theorem 2.9 & Corollary 2.11], in which it is shown that the crystal graph of a certain highest weight module for $U_v(\widehat{\mathfrak{sl}}_e)$ (or for $U_v(\mathfrak{sl}_\infty)$, if $e = \infty$) has vertices indexed by Kleshchev multipartitions and edges corresponding to removal of good nodes. □

In the case $r = 1$, the classification of Kleshchev (multi)partitions is very easy. Say that a partition λ is *e-restricted* if and only if $\lambda_i - \lambda_{i+1} < e$ for all i. (For $e = \infty$, we say that every partition is e-restricted.) The following is then a simple exercise.

Lemma 1.2. *If $r = 1$, then the multipartition (λ) is Kleshchev if and only if λ is e-restricted.*

1.1.2 Specht modules and simple modules

For each multipartition $\boldsymbol{\lambda}$ of n, one defines a *Specht module* $S^{\boldsymbol{\lambda}}$ for $\mathcal{H}_n$. If $\mathcal{H}_n$ is semi-simple, then the Specht modules provide a complete set of irreducible modules for $\mathcal{H}_n$. If $\mathcal{H}_n$ is not semi-simple, then the Specht modules are no longer necessarily irreducible. In this case, for each Kleshchev multipartition $\boldsymbol{\lambda}$, the Specht module $S^{\boldsymbol{\lambda}}$ has an irreducible cosocle $D^{\boldsymbol{\lambda}}$, and the $D^{\boldsymbol{\lambda}}$ provide a complete set of irreducible modules for $\mathcal{H}_n$ as $\boldsymbol{\lambda}$ ranges over the set of Kleshchev multipartitions of n [2, Theorem 4.2].

The *decomposition matrix* of $\mathcal{H}_n$ records the composition multiplicities $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]$. It follows from the cellularity of $\mathcal{H}_n$ that the decomposition matrix is 'triangular'; to state this we need to define the dominance order on multipartitions. Given distinct multipartitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(r)})$ of n, we say that $\boldsymbol{\lambda}$ *dominates* $\boldsymbol{\mu}$ (and write $\boldsymbol{\lambda} \rhd \boldsymbol{\mu}$) if and only if

$$|\lambda^{(1)}| + \cdots + |\lambda^{(k-1)}| + \lambda^{(k)}_1 + \cdots + \lambda^{(k)}_i \geqslant |\mu^{(1)}| + \cdots + |\mu^{(k-1)}| + \mu^{(k)}_1 + \cdots + \mu^{(k)}_i$$

for every $(i, k) \in \mathbb{N} \times \{1, \dots, r\}$. Then we have the following.

Theorem 1.3. *Suppose $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions with $\boldsymbol{\mu}$ Kleshchev.*

1. *If $\boldsymbol{\mu} = \boldsymbol{\lambda}$, then $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}] = 1$.*
2. *If $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}] > 0$, then $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$.*

1.1.3 Induction and restriction

If $n > 1$, then $\mathcal{H}_{n-1}$ is naturally a submodule of $\mathcal{H}_n$, and in fact $\mathcal{H}_n$ is free as an $\mathcal{H}_{n-1}$-module. So there are well-behaved induction and restriction functors between the module categories of $\mathcal{H}_{n-1}$ and $\mathcal{H}_n$. Given modules M, N for $\mathcal{H}_{n-1}$ and $\mathcal{H}_n$ respectively, we write $M\uparrow^{\mathcal{H}_n}$ and $N\downarrow_{\mathcal{H}_{n-1}}$ for the induced and restricted modules. If B and C are blocks (or perhaps direct sums of blocks) of $\mathcal{H}_{n-1}$ and $\mathcal{H}_n$ respectively, then we may write $M\uparrow^C$ and $N\downarrow_B$ for the projections of the induced and restricted modules onto C and B. We need to know the effect of induction and restriction on Specht modules.

Theorem 1.4. [1, Lemma 2.1]

1. *Suppose $\boldsymbol{\lambda}$ is a multipartition of $n-1$, and let $\mathfrak{n}_1, \dots, \mathfrak{n}_s$ be the addable nodes of $[\boldsymbol{\lambda}]$. For each $i = 1, \dots, s$, let $\boldsymbol{\lambda}^{+i}$ be the multipartition of n with $[\boldsymbol{\lambda}^{+i}] = [\boldsymbol{\lambda}] \cup \{\mathfrak{n}_i\}$. Then $S^{\boldsymbol{\lambda}}\uparrow^{\mathcal{H}_n}$ has a filtration in which the factors are $S^{\boldsymbol{\lambda}^{+1}}, \dots, S^{\boldsymbol{\lambda}^{+s}}$.*
2. *Suppose $\boldsymbol{\lambda}$ is a multipartition of n, and let $\mathfrak{n}_1, \dots, \mathfrak{n}_t$ be the removable nodes of $[\boldsymbol{\lambda}]$. For each $i = 1, \dots, t$, let $\boldsymbol{\lambda}^{-i}$ be the multipartition of $n-1$ with $[\boldsymbol{\lambda}^{-i}] = [\boldsymbol{\lambda}] \setminus \{\mathfrak{n}_i\}$. Then $S^{\boldsymbol{\lambda}}\downarrow_{\mathcal{H}_{n-1}}$ has a filtration in which the factors are $S^{\boldsymbol{\lambda}^{-1}}, \dots, S^{\boldsymbol{\lambda}^{-t}}$.*

1.1.4 The blocks of $\mathcal{H}_n$

The blocks of the Iwahori–Hecke algebra are classified by the so-called 'Nakayama conjecture', but the blocks of the Ariki–Koike are rather harder to classify. It follows from the cellularity of $\mathcal{H}_n$ that each Specht module S^λ lies in one block, and so to classify the blocks, one seeks the appropriate partition of the set of multipartitions of n. Indeed, we abuse notation by saying that a multipartition $\boldsymbol{\lambda}$ lies in a block B of $\mathcal{H}_n$ if $S^{\boldsymbol{\lambda}}$ lies in B.

Given a multipartition $\boldsymbol{\lambda}$ and an element f of $\mathbb{F}$, let $c_f(\boldsymbol{\lambda})$ denote the number of nodes in $[\boldsymbol{\lambda}]$ of residue f. Now say that two multipartitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ *lie in the same combinatorial block (of $\mathcal{H}_n$)* if $c_f(\boldsymbol{\lambda}) = c_f(\boldsymbol{\mu})$ for all $f \in \mathbb{F}$. Then we have the following.

Theorem 1.5. [12, Proposition 5.9(ii)] *Suppose $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are two multipartitions of n. Then $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ lie in the same block of $\mathcal{H}_n$ only if they lie in the same combinatorial block.*

Graham and Lehrer conjectured that the converse to this theorem is true, namely that two multipartitions lying in the same combinatorial block actually lie in the same block. As mentioned in the introduction, this has now been proved by Lyle and Mathas, but since their proof uses the results in the present paper, we cannot assume their result here. So this paper will be largely concerned with studying combinatorial blocks of $\mathcal{H}_n$.

1.1.5 The weight of a partition

We now recall the usual notion of e-weight for a partition. If $e = \infty$, then a partition is defined to have e-weight zero. If e is finite, then we define the the weight of λ using *beta-numbers*. Let a be an integer, and define $\beta_1, \beta_2, \dots$ by

$$\beta_i = \lambda_i + a - i.$$

Then $\beta_1, \beta_2, \dots$ are distinct, and the set $B(\lambda) = \{\beta_1, \beta_2, \dots\}$ is a set of integers of which exactly $N + a$ elements are greater than or equal to $-N$, for sufficiently large N. Clearly, any such set of integers is the set of beta-numbers of some partition. We define the weight of λ recursively as follows: if there is some i such that $i \in B(\lambda) \not\ni i - e$, then we let $\hat{\lambda}$ be the partition with $B(\hat{\lambda}) = B(\lambda) \cup \{i - e\} \setminus \{i\}$, and we define weight$(\lambda)$ to be weight$(\hat{\lambda}) + 1$. If there is no such i, then we define weight$(\lambda) = 0$. It is not totally obvious that this definition of weight does not depend on the choice of i, but it is easy to show by induction that, irrespective of the value of i chosen at each stage, we end up with

$$\text{weight}(\lambda) = \left|\left\{(i, j) \in \mathbb{Z} \times \mathbb{Z} \;\middle|\; i > j,\ i \equiv j \,(\text{mod}\, e),\ i \in B(\lambda) \not\ni j\right\}\right|,$$

and so weight is well-defined. A partition of e-weight 0 is referred to as an *e-core*.

The procedure of reducing a beta-number by e is equivalent to 'removing a rim e-hook', which we now define. The *rim* of $[\lambda]$ is defined to be the set of nodes (i, j) of λ such that $(i + 1, j + 1)$ is not a node of $[\lambda]$. A *rim e-hook* is a connected subset of the rim of $[\lambda]$ containing e nodes. Removing a rim e-hook from $[\lambda]$ gives the Young diagram of some partition $\hat{\lambda}$ of $n - e$.

Lemma 1.6. [14, §2.7] *$[\lambda]$ has a rim e-hook whose highest node (that is, the node (i, j) with the smallest value of i) lies in row l if and only if $\beta_l(\lambda) - e \notin B(\lambda)$. If $\hat{\lambda}$ is the partition whose Young diagram is obtained by removing a rim e-hook whose highest node lies in row l, then we have*

$$B(\hat{\lambda}) = B(\lambda) \cup \{\beta_l(\lambda) - e\} \setminus \{\beta_l(\lambda)\}.$$

1.1.6 Miscellaneous notation

Throughout this paper, we shall use the notation $\mathbb{1}_{[S]}$ to denote the indicator function of the truth of a statement S.

2 The weight of a multipartition

2.1 The definition of the weight of a multipartition

Given a multipartition $\boldsymbol{\lambda}$ and given $q, Q_1, \ldots, Q_r$, define the Young diagram $[\boldsymbol{\lambda}]$, the residue of each node in $[\boldsymbol{\lambda}]$ and the integers $c_f(\boldsymbol{\lambda})$ as in Section 1.1. Now define the *weight* $w(\boldsymbol{\lambda})$ of $\boldsymbol{\lambda}$ to be the integer

$$w(\boldsymbol{\lambda}) = \left(\sum_{i=1}^{r} c_{Q_i}(\boldsymbol{\lambda})\right) - \frac{1}{2}\sum_{f\in\mathbb{F}}(c_f(\boldsymbol{\lambda}) - c_{qf}(\boldsymbol{\lambda}))^2.$$

It is immediate that two multipartitions lying in the same combinatorial block have the same weight; hence by Theorem 1.5 any two partitions lying in the same block have the same weight. So we may define the weight of a block or a combinatorial block B to be the weight of any multipartition in B.

Example. Suppose $r = 2$, $(Q_1, Q_2) = (1, q)$ and $\boldsymbol{\lambda} = ((2^2), (2, 1))$. Then the residues of the nodes in $[\boldsymbol{\lambda}]$ are

1	q
q^{-1}	1

q	q^2
1	

.

So we have

$$w(\boldsymbol{\lambda}) = \begin{cases} 6 & (e = 2) \\ 4 & (e = 3) \\ 2 & (e = 4) \\ 1 & (e \geqslant 5). \end{cases}$$

2.2 The case $r = 1$

It seems that the definition of weight given above is the 'right' generalisation of the weight of a partition. In order to justify this assertion, we must show first that it really is a generalisation. That is, we prove the following result.

Proposition 2.1. *Suppose $r = 1$. Let λ be a partition, and let $\boldsymbol{\lambda}$ be the multipartition (λ). Then $w(\boldsymbol{\lambda}) =$* weight$(\lambda)$.

In this section, we may abuse notation by not distinguishing between the partition λ and the multipartition (λ).

Example. Suppose $e = 4$, and that $\lambda = (5, 4, 3^3)$. Then we may remove three rim 4-hooks from $[\lambda]$ to reach the partition $(3, 2, 1)$, which is a 4-core, and so weight$(\lambda) = 3$:

			2	2
		2	2	
	3	1		
3	3	1		
3	1	1		

.

On the other hand, the residues of the nodes in $[\lambda]$ are as follows, assuming $Q_1 = 1$:

1	q	q^2	q^3	1
q^3	1	q	q^2	
q^2	q^3	1		
q	q^2	q^3		
1	q	q^2		

.

So we get $c_1(\lambda) = c_{q^2}(\lambda) = 5, c_q(\lambda) = c_{q^3}(\lambda) = 4$, yielding $w((\lambda)) = 3$.

We shall need the following simple property of e-cores.

Lemma 2.2. *Suppose λ is an e-core. Then there do not exist integers i, l, m such that $\beta_l \equiv i \pmod e$, $\beta_m \equiv i-1 \pmod e$ and neither $\beta_l - 1$ nor $\beta_m + 1$ lies in $B(\lambda)$.*

Proof. We suppose that $\beta_l > \beta_m$; the other case is similar. We may assume $\beta_l \neq \beta_m + 1$, so $\beta_l = \beta_m + 1 + ae$ for some positive integer a. Let $b \geqslant 1$ be minimal such that $\beta_m + 1 + be \in B(\lambda)$. Then $\beta_m + 1 + (b-1)e \notin B(\lambda)$, so λ has positive weight; contradiction. □

It is easily seen that $w((\lambda))$ does not depend on Q_1, and for the proof of Proposition 2.1, we assume that $Q_1 = q^a$, where a is the integer used in the definition of the beta-numbers of λ. Now all residues of nodes are powers of q, and we use the term 'i-node' to mean 'node of residue q^i'.

For each $i \in \mathbb{Z}$, we define $\delta_i(\lambda)$ to be the integer

$$\begin{aligned}\delta_i(\lambda) =& |\{l \mid \beta_l \equiv i \pmod e \text{ and } \beta_l - 1 \notin B(\lambda)\}| \\ &- |\{l \mid \beta_l \equiv i-1 \pmod e \text{ and } \beta_l + 1 \notin B(\lambda)\}|.\end{aligned}$$

It is easy to see that if $\beta_l - 1 \notin B(\lambda)$, then replacing β_l with $\beta_l - 1$ corresponds to removing a removable node of residue q^{β_l} from $[\lambda]$. So $\delta_i(\lambda)$ is the number of removable i-nodes of $[\lambda]$ minus the number of addable i-nodes.

Lemma 2.3. *For each $i = 0, \dots, e-1$, we have*

$$\delta_i(\lambda) = 2c_{q^i}(\lambda) - c_{q^{i-1}}(\lambda) - c_{q^{i+1}}(\lambda) - \mathbb{1}[i \equiv a \pmod e].$$

Proof. Let γ_l^i denote the number of i-nodes in row l of $[\lambda]$. Then it is easy to check that

$$2\gamma_l^i - \gamma_l^{i-1} - \gamma_l^{i+1} = \mathbb{1}[\lambda_l > 0]\left(\mathbb{1}[a+1-l \equiv i] - \mathbb{1}[a+1-l \equiv i+1] + \mathbb{1}[a+\lambda_l - l \equiv i] - \mathbb{1}[a+\lambda_l - l \equiv i-1]\right);$$

here, and for the rest of this proof, all congruences are taken modulo e. The sum

$$\sum_{l \geqslant 1} \mathbb{1}[\lambda_l > 0]\left(\mathbb{1}[a+\lambda_l - l \equiv i] - \mathbb{1}[a+\lambda_l - l \equiv i-1]\right)$$

equals

$$\delta_i(\lambda) + \mathbb{1}[\text{the smallest integer not in } B(\lambda) \text{ is congruent to } i \text{ modulo } e].$$

On the other hand, the sum

$$\sum_{l \geqslant 1} \mathbb{1}[\lambda_l > 0]\left(\mathbb{1}[a+1-l \equiv i] - \mathbb{1}[a+1-l \equiv i+1]\right)$$

equals

$$\mathbb{1}[i \equiv a] - \mathbb{1}[\text{the largest } l \text{ such that } \lambda_l > 0 \text{ satisfies } a+1-l \equiv i+1].$$

The result follows. □

Proof of Proposition 2.1. First suppose that $e = \infty$. It is clear that $c_f(\lambda) = 0$ unless $f = q^i$ for some $i \in \mathbb{Z}$, and that $c_{q^i}(\lambda) = 0$ for i sufficiently large and for i sufficiently small. Now we consider the relationship between $c_{q^i}(\lambda)$ and $c_{q^{i\pm1}}(\lambda)$. If $i > a$, then all the i-nodes lie above the diagonal in the Young diagram of λ. So each i-node has an $(i-1)$-node to its immediate left, and each i-node apart from the one in the top row of $[\lambda]$ has an $i+1$-node immediately above it. Similar statements apply for $i < a$ (in which case the i-nodes lie below the diagonal), and we deduce that:

- if $i > a$, then $c_{q^{i-1}}(\lambda) \geqslant c_{q^i}(\lambda) \geqslant c_{q^{i-1}}(\lambda) - 1$;
- if $i < a$, then $c_{q^{i+1}}(\lambda) \geqslant c_{q^i}(\lambda) \geqslant c_{q^{i+1}}(\lambda) - 1$.

Hence we find that $(c_f(\lambda) - c_{qf}(\lambda))$ equals +1 for exactly $c_{q^a}(\lambda)$ values of f, equals -1 for another $c_{q^a}(\lambda)$ values of f, and equals 0 otherwise. So we have $w((\lambda)) = 0$.

Now suppose that e is finite. If weight$(\lambda) > 0$, then by Lemma 1.6 $[\lambda]$ has a rim e-hook; let $\hat{\lambda}$ be the partition obtained by removing this rim e-hook. It is easily seen that a rim hook contains exactly one node of each of the residues $1, q, \ldots, q^{e-1}$. So we have

$$c_f((\hat{\lambda})) = \begin{cases} c_f(\lambda) - 1 & (f \in \{1, q, \ldots, q^{e-1}\}) \\ c_f(\lambda) = 0 & (f \notin \{1, q, \ldots, q^{e-1}\}), \end{cases}$$

Hence $w((\lambda)) - w((\hat{\lambda})) = 1 = \text{weight}(\lambda) - \text{weight}(\hat{\lambda})$. So by induction we may assume that weight$(\lambda) = 0$.

If $\lambda = \varnothing$, then the result is obvious, so assume otherwise. $[\lambda]$ has a removable i-node for some i. Since weight$(\lambda) = 0$, Lemma 2.2 implies that $[\lambda]$ has no addable i-nodes. So $\delta_i(\lambda)$ is the number of removable i-nodes of λ. If we define the partition λ^- by removing all the removable i-nodes, then we have weight$(\lambda^-) = 0$, and by induction $w((\lambda^-)) = 0$. We also have

$$c_f(\lambda^-) = \begin{cases} c_f(\lambda) - \delta_i(\lambda) & (f = q^i) \\ c_f(\lambda) & (\text{otherwise}), \end{cases}$$

whence

$$\begin{aligned} w((\lambda^-)) - w((\lambda)) = \frac{1}{2}\Big(& (c_{q^{i-1}}(\lambda) - c_{q^i}(\lambda))^2 + (c_{q^i}(\lambda) - c_{q^{i+1}}(\lambda))^2 \\ & - (c_{q^{i-1}}(\lambda) - c_{q^i}(\lambda) + \delta_i(\lambda))^2 - (c_{q^i}(\lambda) - \delta_i(\lambda) - c_{q^{i+1}}(\lambda))^2 \Big) - \delta_i(\lambda)\mathbb{1}[i \equiv a \pmod e] \\ = \delta_i(\lambda) & \left(2c_{q^i}(\lambda) - 2c_{q^{i-1}}(\lambda) - \delta_i(\lambda) - \mathbb{1}[i \equiv a \pmod e]\right) \\ = 0, & \end{aligned}$$

by Lemma 2.3. So

$$w((\lambda)) = w((\lambda^-)) = 0. \qquad \square$$

Example. Beginning with the core $(3, 2, 1)$ of the partition in the last example, we proceed through the following sequence of cores:

	c_1	c_q	c_{q^2}	c_{q^3}
$\begin{array}{\|c\|c\|c\|}\hline 1 & q & q^2 \\ \hline q^3 & 1 \\ \cline{1-2} q^2 \\ \cline{1-1} \end{array}$	2	1	2	1
$\begin{array}{\|c\|c\|}\hline 1 & q \\ \hline q^3 & 1 \\ \cline{1-2} \end{array}$	2	1	0	1
$\begin{array}{\|c\|c\|}\hline 1 & q \\ \hline q^3 \\ \cline{1-1} \end{array}$	1	1	0	1
$\begin{array}{\|c\|c\|}\hline 1 & q \\ \hline \end{array}$	1	1	0	0
$\begin{array}{\|c\|}\hline 1 \\ \hline \end{array}$	1	0	0	0
$\varnothing$	0	0	0	0

.

3 The abacus

In this section, we define the abacus display of a multipartition, analogously with James's definition of the abacus display for a partition, and we show that the weight of a multipartition can be calculated in a simple way from its abacus display. The abacus for a multipartition was briefly introduced in [2].

3.1 q-connected cyclotomic parameters

Suppose $Q_1, \dots, Q_r$ are the cyclotomic parameters of $\mathcal{H}_n$. We say that $\{Q_1, \dots, Q_r\}$ is *q-connected* if there exist integers a_{ij} such that $Q_i = q^{a_{ij}} Q_j$ for all i, j. Dipper and Mathas [7] have shown that if $\{Q_1, \dots, Q_r\}$ is not q-connected, then $\mathcal{H}_n$ is Morita equivalent to a direct sum of tensor products of Ariki–Koike algebras defined for smaller values of r; as a consequence, to understand the representation theory of $\mathcal{H}_n$ in general, it suffices to consider the case where $\{Q_1, \dots, Q_r\}$ is q-connected.

It will also suffice for our purposes – calculating and working with weights of multipartitions – to consider the case where the set of cyclotomic parameters of $\mathcal{H}_n$ is q-connected. For if it is not, then we have $\{1, \dots, r\} = I \sqcup J$, where I and J are non-empty sets such that for $i \in I$ and $j \in J$, there is no integer a such that $Q_i = q^a Q_j$. Now write

$$I = \{i_1 < \cdots < i_s\}, \qquad J = \{j_1 < \cdots < j_t\}$$

and given a multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$, define the multipartitions

$$\boldsymbol{\lambda}^I = (\lambda^{(i_1)}, \dots, \lambda^{(i_s)}), \qquad \boldsymbol{\lambda}^J = (\lambda^{(j_1)}, \dots, \lambda^{(j_t)}),$$

and calculate the weights $w(\boldsymbol{\lambda}^I)$ and $w(\boldsymbol{\lambda}^J)$ using the parameter sets $(Q_{i_1}, \dots, Q_{i_s})$ and $(Q_{j_1}, \dots, Q_{j_t})$ respectively. Then it follows immediately from the definition of w that $w(\boldsymbol{\lambda}) = w(\boldsymbol{\lambda}^I) + w(\boldsymbol{\lambda}^J)$.

So we assume from now on that the set $\{Q_1, \dots, Q_r\}$ is q-connected. In fact, by re-scaling the generator T_0, we may assume that each Q_j is a power of q, so we let $a_1, \dots, a_r$ be integers such that $Q_j = q^{a_j}$ for all j. We continue to refer to a node of residue q^i as an i-node.

3.2 Beta-numbers and the abacus

Given a multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$, we define $\beta_1^{(j)}, \beta_2^{(j)}, \dots$ to be the beta-numbers

$$\beta_i^{(j)} = \lambda_i^{(j)} + a_j - i$$

of $\lambda^{(j)}$, for $j = 1, \dots, r$, and as above we define $B(\lambda^{(j)}) = \{\beta_i^{(j)} \mid i = 1, 2, \dots\}$.

As above, we can see that if $\beta_l^{(j)}$ is congruent to i modulo e and $\beta_l^{(j)} - 1 \notin B(\lambda^{(j)})$, then replacing $\beta_l^{(j)}$ with $\beta_l^{(j)} - 1$ corresponds to removing an i-node from $[\lambda^{(j)}]$. Recalling the definition of $\delta_i(\lambda)$ from above, we define $\delta_i(\boldsymbol{\lambda}) = \sum_{j=1}^r \delta_i(\lambda^{(j)})$.

The crucial property of the integers $\delta_i(\boldsymbol{\lambda})$ is that (together with n) they determine the combinatorial block in which a multipartition lies. First we obtain an expression for $\delta_i(\boldsymbol{\lambda})$ in terms of the integers $c_{q_k}(\boldsymbol{\lambda})$.

Lemma 3.1.

$$\delta_i(\boldsymbol{\lambda}) = 2c_{q^i}(\boldsymbol{\lambda}) - c_{q^{i-1}}(\boldsymbol{\lambda}) - c_{q^{i+1}}(\boldsymbol{\lambda}) - |\{j \mid Q_j = q^i\}|.$$

Proof. If we put $c_f(\boldsymbol{\lambda}) = c_f(\lambda^{(1)}) + \cdots + c_f(\lambda^{(r)})$, then by Lemma 2.3, we have

$$\delta_i(\lambda^{(j)}) = 2c_{q^i}(\lambda^{(j)}) - c_{q^{i-1}}(\lambda^{(j)}) - c_{q^{i+1}}(\lambda^{(j)}) - \mathbb{1}_{[Q_j = q^i]},$$

and the result follows by summing. □

Proposition 3.2. *Suppose $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(r)})$ are two multipartitions of n, and define $\delta_i(\boldsymbol{\lambda})$ and $\delta_i(\boldsymbol{\mu})$ as above. Then $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ lie in the same combinatorial block of $\mathcal{H}_n$ if and only if $\delta_i(\boldsymbol{\lambda}) = \delta_i(\boldsymbol{\mu})$ for all i.*

Proof. By Lemma 3.1, the integers $c_f(\boldsymbol{\lambda})$ determine the integers $\delta_i(\boldsymbol{\lambda})$, and the 'only if' part of the proposition is proved. On the other hand, it is a standard fact from Lie theory that if $e < \infty$ then the matrix with entries $(a_{ij})_{0 \leqslant i,j \leqslant e-1}$ given by

$$a_{ij} = 2\mathbb{1}[i \equiv j \,(\mathrm{mod}\, e)] - \mathbb{1}[i \equiv j-1 \,(\mathrm{mod}\, e)] - \mathbb{1}[i \equiv j+1 \,(\mathrm{mod}\, e)]$$

has nullity 1, and so the $\delta_i(\boldsymbol{\lambda})$ determine the $c_{q^i}(\boldsymbol{\lambda})$ up to addition of a scalar; a similar statement applies when $e = \infty$. Hence the $\delta_i(\boldsymbol{\lambda})$ together with $n = \sum c_{q^i}(\boldsymbol{\lambda})$ determine the $c_{q^i}(\boldsymbol{\lambda})$ and hence the combinatorial block in which $\boldsymbol{\lambda}$ lies. □

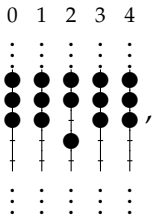
Given a multipartition $\boldsymbol{\lambda}$, we refer to the sequence of integers $\delta_i(\boldsymbol{\lambda})$ as the *hub* of $\boldsymbol{\lambda}$. If $\boldsymbol{\lambda}$ lies in a combinatorial block B, then in view of Proposition 3.2, we may also refer to this sequence as the hub of B, or indeed of any block contained in B.

Given a set of beta-numbers for a partition λ, we can create an *abacus display*. We take an abacus with e infinite vertical runners, which we label $0, 1, \ldots, e-1$ from left to right (or $\ldots, -1, 0, 1, \ldots$ from left to right, if $e = \infty$), and we mark positions on runner i and label them with the integers congruent to i modulo e, so that (if $e < \infty$) position $(x+1)e + i$ lies immediately below position $xe + i$, for each x. For each l, we then place a bead at position β_l. The resulting diagram is referred to as the abacus display for λ.

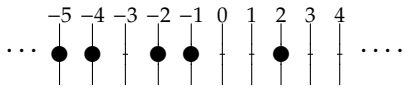
Example. Suppose $\lambda = (3, 1, 1)$, and $a = 0$. Then we have

$$B(\lambda) = \{2, -1, -2, -4, -5, -6, \ldots\},$$

so the abacus display with $e = 5$ is



while the abacus display with $e = \infty$ is



Given a multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ (and assuming still that $Q_j = q^{a_j}$ for $j = 1, \ldots, r$), we calculate the beta-numbers of $\lambda^{(1)}, \ldots, \lambda^{(r)}$ using $a_1, \ldots, a_r$ as above, and form the corresponding abacus displays. This r-tuple of abacus displays is then referred to as the abacus display for $\boldsymbol{\lambda}$.

3.3 The weight of a multipartition and the abacus

The aim of the rest of this section is to find a simple way to calculate $w(\boldsymbol{\lambda})$ from the abacus display for $\boldsymbol{\lambda}$. We begin with a simple lemma which relates the weights of two multipartitions with the same hub.

Lemma 3.3. *Suppose $\boldsymbol{\lambda}$ is a multipartition of n and $\boldsymbol{\mu}$ a multipartition of m, and that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have the same hub. If $e = \infty$, then $n = m$ and $w(\boldsymbol{\lambda}) = w(\boldsymbol{\mu})$. If e is finite, then*

$$w(\boldsymbol{\lambda}) - w(\boldsymbol{\mu}) = \frac{r(n-m)}{e}.$$

Proof. As noted in the proof of Proposition 3.2, the hub of $\boldsymbol{\lambda}$ determines the integers $c_{q^i}(\boldsymbol{\lambda})$ up to addition of a scalar, so there exists $C \in \mathbb{Z}$ such that

$$c_{q^i}(\boldsymbol{\lambda}) = c_{q^i}(\boldsymbol{\mu}) + C$$

for all i, which gives $w(\boldsymbol{\lambda}) = w(\boldsymbol{\mu}) + rC$. But we also have

$$m = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} c_{q^i}(\boldsymbol{\lambda}), \qquad n = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} c_{q^i}(\boldsymbol{\mu}).$$

Hence when $e = \infty$, we must have $C = 0$ (since $c_{q^i}(\boldsymbol{\lambda}) = c_{q^i}(\boldsymbol{\mu}) = 0$ for sufficiently large i), while for finite e, we must have $C = \frac{n-m}{e}$. The result follows. □

Next we consider removing rim e-hooks, if e is finite. Suppose that in the abacus display for $\lambda^{(j)}$, there is a bead with an empty space immediately above it. Sliding this bead up into the empty space corresponds to reducing by e the corresponding beta-number for $\lambda^{(j)}$, and, as we saw above, this corresponds to removing a rim e-hook from $[\lambda^{(j)}]$. Lemma 3.3 tells us the change in weight.

Corollary 3.4. *Suppose $\boldsymbol{\lambda}$ is a multipartition, and that $\boldsymbol{\lambda}^-$ is obtained from $\boldsymbol{\lambda}$ by removing a rim e-hook from some $\lambda^{(j)}$. Then $w(\boldsymbol{\lambda}) = w(\boldsymbol{\lambda}^-) + r$.*

Now we consider the case where each $\lambda^{(j)}$ is an e-core, so that no rim e-hooks can be removed from $[\boldsymbol{\lambda}]$. We refer to such a multipartition as a *multicore*. When $e = \infty$, every multipartition is a multicore. Of course, when $r = 1$ a multicore is simply a core, and has weight zero. But when $r \geqslant 2$, calculating the weight of a multicore is non-trivial. Our next result shows that we can reduce to the case $r = 2$.

Proposition 3.5. *Suppose that $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is a multicore. For each $1 \leqslant k < l \leqslant r$, define the weight $w((\lambda^{(k)}, \lambda^{(l)}))$ using the cyclotomic parameters Q_k, Q_l. Then*

$$w(\boldsymbol{\lambda}) = \sum_{1 \leqslant k < l \leqslant r} w((\lambda^{(k)}, \lambda^{(l)})).$$

Example. Suppose $r = 3$, $(Q_1, Q_2, Q_3) = (q, 1, q^2)$, and let $\boldsymbol{\lambda} = ((1^2), (2), (2,1))$, which is a multicore as long as $e \geqslant 4$. The e-residue diagram of $\boldsymbol{\lambda}$ is

$$\begin{array}{|c|}\hline q \\ \hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & q \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline q^2 & q^3 \\ \hline q \\ \cline{1-1} \end{array},$$

and we may easily calculate the following:

	$w((\lambda^{(1)}, \lambda^{(2)}))$	$w((\lambda^{(1)}, \lambda^{(3)}))$	$w((\lambda^{(2)}, \lambda^{(3)}))$	$w(\boldsymbol{\lambda})$
$e = 4$	0	2	1	3
$e \geqslant 5$	0	1	0	1

.

The reader may care to check that for $e = 2, 3$, the relation $w(\boldsymbol{\lambda}) = w((\lambda^{(1)}, \lambda^{(2)})) + w((\lambda^{(1)}, \lambda^{(3)})) + w((\lambda^{(2)}, \lambda^{(3)}))$ does not hold – so it is essential in Proposition 3.5 that $\boldsymbol{\lambda}$ is a multicore.

In order to prove Proposition 3.5, we prove an important lemma which we shall use later. It tells us the effect on the weight of a multipartition of removing nodes of a given residue.

Lemma 3.6. *Suppose $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions, and that the Young diagram of $\boldsymbol{\mu}$ is obtained from that of $\boldsymbol{\lambda}$ by removing some u removable i-nodes. Then*

$$w(\boldsymbol{\mu}) - w(\boldsymbol{\lambda}) = u(\delta_i(\boldsymbol{\lambda}) - u).$$

Proof. By performing a similar calculation to that used in the proof of Proposition 2.1, we find that

$$w(\boldsymbol{\mu}) - w(\boldsymbol{\lambda}) = u\left(2c_{q^i}(\boldsymbol{\lambda}) - c_{q^{i-1}}(\boldsymbol{\lambda}) - c_{q^{i+1}}(\boldsymbol{\lambda}) - u - |\{j \mid Q_j = q^i\}|\right).$$

The result now follows by Lemma 3.1. □

Proof of Proposition 3.5. We proceed by induction on n. The unique multipartition of 0 has weight 0 whatever the values of $Q_1, \dots, Q_r$, and so we suppose that $n > 0$. Then we can choose some i, j such that $[\lambda^{(j)}]$ has removable i-nodes; by Lemma 2.2 $[\lambda^{(j)}]$ cannot also have addable i-nodes, so $\delta_i(\lambda^{(j)}) > 0$. Let $\boldsymbol{\lambda}^- = (\lambda^{(1)}, \dots, \lambda^{(j-1)}, \lambda^{(j)-}, \lambda^{(j+1)}, \dots, \lambda^{(r)})$ be the multipartition obtained by removing all the removable i-nodes from $[\lambda^{(j)}]$. By Lemma 3.6, we have

$$w(\boldsymbol{\lambda}^-) - w(\boldsymbol{\lambda}) = \delta_i(\lambda^{(j)})\left(\delta_i(\boldsymbol{\lambda}) - \delta_i(\lambda^{(j)})\right).$$

On the other hand, writing $w_{kl}(\boldsymbol{\lambda})$ for $w((\lambda^{(k)}, \lambda^{(l)}))$, and similarly $w_{kl}(\boldsymbol{\lambda}^-)$, Lemma 3.6 also gives

$$w_{kl}(\boldsymbol{\lambda}^-) - w_{kl}(\boldsymbol{\lambda}) = \begin{cases} \delta_i(\lambda^{(j)})(\delta_i((\lambda^{(k)}, \lambda^{(l)})) - \delta_i(\lambda^{(j)})) & (\text{if } j = k \text{ or } l) \\ 0 & (\text{otherwise}). \end{cases}$$

We obtain

$$\begin{aligned} \sum_{1 \leqslant k < l \leqslant r} (w_{kl}(\boldsymbol{\lambda}^-) - w_{kl}(\boldsymbol{\lambda})) &= \sum_{k \neq j} \delta_i(\lambda^{(j)})(\delta_i(\lambda^{(k)}) + \delta_i(\lambda^{(j)}) - \delta_i(\lambda^{(j)})) \\ &= \delta_i(\lambda^{(j)})\left(\sum_{k \neq j} \delta_i(\lambda^{(k)})\right) \\ &= \delta_i(\lambda^{(j)})(\delta_i(\boldsymbol{\lambda}) - \delta_i(\lambda^{(j)})) \\ &= w(\boldsymbol{\lambda}^-) - w(\boldsymbol{\lambda}). \end{aligned}$$

By induction we have

$$w(\boldsymbol{\lambda}^-) = \sum_{k,l} w_{kl}(\boldsymbol{\lambda}^-),$$

and the result follows. □

3.4 The case $r = 2$

In view of Proposition 3.5, we examine the case $r = 2$. A multipartition when $r = 2$ is usually referred to as a *bipartition*, and we shall refer to a multicore with $r = 2$ as a *bicore*.

Suppose $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ is a bicore, and define integers $\gamma_i(\boldsymbol{\lambda})$ for $i \in \mathbb{Z}$ as follows:

$$\gamma_i(\boldsymbol{\lambda}) = \sum_{j \in \mathbb{Z}} \mathbb{1}[q^j = q^i](\mathbb{1}[j \in B(\lambda^{(1)})] - \mathbb{1}[j \in B(\lambda^{(2)})]).$$

(Informally, $\gamma_i(\boldsymbol{\lambda})$ is the 'difference between the number of beads on runner i of the abacus for $\lambda^{(1)}$ and the number of beads on runner i of the abacus for $\lambda^{(2)}$'.) Now suppose we are given $i \neq j \in \mathbb{Z}$ such that $i \not\equiv j \pmod{e}$ and such that if $e = \infty$, then $i \in B(\lambda^{(1)}) \not\ni j \in B(\lambda^{(2)}) \not\ni i$. Let l_1 be the largest element of $B(\lambda^{(1)})$ which is congruent to i modulo e, and let l_2 be the largest element of $B(\lambda^{(2)})$ which is congruent to j modulo e. Let m_1 be the smallest integer not in $B(\lambda^{(1)})$ which is congruent to j modulo e, and let m_2 be the smallest integer not in $B(\lambda^{(2)})$ which is congruent to i modulo e (note that the extra assumption on i, j in the case where $e = \infty$ guarantees that l_1, l_2, m_1, m_2 are defined). Define $s_{ij}(\boldsymbol{\lambda})$ to be the bipartition obtained by moving a bead from position l_k to position m_k in the abacus display for $\lambda^{(k)}$, for $k = 1$ and 2. If $e < \infty$, then we may regard $s_{ij}(\boldsymbol{\lambda})$ as being obtained by moving all the beads on runner i up one position and all the beads on runner j down one position on the abacus for $\lambda^{(1)}$, and vice versa on the abacus for $\lambda^{(2)}$. Then we have the following.

Lemma 3.7.

1. *$s_{ij}(\boldsymbol{\lambda})$ has the same hub as $\boldsymbol{\lambda}$.*

2. $$\gamma_l(s_{ij}(\boldsymbol{\lambda})) = \begin{cases} \gamma_l(\boldsymbol{\lambda}) - 2 & (l = i) \\ \gamma_l(\boldsymbol{\lambda}) + 2 & (l = j) \\ \gamma_l(\boldsymbol{\lambda}) & (\textit{otherwise}). \end{cases}$$

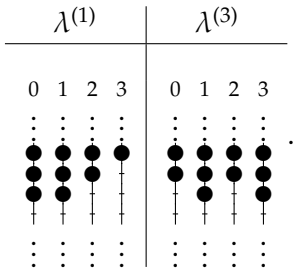
3. $$w(s_{ij}(\boldsymbol{\lambda})) = w(\boldsymbol{\lambda}) + 2(\gamma_j(\boldsymbol{\lambda}) - \gamma_i(\boldsymbol{\lambda}) + 2).$$

Proof. (1) and (2) are obvious. For (3), we need to apply Lemma 3.3: for $e = \infty$, the result is immediate, since by assumption we have $\gamma_i(\boldsymbol{\lambda}) = 1$, $\gamma_j(\boldsymbol{\lambda}) = -1$. If $e < \infty$, we find

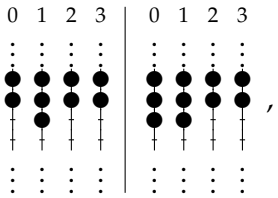
$$\begin{aligned} |s_{ij}(\boldsymbol{\lambda})| - |\boldsymbol{\lambda}| &= (m_1 - l_2) + (m_2 - l_1) \\ &= e\Big((1 + \gamma_j(\boldsymbol{\lambda})) + (1 - \gamma_i(\boldsymbol{\lambda}))\Big) \end{aligned}$$

and the result follows. □

Example. Recalling the multipartition $\boldsymbol{\lambda}$ from the last example, we examine the bicore $(\lambda^{(1)}, \lambda^{(3)}) = ((1^2), (2, 1))$ when $e = 4$. An abacus display for this bicore is



We may read off $(\gamma_0, \gamma_1, \gamma_2, \gamma_3) = (1, 0, 0, -2)$; in particular, $\gamma_0 - \gamma_3 = -3$; the bipartition $s_{03}((\lambda^{(1)}, \lambda^{(3)})) = ((1), \varnothing)$ has abacus display



and we have $w((\lambda^{(1)}, \lambda^{(3)})) = w(s_{03}((\lambda^{(1)}, \lambda^{(3)}))) + 2$.

Using Lemma 3.7, we may reduce the calculation of the weight of a bicore $\boldsymbol{\lambda}$ to the case where $\gamma_i(\boldsymbol{\lambda}) - \gamma_j(\boldsymbol{\lambda}) \leqslant 2$ for all i, j: for if we have $\gamma_i(\boldsymbol{\lambda}) - \gamma_j(\boldsymbol{\lambda}) \geqslant 3$, then by induction on $\sum_{k,l} |\gamma_k(\boldsymbol{\lambda}) - \gamma_l(\boldsymbol{\lambda})|$ we know the weight of $s_{ij}(\boldsymbol{\lambda})$, and so we can calculate the weight of $\boldsymbol{\lambda}$. So we are left with the case where $\gamma_i(\boldsymbol{\lambda}) - \gamma_j(\boldsymbol{\lambda}) \leqslant 2$ for all i, j. The weight in this case is given by the following proposition.

Proposition 3.8. *Suppose $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ is a bicore, and that the integers $\gamma_i(\boldsymbol{\lambda})$ defined above satisfy*

$$\gamma_i(\boldsymbol{\lambda}) - \gamma_j(\boldsymbol{\lambda}) \leqslant 2$$

for all i, j. Define

$$I = \{i \mid \gamma_i(\boldsymbol{\lambda}) - \gamma_j(\boldsymbol{\lambda}) = 2 \text{ for some } j\}, \qquad J = \{j \mid \gamma_i(\boldsymbol{\lambda}) - \gamma_j(\boldsymbol{\lambda}) = 2 \text{ for some } i\}.$$

Then $w(\boldsymbol{\lambda}) = \min\{|I|, |J|\}$.

Proof. We proceed by induction on n, with the case $n = 0$ being trivial. For $n > 0$, suppose that for some i, $[\boldsymbol{\lambda}]$ has at least one removable i-node but no addable i-nodes. Then the number of removable i-nodes is $\delta_i(\boldsymbol{\lambda})$; let $\boldsymbol{\lambda}^-$ be the multicore obtained by removing all these nodes. Then by Lemma 3.6 we have $w(\boldsymbol{\lambda}^-) = w(\boldsymbol{\lambda})$, and

$$\gamma_k(\boldsymbol{\lambda}^-) = \begin{cases} \gamma_i(\boldsymbol{\lambda}) & (k = i-1) \\ \gamma_{i-1}(\boldsymbol{\lambda}) & (k = i) \\ \gamma_k(\boldsymbol{\lambda}) & (k \neq i, i-1). \end{cases}$$

So the result follows by induction.

So we may assume that there is no i such that $[\boldsymbol{\lambda}]$ has removable i-nodes but no addable i-nodes. We now proceed by induction on $|\lambda^{(1)}|$, supposing first that $|\lambda^{(1)}| > 0$. Choose some i such that $[\lambda^{(1)}]$ has removable i-nodes. $\lambda^{(1)}$ is an e-core, so it cannot also have addable i-nodes, and so $[\lambda^{(2)}]$ must have addable i-nodes. If $[\lambda^{(1)}]$ has t removable i-nodes and $[\lambda^{(2)}]$ has s addable i-nodes, then we find that

$$2 \leqslant s + t = \gamma_i(\boldsymbol{\lambda}) - \gamma_{i-1}(\boldsymbol{\lambda}) \leqslant 2,$$

so we must have $s = t = 1$ and $\gamma_i(\boldsymbol{\lambda}) - \gamma_{i-1}(\boldsymbol{\lambda}) = 2$. Lemma 3.7 then implies that $w(s_{i(i-1)}(\boldsymbol{\lambda})) = w(\boldsymbol{\lambda})$, while

$$\gamma_k(s_{i(i-1)}(\boldsymbol{\lambda})) = \begin{cases} \gamma_i(\boldsymbol{\lambda}) & (k = i-1) \\ \gamma_{i-1}(\boldsymbol{\lambda}) & (k = i) \\ \gamma_k(\boldsymbol{\lambda}) & (k \neq i, i-1). \end{cases}$$

The result follows by induction on $|\lambda^{(1)}|$.

So we are left with the case where $\lambda^{(1)} = \varnothing$, and where for each i either $[\boldsymbol{\lambda}]$ has addable i-nodes or $[\boldsymbol{\lambda}]$ has no removable i-nodes. If $\lambda^{(2)} \neq \varnothing$, then there is some i such that $[\lambda^{(2)}]$ has at least one removable i-node. $[\lambda^{(2)}]$ cannot have any addable i-nodes, so $[\lambda^{(1)}]$ must have at least one addable i-node, i.e. $q^i = Q_1$. So $[\lambda^{(2)}]$ has removable nodes of only one residue, and in fact, calculating as above and using the fact that $\gamma_i(\boldsymbol{\lambda}) - \gamma_{i-1}(\boldsymbol{\lambda}) \leqslant 2$, we find that $[\lambda^{(2)}]$ has only one removable node. So $[\lambda^{(2)}]$ is a rectangular partition, say $\lambda^{(2)} = (v^w)$, with the top left corner residue being Q_2 and the bottom right corner residue being Q_1. In addition, $\lambda^{(2)}$ is an e-core, so $v + w \leqslant e$. So we may calculate $w(\boldsymbol{\lambda})$ easily: we have $c_{Q_1}(\boldsymbol{\lambda}) = c_{Q_2}(\boldsymbol{\lambda}) = \min\{v, w\}$ while $(c_f(\boldsymbol{\lambda}) - c_{qf}(\boldsymbol{\lambda}))$ equals $+1$ for exactly $\min\{v, w\}$ values of f, equals -1 for another $\min\{v, w\}$ value of f, and equals 0 otherwise. So

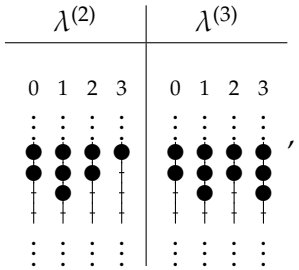
$$w(\boldsymbol{\lambda}) = \min\{v, w\}.$$

On the other hand, we may calculate

$$\gamma_j(\boldsymbol{\lambda}) = \begin{cases} C+1 & (j = i-v, \ldots, i-1) \\ C-1 & (j = i, \ldots, i+w-1) \\ C & (\text{otherwise}) \end{cases}$$

for some constant C (depending on the values of a_1, a_2), and the result follows. □

Example. Returning to the example $\boldsymbol{\lambda} = ((1^2), (2), (2,1))$ after Proposition 3.5, we calculate the weight of $((\lambda^{(2)}, \lambda^{(3)}))$ when $e = 4$. An abacus display for this bicore is



from which we can read off $(\gamma_0, \gamma_1, \gamma_2, \gamma_3) = (0, 0, 0, -2)$. So $\gamma_i - \gamma_j \leqslant 2$ for all i, j, and the sets I, J defined in Proposition 3.8 are

$$I = \{0, 1, 2\}, \qquad J = \{3\}.$$

Hence $w((\lambda^{(2)}, \lambda^{(3)})) = 1$.

3.5 The weight of a multipartition is non-negative

Corollary 3.4 and Propositions 3.5 and 3.8 give an us an algorithm for computing the weight of a multipartition recursively from its abacus display. This may be summarised as follows.

1. If e is finite, slide all the beads up their runners as far as they will go, and calculate the weight change using Corollary 3.4.
2. For each $k < l$, calculate the weight of the bicore $\boldsymbol{\lambda}_{kl} = (\lambda^{(k)}, \lambda^{(l)})$:
   (a) calculate the integers $\gamma_i(\boldsymbol{\lambda}_{kl})$;
   (b) if $\gamma_i(\boldsymbol{\lambda}_{kl}) - \gamma_j(\boldsymbol{\lambda}_{kl}) \geqslant 3$ for any i, j, replace $\boldsymbol{\lambda}_{kl}$ with $s_{ij}(\boldsymbol{\lambda}_{kl})$ (and calculate the weight change using Lemma 3.7); repeat until $s_{ij}(\boldsymbol{\lambda}_{kl}) - s_{ij}(\boldsymbol{\lambda}_{kl}) \leqslant 2$ for all i, j;
   (c) now calculate $w(\boldsymbol{\lambda}_{kl})$ using Proposition 3.8.
3. Finally, add together all the $w(\boldsymbol{\lambda}_{kl})$ and appeal to Proposition 3.5.

This enables us to prove the following result, which gives us further reassurance that our definition of weight is an appropriate generalisation of the weight of a partition.

Corollary 3.9. *For any multipartition $\boldsymbol{\lambda}$, we have $w(\boldsymbol{\lambda}) \geqslant 0$.*

Proof. By Proposition 3.8, any bicore $\boldsymbol{\lambda}$ with $\gamma_i(\boldsymbol{\lambda}) - \gamma_j(\boldsymbol{\lambda}) \leqslant 2$ for all i, j has non-negative weight. Lemma 3.7 then guarantees that any bicore has non-negative weight. Proposition 3.5 then shows that any multicore has non-negative weight, and then Corollary 3.4 then shows that the same is true of any multipartition. □

4 Blocks of small weight

In this section, we examine blocks of $\mathcal{H}_n$ of weight 0 and 1; we describe the partitions and simple modules in a combinatorial block of weight 0 or 1, and calculate the decomposition numbers. As a consequence, we find that the converse of Theorem 1.5 holds for blocks of weight 0 and 1; that is, a combinatorial block is a block. The behaviour of blocks of weight 0 and 1 is analogous to the case $r = 1$, except that, as we shall see, the number of partitions in a block of weight 1 is not always e.

We continue to assume that the parameter set $\{Q_1, \dots, Q_r\}$ is q-connected – the corresponding results when it is not follow easily from the discussion at the start of Section 3.

4.1 Blocks of weight 0

For blocks of weight zero, we prove the following theorem, which is well known in the case $r = 1$ (that is, for the Iwahori–Hecke algebras).

Theorem 4.1. *Let B be a combinatorial block of $\mathcal{H}_n$. Then B contains exactly one multipartition if and only if it has weight 0.*

Note that Theorem 4.1 implies that every multipartition of weight 0 is Kleshchev, which is not at all obvious from the definitions. Theorem 4.1 also implies that a combinatorial block of $\mathcal{H}_n$ of weight 0 is also a block of $\mathcal{H}_n$, and that this block is simple. The same statement applies for the cyclotomic q-Schur algebra $\mathcal{S}_n$.

Proof.

($\Leftarrow$) We use induction on n, with the case $n = 0$ being trivial. Assuming $n > 0$, suppose that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are partitions of weight 0 lying in the same combinatorial block. The fact that $n > 0$ guarantees that $[\boldsymbol{\lambda}]$ has at least one removable i-node, for some i. Take such an i, and suppose $[\boldsymbol{\lambda}]$ has t removable i-nodes and s addable i-nodes; let $\boldsymbol{\lambda}^-$ be the partition obtained by removing all the removable i-nodes. Then Lemma 3.6 implies that

$$w(\boldsymbol{\lambda}^-) = w(\boldsymbol{\lambda}) + t(t - s - t) = -st,$$

and so by Corollary 3.9, we must have $s = 0$, whence $t = \delta_i(\boldsymbol{\lambda})$ and $w(\boldsymbol{\lambda}^-) = 0$.

Now suppose $[\boldsymbol{\mu}]$ has v removable i-nodes and u addable i-nodes, and define $\boldsymbol{\mu}^-$ by removing all the removable i-nodes. As above, if $uv > 0$ then we find that $w(\boldsymbol{\mu}^-) < 0$, so we must have u or v equal to 0. But $v - u = \delta_i(\boldsymbol{\mu}) = \delta_i(\boldsymbol{\lambda})$ by Proposition 3.2, and so we have $v = \delta_i(\boldsymbol{\lambda})$, $u = 0$. So $\boldsymbol{\lambda}^-$ and $\boldsymbol{\mu}^-$ both lie in the same combinatorial block; they both have weight 0, and so $\boldsymbol{\lambda}^- = \boldsymbol{\mu}^-$ by induction. $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ can easily be recovered from $\boldsymbol{\lambda}^-$ and $\boldsymbol{\mu}^-$, and we find that $\boldsymbol{\lambda} = \boldsymbol{\mu}$.

($\Rightarrow$) Suppose $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ is a multipartition of positive weight; we shall construct a different multipartition in the same combinatorial block as $\boldsymbol{\lambda}$.

If $\boldsymbol{\lambda}$ is not a multicore, then some bead in the abacus display for $\boldsymbol{\lambda}$ can be moved up one space. This gives a multipartition of $n - e$ with the same hub as $\boldsymbol{\lambda}$. Now we can move a different bead down one space to get a multipartition of n distinct from $\boldsymbol{\lambda}$ but with the same hub, and we are done.

So we assume that $\boldsymbol{\lambda}$ is a multicore. By Proposition 3.5, we then have $w((\lambda^{(k)}, \lambda^{(l)})) > 0$ for some k, l (where $w((\lambda^{(k)}, \lambda^{(l)}))$ is calculated using the cyclotomic parameters Q_k, Q_l). If we can construct some other bipartition $(\mu^{(k)}, \mu^{(l)})$ in the same combinatorial block as $(\lambda^{(k)}, \lambda^{(l)})$, then the multipartition

$$\boldsymbol{\mu} = (\lambda^{(1)}, \ldots, \lambda^{(k-1)}, \mu^{(k)}, \lambda^{(k+1)}, \ldots, \lambda^{(l-1)}, \mu^{(l)}, \lambda^{(l+1)}, \ldots, \lambda^{(r)})$$

will be distinct from $\boldsymbol{\lambda}$ but will lie in the same combinatorial block. Hence it suffices to consider the case $r = 2$.

Assuming $r = 2$, we define the integers $\gamma_i(\boldsymbol{\lambda})$ as in Section 3. Since $\boldsymbol{\lambda}$ has positive weight, we must have $\gamma_i(\boldsymbol{\lambda}) - \gamma_j(\boldsymbol{\lambda}) \geqslant 2$ for some i, j. We define the bipartition $s_{ij}(\boldsymbol{\lambda})$ as in the last section. By Lemma 3.7, $s_{ij}(\boldsymbol{\lambda})$ has the same hub as $\boldsymbol{\lambda}$, and weight $w(\boldsymbol{\lambda}) - 2(\gamma_i(\boldsymbol{\lambda}) - \gamma_j(\boldsymbol{\lambda}) - 2)$. Now we take any runner in the abacus display for $s_{ij}(\boldsymbol{\lambda})$, and move the lowest bead down $\gamma_i(\boldsymbol{\lambda}) - \gamma_j(\boldsymbol{\lambda}) - 2$ spaces. The resulting bipartition $(\mu^{(1)}, \mu^{(2)})$ has the same weight and hub as $\boldsymbol{\lambda}$. □

4.2 Blocks of weight 1

Weight 1 blocks of Iwahori–Hecke algebras are very well understood, and their properties may be summarised in the following theorem.

Theorem 4.2. *Suppose $r = 1$, and that B is a block of $\mathcal{H}_n$ of weight 1. Then $e < \infty$ and B contains exactly e partitions, totally ordered by dominance: $\lambda^{(1)} \lhd \ldots \lhd \lambda^{(e)}$. $\lambda^{(x)}$ is e-restricted if and only if $x < e$, and the decomposition number $[S^{\lambda^{(x)}} : D^{\lambda^{(y)}}]$ equals 1 if $x = y$ or $y + 1$, and 0 otherwise.*

We shall see that a corresponding result is true for arbitrary r, except that e need not be finite, and the number of multipartitions in B need not equal e.

4.2.1 The set of multipartitions in a combinatorial block of weight 1

To begin with, we need to describe the multipartitions in a given combinatorial block of weight 1. Given Theorem 4.2, we may assume for the rest of this section that $r \geqslant 2$. From Corollary 3.4, we can see that all multipartitions of weight 1 are multicores. The next result now follows from Proposition 3.5 and Proposition 3.8.

Lemma 4.3. *Suppose $\boldsymbol{\lambda}$ is a multipartition of weight 1. Then there are unique $1 \leqslant k_\lambda < l_\lambda \leqslant r$ such that, for $i < j$,*

$$w((\lambda^{(i)}, \lambda^{(j)})) = \begin{cases} 1 & (i = k_\lambda, j = l_\lambda) \\ 0 & (\textit{otherwise}). \end{cases}$$

Writing $\hat{\boldsymbol{\lambda}} = (\lambda^{(k_\lambda)}, \lambda^{(l_\lambda)})$ and calculating the integers $\gamma_i(\hat{\boldsymbol{\lambda}})$ as above, there exist unique $V_\lambda, W_\lambda \subset \mathbb{Z}/e\mathbb{Z}$ such that

$$\gamma_i(\hat{\boldsymbol{\lambda}}) - \gamma_j(\hat{\boldsymbol{\lambda}}) \begin{cases} = 2 & (i \in V_\lambda, j \in W_\lambda) \\ \leqslant 1 & (\textit{otherwise}) \end{cases}$$

and either $|V_\lambda| = 1 \leqslant |W_\lambda|$ or $|V_\lambda| \geqslant 1 = |W_\lambda|$.

For $v \in V_\lambda, w \in W_\lambda$ we define $s_{vw}(\hat{\boldsymbol{\lambda}})$ as in §3.4, writing

$$s_{vw}(\hat{\boldsymbol{\lambda}}) = (\lambda^{(k_\lambda)}_{(vw)}, \lambda^{(l_\lambda)}_{(vw)}).$$

Now we write

$$\boldsymbol{\lambda}_{vw} = (\lambda^{(1)}, \dots, \lambda^{(k_\lambda - 1)}, \lambda^{(k_\lambda)}_{(vw)}, \lambda^{(k_\lambda + 1)}, \dots, \lambda^{(l_\lambda - 1)}, \lambda^{(l_\lambda)}_{(vw)}, \lambda^{(l_\lambda + 1)}, \dots, \lambda^{(r)}).$$

By Lemma 3.7, $s_{vw}(\hat{\boldsymbol{\lambda}})$ has the same weight and hub as $\hat{\boldsymbol{\lambda}}$, and so $\boldsymbol{\lambda}_{vw}$ has the same weight and hub as $\boldsymbol{\lambda}$. Now we can describe the set of multipartitions in a combinatorial block of weight 1.

Theorem 4.4. *Suppose $\boldsymbol{\lambda}$ is a multipartition in a combinatorial block B of $\mathcal{H}_n$ of weight 1, and define $\boldsymbol{\lambda}_{vw}$ for $v \in V_\lambda, w \in W_\lambda$ as above. Then the set of multipartitions in B is precisely*

$$\{\boldsymbol{\lambda}\} \cup \{\boldsymbol{\lambda}_{vw} \mid v \in V_\lambda, w \in W_\lambda\}.$$

We shall prove Theorem 4.4 by induction on n. First we want to show that we can replace $\boldsymbol{\lambda}$ with any of the $\boldsymbol{\lambda}_{vw}$.

Lemma 4.5. *Suppose $\boldsymbol{\lambda}$ is a multipartition of weight 1, and define the multipartitions $\boldsymbol{\lambda}_{vw}$ as above. Then, for any v, w, Theorem 4.4 holds for $\boldsymbol{\lambda}$ if and only if it holds for $\boldsymbol{\lambda}_{vw}$.*

Proof. $\boldsymbol{\lambda}_{vw}$ has weight 1, and so we may calculate the data $k_{\boldsymbol{\lambda}_{vw}}, l_{\boldsymbol{\lambda}_{vw}}, V_{\boldsymbol{\lambda}_{vw}}, W_{\boldsymbol{\lambda}_{vw}}$ defined by Lemma 4.3. We have $w((\lambda^{(k_\lambda)}_{(vw)}, \lambda^{(l_\lambda)}_{(vw)})) > 0$, so we must have $k_{\boldsymbol{\lambda}_{vw}} = k_\lambda$, $l_{\boldsymbol{\lambda}_{vw}} = l_\lambda$. By examining the differences between the values $\gamma_i(s_{vw}(\hat{\boldsymbol{\lambda}}))$ and $\gamma_i(\hat{\boldsymbol{\lambda}})$ using Lemma 3.7, we find

$$V_{\boldsymbol{\lambda}_{vw}} = V_\lambda \cup \{w\} \setminus \{v\}, \qquad W_{\boldsymbol{\lambda}_{vw}} = W_\lambda \cup \{v\} \setminus \{w\}.$$

Furthermore, for $x \in V_{\boldsymbol{\lambda}_{vw}}, y \in W_{\boldsymbol{\lambda}_{vw}}$,

$$s_{xy}(s_{vw}(\hat{\boldsymbol{\lambda}})) = \begin{cases} s_{vy}(\hat{\boldsymbol{\lambda}}) & (x = w,\ y \neq v) \\ s_{xw}(\hat{\boldsymbol{\lambda}}) & (x \neq w,\ y = v) \\ \hat{\boldsymbol{\lambda}} & (x = w,\ y = v). \end{cases}$$

Hence the set

$$\{\boldsymbol{\lambda}_{vw}\} \cup \{(\boldsymbol{\lambda}_{vw})_{xy} \mid x \in V_{\boldsymbol{\lambda}_{vw}}, y \in W_{\boldsymbol{\lambda}_{vw}}\}$$

equals the set

$$\{\boldsymbol{\lambda}\} \cup \{\boldsymbol{\lambda}_{xy} \mid x \in V_\lambda, y \in W_\lambda\}.$$

□

We also need the following more powerful inductive device, analogous to the 'Scopes bijections' [19] for Iwahori–Hecke algebras. Recall that two multipartitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$ of n lie in the same combinatorial block if and only if $\delta_i(\boldsymbol{\lambda}) = \delta_i(\boldsymbol{\mu})$ for all i; accordingly, for a combinatorial block B we may define $\delta_i(B)$ to equal $\delta_i(\boldsymbol{\lambda})$ for any $\boldsymbol{\lambda}$ in B. For any multipartition $\boldsymbol{\lambda}$, define $\Phi_i(\boldsymbol{\lambda})$ by replacing any beta-number $\beta_k^{(j)}$ for $\boldsymbol{\lambda}$ congruent to i modulo e with $\beta_k^{(j)} - 1$, and replacing any $\beta_k^{(j)}$ congruent to $i-1$ modulo e with $\beta_k^{(j)} + 1$. Informally, $\Phi_i(\boldsymbol{\lambda})$ is obtained from $\boldsymbol{\lambda}$ by simultaneously removing all removable i-nodes and adding all addable i-nodes, or by swapping the $(i-1)$th and ith runners of each abacus in the abacus display for $\boldsymbol{\lambda}$.

Proposition 4.6. *Suppose B is a combinatorial block of $\mathcal{H}_n$, and $i \in \mathbb{Z}/e\mathbb{Z}$. Then there is a combinatorial block C of $\mathcal{H}_{n-\delta_i(B)}$ with*

$$\delta_j(C) = \delta_j(B) + \delta_i(B)(\mathbb{1}[j \equiv i-1 \pmod e] + \mathbb{1}[j \equiv i+1 \pmod e] - 2\mathbb{1}[j \equiv i \pmod e])$$

for each j. C has the same weight as B, and Φ_i gives a bijection between the set of multipartitions in B and the set of multipartitions in C.

Proof. Suppose $\boldsymbol{\lambda} \in B$. Then, using the coefficients $c_f(\boldsymbol{\lambda})$ from Section 1.1, we have

$$c_f(\Phi_i(\boldsymbol{\lambda})) = \begin{cases} c_{q^i}(\boldsymbol{\lambda}) - \delta_i(\boldsymbol{\lambda}) & (f = q^i) \\ c_f(\boldsymbol{\lambda}) & (\text{otherwise}). \end{cases}$$

Lemma 3.1 then gives

$$\delta_j(\Phi_i(\boldsymbol{\lambda})) = \delta_j(\boldsymbol{\lambda}) + \delta_i(\boldsymbol{\lambda})(\mathbb{1}[j \equiv i-1 \pmod e] + \mathbb{1}[j \equiv i+1 \pmod e] - 2\mathbb{1}[j \equiv i \pmod e])$$

for each j. So $\Phi_i(\boldsymbol{\lambda})$ lies in C. $\Phi_i(\lambda)$ has the same weight as $\boldsymbol{\lambda}$, by Lemma 3.6. By interchanging the rôles of B and C, we find that Φ_i also maps the set of multipartitions in C to the set of multipartitions in B. Φ_i is also clearly an involution, so we are done. □

We apply this to proving Theorem 4.4.

Lemma 4.7. *Suppose that $\boldsymbol{\lambda}$ is a multipartition of weight 1, and $i \in \mathbb{Z}/e\mathbb{Z}$. Then Theorem 4.4 holds for $\boldsymbol{\lambda}$ if and only if it holds for $\Phi_i(\boldsymbol{\lambda})$.*

Proof. We calculate the data specified by Lemma 4.3 for $\Phi_i(\boldsymbol{\lambda})$. Defining $\phi_i : \mathbb{Z}/e\mathbb{Z} \to \mathbb{Z}/e\mathbb{Z}$ to be the involution given by

$$j \longmapsto \begin{cases} i & (j = i-1) \\ i-1 & (j = i) \\ j & (\text{otherwise}), \end{cases}$$

we find

$$k_{\Phi_i(\boldsymbol{\lambda})} = k_{\boldsymbol{\lambda}}, \quad l_{\Phi_i(\boldsymbol{\lambda})} = l_{\boldsymbol{\lambda}}, \quad V_{\Phi_i(\boldsymbol{\lambda})} = \{\phi_i(v) \mid v \in V_{\boldsymbol{\lambda}}\}, \quad W_{\Phi_i(\boldsymbol{\lambda})} = \{\phi_i(w) \mid w \in W_{\boldsymbol{\lambda}}\}.$$

We obtain

$$\Phi_i(\boldsymbol{\lambda}_{vw}) = (\Phi_i(\boldsymbol{\lambda}))_{\phi_i(v),\phi_i(w)}$$

for $v \in V_{\boldsymbol{\lambda}}, w \in W_{\boldsymbol{\lambda}}$, and so Φ_i gives a bijection between

$$\{\boldsymbol{\lambda}\} \cup \{\boldsymbol{\lambda}_{vw} \mid v \in V_{\boldsymbol{\lambda}}, w \in W_{\boldsymbol{\lambda}}\}$$

and

$$\{\Phi_i(\boldsymbol{\lambda})\} \cup \{(\Phi_i(\boldsymbol{\lambda}))_{vw} \mid v \in V_{\Phi_i(\boldsymbol{\lambda})}, w \in W_{\Phi_i(\boldsymbol{\lambda})}\}.$$

□

We can now prove Theorem 4.4; the structure of the proof is the same as for Proposition 3.8.

Proof of Theorem 4.4. Given $\boldsymbol{\lambda}$, write $k = k_{\boldsymbol{\lambda}}, l = l_{\boldsymbol{\lambda}}$. We proceed by induction on n and on $|\lambda^{(k)}|$. Note that if $\delta_i(B) > 0$ for any i, then we may replace B with a combinatorial block of $\mathcal{H}_{n-\delta_i(B)}$ using Proposition 4.6 and appeal to the inductive hypothesis. So we may assume that $\delta_i(B) \leqslant 0$ for each i.

Suppose that $\lambda^{(m)} \neq \varnothing$ for some $m \neq k, l$. Then $[\lambda^{(m)}]$ has a removable i-node for some i. Since $\delta_i(B) \leqslant 0$, some $[\lambda^{(p)}]$ must have an addable i-node. But then $(\lambda^{(m)}, \lambda^{(p)})$ has positive weight by Lemma 3.7 and Proposition 3.8; contradiction. So our assumption means that $\lambda^{(m)} = \varnothing$ for all $m \neq k, l$.

Next suppose that $\lambda^{(k)} \neq \varnothing$. Then $[\lambda^{(k)}]$ has a removable i-node for some i. Arguing as above, we find that $[\lambda^{(l)}]$ must have an addable i-node. Supposing that $[\lambda^{(k)}]$ has t removable i-nodes and $[\lambda^{(l)}]$ has s addable i-nodes, we find that

$$\gamma_i(\hat{\boldsymbol{\lambda}}) - \gamma_{i-1}(\hat{\boldsymbol{\lambda}}) = s + t.$$

Lemma 3.7 then implies that $\hat{\boldsymbol{\lambda}}$ has weight at least $2(s+t-2)$; since we are assuming that $w(\hat{\boldsymbol{\lambda}}) = 1$, we must have $s = t = 1$. So we have $i \in V_{\boldsymbol{\lambda}}$, $i-1 \in W_{\boldsymbol{\lambda}}$, and $s_{i(i-1)}(\boldsymbol{\lambda})$ is obtained from $\boldsymbol{\lambda}$ by removing the removable i-node from $[\lambda^{(k)}]$ and adding the addable i-node to $[\lambda^{(l)}]$. By Lemma 4.5 we may replace $\boldsymbol{\lambda}$ with $s_{i(i-1)}(\boldsymbol{\lambda})$, and we are done by induction on $|\lambda^{(k)}|$.

We are left with case where $\lambda^{(m)} = \varnothing$ for all $m \neq l$. We cannot have $\lambda^{(l)} = \varnothing$ (since then we should get $w(\boldsymbol{\lambda}) = 0$), so $[\lambda^{(l)}]$ has a removable i-node, for some i. Again, the fact that $\delta_i(\boldsymbol{\lambda}) \leqslant 0$ implies that $[\lambda^{(m)}]$ has an addable i-node for some $m \neq l$; that is, $Q_m = q^i$. We then get $w((\lambda^{(l)}, \lambda^{(m)})) > 0$, and so in fact $m = k$. So we find that $[\lambda^{(l)}]$ has removable nodes of a unique residue, namely Q_k, and in fact since $\delta_i(\boldsymbol{\lambda}) \leqslant 0$, there can only be one removable node. So $[\lambda^{(l)}]$ is a rectangular partition, say $\lambda^{(l)} = (a^b)$, with $Q_k = q^i$ and $Q_l = q^{i+b-a}$.

Letting $c = \min\{a, b\}$ and examining the residues of the nodes of $[\boldsymbol{\lambda}]$, we find that $c_{q^i}(\boldsymbol{\lambda}) = c_{q^{i+b-a}}(\boldsymbol{\lambda}) = c$, and that $(c_f(\boldsymbol{\lambda}) - c_{qf}(\boldsymbol{\lambda}))^2$ equals 1 for precisely $2c$ values of f, and 0 for all remaining values (note that since $\lambda^{(l)}$ is a core, we have $a + b \leqslant e$). So the fact that $w(\boldsymbol{\lambda}) = 1$ implies that $c = 1$ and $c_{Q_m}(\boldsymbol{\lambda}) = 0$ for $m \neq k, l$. Hence we find that there is some $0 < t < e$ such that either

1. $\lambda^{(l)} = (1^t)$, $Q_l = q^{i+t-1}$ and $[\boldsymbol{\lambda}]$ contains exactly one node of each of the residues $q^i, q^{i+1}, \ldots, q^{i+t-1}$ (none of which equals Q_m for any $m \neq k, l$)

or

2. $\lambda^{(l)} = (t)$, $Q_l = q^{i-t+1}$ and $[\boldsymbol{\lambda}]$ contains exactly one node of each of the residues $q^{i-t+1}, q^{i-t+2}, \ldots, q^i$ (none of which equals Q_m for any $m \neq k, l$).

We assume that we are in case (1), the other case being similar. By examining possible Young diagrams, we then easily find that the multipartitions in B are precisely the multipartitions $\boldsymbol{\mu}(g) = (\mu^{(1)}(g), \ldots, \mu^{(r)}(g))$ given by

$$\mu^{(m)}(g) = \begin{cases} (g) & (m = k) \\ (1^{t-g}) & (m = l) \\ \varnothing & (\text{otherwise}), \end{cases}$$

for $g = 0, \ldots, t$, where $\boldsymbol{\lambda} = \boldsymbol{\mu}(0)$. On the other hand, it is easy to calculate

$$\gamma_j(\hat{\boldsymbol{\lambda}}) = \begin{cases} C+1 & (j = i-1) \\ C-1 & (j = i, i+1, \ldots, i+t-1) \\ C & (\text{otherwise}) \end{cases}$$

for some constant C, so we find that $V_{\boldsymbol{\lambda}} = \{i-1\}$ and $W_{\boldsymbol{\lambda}} = \{i, \ldots, i+t-1\}$. We have $\boldsymbol{\lambda} = \boldsymbol{\mu}(0)$ and we find that $\boldsymbol{\lambda}_{(i-1)(i+x)} = \boldsymbol{\mu}(x+1)$ for $x = 0, \ldots, t-1$, and the result follows. □

We next prove that the multipartitions in a combinatorial block of weight 1 are totally ordered by dominance. Theorem 4.4 provides a description of the multipartitions in a weight 1 combinatorial block, but we want to make this independent of the choice of $\boldsymbol{\lambda}$. Given a weight 1 combinatorial block B containing a multipartition $\boldsymbol{\lambda}$, observe from the proof of Lemma 4.5 that k_λ, l_λ, $|V_\lambda|$, $|W_\lambda|$ and $X = V_\lambda \cup W_\lambda$ are independent of $\boldsymbol{\lambda}$; accordingly, we write these as k_B, l_B, v_B, w_B, X_B. Our description of the multipartitions in B will depend upon which of v_B or w_B equals 1; if both equal 1, then both descriptions apply.

- If $v_B = 1$, then we choose $\boldsymbol{\lambda}$ in B, write $V_\lambda = \{y\}$ and for $x \in X_B$ define

$$\boldsymbol{\nu}_x = \begin{cases} \boldsymbol{\lambda}_{vx} & (x \in W_\lambda) \\ \boldsymbol{\lambda} & (x = y). \end{cases}$$

  Then by the proof of Lemma 4.5, $\boldsymbol{\nu}_x$ is independent of the choice of $\boldsymbol{\lambda}$. We shall write $\boldsymbol{\nu}_x(B)$ when there is a danger of ambiguity.

- If $w_B = 1$, then we choose $\boldsymbol{\lambda}$ in B, write $W_\lambda = \{z\}$ and for $x \in X_B$ define

$$\boldsymbol{\xi}_x = \begin{cases} \boldsymbol{\lambda}_{xw} & (x \in V_\lambda) \\ \boldsymbol{\lambda} & (x = z); \end{cases}$$

  again, $\boldsymbol{\xi}_x$ is independent of the choice of $\boldsymbol{\lambda}$.

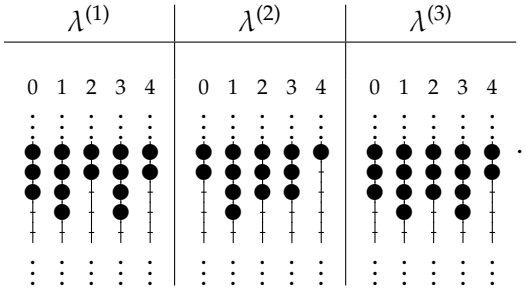
Lemma 4.8. *Suppose B is a combinatorial block of weight* 1.

1. *If $v_B = 1$, define $\boldsymbol{\nu}_x = (\nu_x^{(1)}, \dots, \nu_x^{(r)})$ for $x \in X_B$ as above, and let $\beta^B(x)$ be the largest beta-number of $\nu_x^{(k_B)}$ which is congruent to x modulo e. Then $\boldsymbol{\nu}_x \rhd \boldsymbol{\nu}_y$ if and only if $\beta^B(x) \geqslant \beta^B(y)$.*

2. *If $w_B = 1$, define $\boldsymbol{\xi}_x = (\xi_x^{(1)}, \dots, \xi_x^{(r)})$ for $x \in X_B$ as above, and let $\beta^B(x)$ be the largest beta-number of $\xi_x^{(l_B)}$ which is congruent to x modulo e. Then $\boldsymbol{\xi}_x \rhd \boldsymbol{\xi}_y$ if and only if $\beta^B(x) \leqslant \beta^B(y)$.*

Proof. Suppose that $v_B = 1$; the other case is similar. In order to change $\boldsymbol{\nu}_x$ into $\boldsymbol{\nu}_y$, we move a bead from runner x to runner y in the abacus for $\nu_x^{(k_B)}$, and we move a bead from runner y to runner x in the abacus for $\nu_x^{(l)}$. Moving a single bead corresponds to either increasing or decreasing a beta-number, which corresponds to adding or removing a rim hook from the Young diagram, respectively. Since $k_B < l_B$, we have $\boldsymbol{\nu}_x \rhd \boldsymbol{\nu}_y$ if we remove a rim hook from $\nu_x^{(k_B)}$ (and add a rim hook to $\nu_x^{(l)}$), or $\boldsymbol{\nu}_x \lhd \boldsymbol{\nu}_y$ if we add a rim hook to $\nu_x^{(k_B)}$ and remove a rim hook from $\nu_x^{(l_B)}$. Now the bead we move in the abacus for $\nu_x^{(k_B)}$ is moved from position $\beta^B(x)$ to position $\beta^B(y)$, and so the corresponding beta-number is reduced if and only if $\beta^B(x) \geqslant \beta^B(y)$. □

Example. Suppose $e = 5, r = 3$ and $(Q_1, Q_2, Q_3) = (1, q^3, q)$. Let $\boldsymbol{\lambda}$ be the multipartition $((4,3,1),(4,2^3),(3,2))$, which has an abacus display



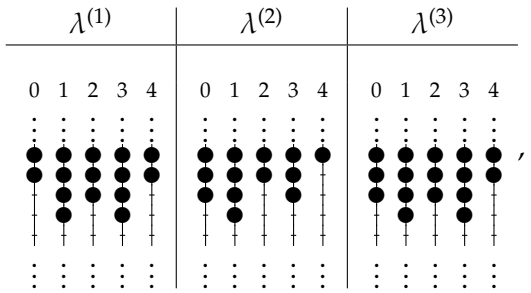
$\boldsymbol{\lambda}$ has weight 1, and

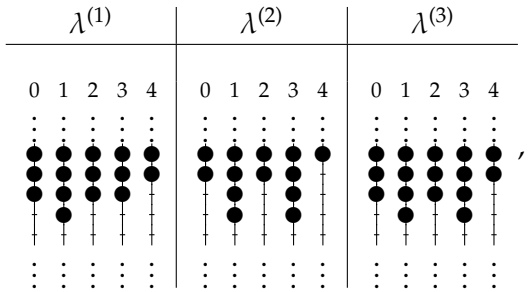
$$k_\lambda = 1, \quad l_\lambda = 2, \quad V_\lambda = \{0,3,4\}, \quad W_\lambda = \{2\}.$$

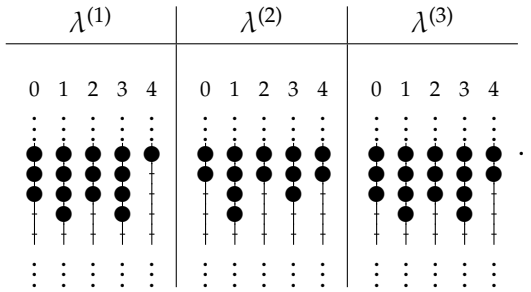
So the multipartitions in the same combinatorial block as $\boldsymbol{\lambda}$ are

$$\xi_3 \lhd \xi_2 \lhd \xi_0 \lhd \xi_4,$$

where $\xi_2 = \boldsymbol{\lambda}$ and

$\xi_0 =$ 

$\xi_3 =$ 

$\xi_4 =$ 

4.2.2 Decomposition numbers for blocks of weight 1

We are now in a position to prove a generalisation of Theorem 4.2. We begin with the combinatorial blocks discussed at the end of the proof of Theorem 4.4. Assume that for some $1 \leqslant k < l \leqslant r$ and some $1 \leqslant t < e$ we have $Q_l = q^{t-1}Q_k$, and

$$Q_m \notin \{Q_k, qQ_k, \ldots, q^{t-1}Q_k\}$$

for $m \neq k, l$. We then let B_0 be the combinatorial block in which all multipartitions $\boldsymbol{\mu}$ have

$$c_f(\boldsymbol{\mu}) = \mathbb{1}[f \in \{Q_k, qQ_k, \ldots, q^{t-1}Q_k\}].$$

So B_0 contains all of the multipartitions $\boldsymbol{\mu}(g)$ defined in the proof of Theorem 4.4.

Lemma 4.9.

1. *$\boldsymbol{\mu}(g)$ is Kleshchev if and only if $g < t$.*
2. *$\boldsymbol{\mu}(g) \trianglerighteq \boldsymbol{\mu}(h)$ if and only if $g \geqslant h$.*
3. *The decomposition number $[S^{\boldsymbol{\mu}(g)} : D^{\boldsymbol{\mu}(h)}]$ equals 1 if $g = h$ or $h + 1$, and 0 otherwise.*

Proof.

1. $[\boldsymbol{\mu}(t)]$ has only one removable node, which is not good, and so $\boldsymbol{\mu}(t)$ cannot be Kleshchev. For $g < t$, we can remove the nodes of $\mu^{(l)}(g)$ from bottom to top, and then remove the nodes of $\mu^{(k)}(g)$ from right to left; the removed node at each stage will be good, and we reach the empty multipartition.

2. This is obvious.

3. The decomposition numbers may be calculated using the cyclotomic Jantzen–Schaper theorem [13], or (in infinite characteristic) by calculating canonical basis vectors in the Fock space for $U_v(\widehat{\mathfrak{sl}}_e)$ and appealing to Ariki's theorem [1]. Although applying these results is normally quite cumbersome, it turns out to be fairly straightforward for these multipartitions. We do not include the details here. □

By entirely similar arguments, we obtain the same result for the combinatorial blocks in the other 'initial case' of Theorem 4.4.

Lemma 4.10. *Suppose that for some $1 \leqslant k < l \leqslant r$ and some $1 \leqslant t < e$ we have $Q_k = q^{t-1}Q_l$, and $Q_m \notin \{Q_l, qQ_l, \dots, q^{t-1}Q_l\}$ for $m \neq k, l$. Let B be the combinatorial block of $\mathcal{H}_t$ in which all multipartitions $\boldsymbol{\mu}$ satisfy*

$$c_f(\boldsymbol{\mu}) = \mathbb{1}[f \in \{Q_l, qQ_l, \dots, q^{t-1}Q_l\}].$$

Then there are $t+1$ multipartitions in B, which are totally ordered by dominance:

$$\boldsymbol{\mu}(0) \lhd \dots \lhd \boldsymbol{\mu}(t).$$

$\boldsymbol{\mu}(g)$ is Kleshchev if and only if $g < t$, and the decomposition number $[S^{\boldsymbol{\mu}(g)} : D^{\boldsymbol{\mu}(h)}]$ equals 1 if $g = h$ or $h+1$, and 0 otherwise.

Now we prove an inductive step towards our main result.

Proposition 4.11. *Suppose that B is a combinatorial block of $\mathcal{H}_n$ of weight 1, and that $\delta_i(B) > 0$ for some i. Let C be the combinatorial block of $\mathcal{H}_{n-\delta_i(B)}$ defined in Proposition 4.6.*

1. *If $\boldsymbol{\lambda}$ is a multipartition in B then $[\boldsymbol{\lambda}]$ has exactly $\delta_i(B)$ removable i-nodes, and no addable i-nodes.*

2. *$\Phi_i(\boldsymbol{\lambda})$ is obtained by removing all the removable i-nodes, and $\Phi_i(\boldsymbol{\lambda})$ is Kleshchev if and only if $\boldsymbol{\lambda}$ is.*

3. *There exists a bijection σ between the set of Kleshchev multipartitions in B and the set of Kleshchev multipartitions in C such that*

$$S^{\boldsymbol{\lambda}}\downarrow_C^B \sim \delta_i(B)!S^{\Phi_i(\boldsymbol{\lambda})}, \qquad S^{\Phi_i(\boldsymbol{\lambda})}\uparrow_C^B \sim \delta_i(B)!S^{\boldsymbol{\lambda}},$$
$$D^{\boldsymbol{\mu}}\downarrow_C^B \sim \delta_i(B)!D^{\sigma(\boldsymbol{\mu})}, \qquad D^{\sigma(\boldsymbol{\mu})}\uparrow_C^B \sim \delta_i(B)!D^{\boldsymbol{\mu}}$$

   and

$$[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}] = [S^{\Phi_i(\boldsymbol{\lambda})} : D^{\sigma(\boldsymbol{\mu})}]$$

   for any multipartition $\boldsymbol{\lambda}$ and any Kleshchev multipartition $\boldsymbol{\mu}$ in B.

4. *Φ_i preserves the dominance order of multipartitions in B.*

Proof.

1. $\delta_i(B)$ is the number of removable i-nodes of $[\boldsymbol{\lambda}]$ minus the number of addable i-nodes of $[\boldsymbol{\lambda}]$. So if $[\boldsymbol{\lambda}]$ has an addable i-node, then it has at least two removable i-nodes. Since $\boldsymbol{\lambda}$ is a multicore, no $\lambda^{(k)}$ can have both addable and removable i-nodes, so we find that either there are distinct l, m such that $[\lambda^{(l)}]$ has at least two removable i-nodes and $[\lambda^{(m)}]$ has at least one addable i-node, or there are distinct k, l, m such that $[\lambda^{(k)}]$ and $[\lambda^{(l)}]$ each have a removable i-node and $[\lambda^{(m)}]$

has an addable i-node. In the first case, we find that $\gamma_i((\lambda^{(l)},\lambda^{(m)})) - \gamma_{i-1}((\lambda^{(l)},\lambda^{(m)})) \geqslant 3$, which gives $w((\lambda^{(l)},\lambda^{(m)})) \geqslant 2$ by Lemma 3.7, and hence $w(\boldsymbol{\lambda}) \geqslant 2$ by Lemma 3.5; contradiction. In the second case, we find that $\gamma_i((\lambda^{(k)},\lambda^{(m)})) - \gamma_{i-1}((\lambda^{(k)},\lambda^{(m)}))$ and $\gamma_i((\lambda^{(l)},\lambda^{(m)})) - \gamma_{i-1}((\lambda^{(l)},\lambda^{(m)}))$ are both at least 2, giving $w((\lambda^{(k)},\lambda^{(m)})), w((\lambda^{(l)},\lambda^{(m)})) \geqslant 1$ by Lemma 3.7 and Lemma 3.8; again, this contradicts the assumption $w(\boldsymbol{\lambda}) = 1$. So $[\boldsymbol{\lambda}]$ has no addable i-nodes.

2. If the removable i-nodes of $[\boldsymbol{\lambda}]$ are $\mathfrak{n}_1, \ldots, \mathfrak{n}_{\delta_i(B)}$ from top to bottom, define $\boldsymbol{\lambda}(0), \boldsymbol{\lambda}(1), \ldots, \boldsymbol{\lambda}(\delta_i(B))$ recursively by setting $\boldsymbol{\lambda}(0) = \boldsymbol{\lambda}$ and obtaining $\boldsymbol{\lambda}(x)$ from $\boldsymbol{\lambda}(x-1)$ by removing $\mathfrak{n}_x$. Then the q^i-signature of $\boldsymbol{\lambda}(x)$ consists of x plus signs followed by $\delta_i(B) - x$ minus signs, and so $\mathfrak{n}_{x+1}$ is a good node of $\boldsymbol{\lambda}(x)$. Hence by Proposition 1.1 $\boldsymbol{\lambda}$ is Kleshchev if and only if $\boldsymbol{\lambda}(\delta_i(B)) = \Phi_i(\boldsymbol{\lambda})$ is Kleshchev.

3. The equations concerning induction and restriction of Specht modules follow from Theorem 1.4, and the statements about induction and restriction of simple modules and about decomposition numbers may be obtained by mimicking the arguments from [19], using the fact that induction and restriction are exact functors together with Frobenius reciprocity.

4. We suppose $v_B = 1$ (which implies $v_C = 1$); the case where $w_B = 1$ is similar. Recalling the function ϕ_i from the proof of Lemma 4.7, we find from that proof that $X_C = \phi_i(X_B)$, and $\boldsymbol{\nu}_{\phi_i(x)}(C) = \Phi_i(\boldsymbol{\nu}_x)(B)$ for $x \in X_B$. Hence by Lemma 4.8 Φ_i preserves the dominance order of the multipartitions in B if and only if ϕ_i preserves the order of the integers β^B_x, i.e.

$$\left(\beta^B(x) \leqslant \beta^B(y)\right) \iff \left(\beta^C(\phi_i(x)) \leqslant \beta^C(\phi_i(y))\right)$$

for $x \in X_B$. The only way this can fail is if $i-1$ and i are both elements of X_B and $\beta^B(i-1) = \beta^B(i)-1$. But this means that $\nu_x^{(k_B)}$ has exactly one removable i-node. The fact that $i-1$ and i are both in X_B means that $\gamma_i(\widehat{\boldsymbol{\nu}_i(B)}) - \gamma_i(\widehat{\boldsymbol{\nu}_{i-1}(B)}) = 2$, which then implies that $\nu_x^{(l_B)}$ has an addable i-node. But this contradicts part (1) of the present proposition. □

Now we can prove our main result.

Theorem 4.12. *Suppose B is a combinatorial block of $\mathcal{H}_n$ of weight 1. Then the multipartitions in B are*

$$\boldsymbol{\lambda}_1 \lhd \ldots \lhd \boldsymbol{\lambda}_s,$$

for some $s \leqslant e$. $\boldsymbol{\lambda}_x$ is Kleshchev if and only if $x < s$, and the decomposition number $[S^{\boldsymbol{\lambda}_x} : D^{\boldsymbol{\lambda}_y}]$ equals 1 if $x = y$ or $y+1$, and 0 otherwise.

As an immediate consequence, we see that a combinatorial block of weight 1 is a block. A very similar theorem applies to the cyclotomic q-Schur algebra $\mathcal{S}_n$. In that case, there are simple modules $L(\boldsymbol{\lambda}_y)$ for all y, but the decomposition number result is just the same: the multiplicity of $L(\boldsymbol{\lambda}_y)$ as a composition factor of the Weyl module $\Delta(\boldsymbol{\lambda}_x)$ is 1 if $x = y$ or $y+1$, and 0 otherwise. This may be proved by a direct application of the cyclotomic Jantzen–Schaper formula, and in fact it yields a quicker proof of Theorem 4.12 (given the results that $\mathcal{H}_n$ is symmetric and $\mathcal{S}_n$ is quasi-hereditary). We do not reproduce this proof here, but we are grateful to the referee for pointing it out.

Proof of Theorem 4.12. We use induction on n. Suppose first that $\delta_i(B) > 0$ for some i, and let C be the combinatorial block defined in Proposition 4.6. C has weight 1, and by induction we assume that the theorem holds for C. By Theorem 4.4, B contains at most e multipartitions, and by Proposition 4.11 and by induction, all but one of these is Kleshchev. Proposition 4.11 also implies that the multipartitions in B are totally ordered by dominance, say $\boldsymbol{\lambda}_1 \lhd \ldots \lhd \boldsymbol{\lambda}_s$, and that $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{s-1}$ are Kleshchev. Furthermore, there is some permutation τ of $\{1, \ldots, s-1\}$ such that

$$[S^{\boldsymbol{\lambda}_x} : D^{\boldsymbol{\lambda}_y}] = \begin{cases} 1 & (x = \tau(y) \text{ or } \tau(y)+1) \\ 0 & (\text{otherwise}). \end{cases}$$

Now Theorem 1.3 implies that τ is the identity permutation, and we are done.

So we suppose that $\delta_i(B) \leqslant 0$ for all i. But, as in the proof of Theorem 4.4, we find that B must be one of the combinatorial blocks described by Lemmata 4.9 and 4.10, and so the theorem is proved. □

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