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Weight two blocks of Iwahori–Hecke algebras of type B

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Abstract

In the representation theory of Iwahori–Hecke algebras of type A (and in particular for representations of symmetric groups) the notion of the weight of a block, introduced by James, plays a central rôle. Richards determined the decomposition numbers for blocks of weight 2, and here the same task is undertaken for weight two blocks of Iwahori–Hecke algebras of type B, using the author's own definition of the weight of a bipartition.

1 Introduction

1.1 Iwahori–Hecke algebras and Ariki–Koike algebras

Let n be a positive integer, let $\mathbb{F}$ be a field, suppose q, Q_1, Q_2 are elements of $\mathbb{F}$. The *Iwahori–Hecke algebra* $\mathcal{H}_n$ of type B is the unital associative $\mathbb{F}$-algebra with generators $T_0, \dots, T_{n-1}$ and relations

$$\begin{aligned}
(T_i + q)(T_i - 1) &= 0 && (1 \leqslant i \leqslant n-1)\\
(T_0 - Q_1)(T_0 - Q_2) &= 0 &&\\
T_iT_j &= T_jT_i && (0 \leqslant i, j \leqslant n-1, |i-j| > 1)\\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && (1 \leqslant i \leqslant n-2)\\
T_0T_1T_0T_1 &= T_1T_0T_1T_0. &&
\end{aligned}$$

The subalgebra H_n generated by $T_1, \dots, T_{n-1}$ is the Iwahori–Hecke algebra of type A, and both these Iwahori–Hecke algebras are special cases of the *Ariki–Koike algebra*, which corresponds to the complex reflection group $C_r \wr \mathfrak{S}_n$. These Iwahori–Hecke algebras arise in the study of groups with BN-pairs, and their representation theory bears a close relationship to the representation theory of the corresponding Coxeter groups. This relationship has been exploited to great effect in type A: a great deal of the very rich theory of representations of symmetric groups has been generalised to the representation theory of H_n, and in some instances (such as the author's proof [10] that the decomposition numbers for weight 3 blocks of symmetric groups are at most 1) information has passed in the opposite direction, with the representation theory of (particular instances of) H_n answering questions about the symmetric groups.

$\mathcal{H}_n$ is less well studied than H_n; it was first examined by Dipper, James and Murphy [7, 8], who defined Specht modules for $\mathcal{H}_n$ and classified the simple modules. Since then, much of the representation theory of $\mathcal{H}_n$ has been deduced as a special case of the representation theory of the Ariki–Koike algebra. This algebra was introduced by Ariki and Koike [3], and independently by Broué and Malle [6], and various facts are known about it. Ariki gave a necessary and sufficient criterion for the Ariki–Koike algebra to be semi-simple, and described the simple modules in this case. These are indexed by *multipartitions* of n with r components, and in general the combinatorics underpinning the representation theory of the Ariki–Koike algebra seem to be analogous to those of the Iwahori–Hecke algebra H_n, but with partitions replaced by multipartitions; for the Iwahori–Hecke algebra of Type B, one uses multipartitions with two components, or *bipartitions*. It has been shown that the Ariki–Koike algebra is cellular, and this provides a great deal of information about its representation theory. In particular, we have a classification of the simple modules, in terms of 'Kleshchev multipartitions'. One of the central problems in the study of algebras such as the Iwahori–Hecke algebra and the Ariki–Koike algebra is the determination of the decomposition numbers, i.e. the composition multiplicities of the simple modules in the Specht modules. In this paper, we do this in a special case.

1.2 Blocks of weight 2

One of the most useful notions in the representation theory of Iwahori–Hecke algebras in type A is that of the *e-weight* of a partition, defined by James, where e is the least positive integer such that $1 + q + \cdots + q^{e-1} = 0$ in $\mathbb{F}$. To each partition λ of n, one associates a Specht module S^λ for H_n, and if we define the weight of S^λ to be the weight of λ, then weight is a block invariant, and gives a useful notion of how 'complicated' a block is. Much of the representation theory of the symmetric groups and Iwahori–Hecke algebras of type A has taken a 'bottom up' approach, by studying blocks of a given small weight. This was done to great effect in weight 2 by Richards [18], who described the decomposition numbers for these blocks in terms of the combinatorics of weight 2 partitions. His result may be summarised as follows. Assume e is finite. If λ is an e-restricted partition (that is, if $\lambda_i - \lambda_{i+1} < e$ for all i), then S^λ has a simple cosocle D^λ. The D^λ give all the irreducible modules for H_n as λ ranges over the set of e-restricted partitions of n. To each e-restricted partition μ is associated an e-regular partition $\mu^\diamond$, with the property that the decomposition number $[S^\lambda : D^\mu]$ is zero unless $\mu^\diamond \trianglerighteq \lambda \trianglerighteq \mu$, where $\trianglerighteq$ is the usual dominance order on partitions.

Theorem 1.1. [18, Theorem 4.4] *Assume that the characteristic of $\mathbb{F}$ is not 2, and let B be a block of H_n of e-weight 2. To each partition λ in B, one may associate a non-negative integer $\partial\lambda$, and to each λ for which $\partial\lambda = 0$, one may associate a colour (black or white) such that the following hold.*

1. *The partitions λ in B with a given value of $\partial\lambda$ are totally ordered by $\trianglerighteq$.*
2. *A partition λ in B is e-restricted if and only if there is a partition ν in B such that $\nu \triangleright \lambda$, $\partial\nu = \partial\lambda$ and (if $\partial\lambda = 0$) ν has the same colour as λ. In this case, $\lambda^\diamond$ is the least dominant such ν.*
3. *If λ and μ are partitions in B with μ e-restricted, then*

$$[S^\lambda : D^\mu] = \begin{cases} 1 & (\mu = \lambda) \\ 1 & (\mu = \lambda^\diamond) \\ 1 & (\mu^\diamond \trianglerighteq \lambda \trianglerighteq \mu \text{ and } |\partial\lambda - \partial\mu| = 1) \\ 0 & (\text{otherwise}). \end{cases}$$

Richards's theorem is proved by extensive use of the Jantzen–Schaper formula, a tool for calculating decomposition numbers of Iwahori–Hecke algebras. Richards classifies all possible 'cases' of this formula in weight 2. Using the same techniques, the present author has extended Richards's work to characteristic 2 [9].

The purpose of the present paper is to prove a version of Richards's theorem for the algebra $\mathcal{H}_n$. In [11], the author introduced a definition of weight for multipartitions, and hence for representations of

Ariki–Koike algebras, and demonstrated some basic properties of this weight function. In particular, blocks of weight 0 and 1 were studied, and found to behave in similar ways to blocks of H_n of weight 0 and 1. Here we examine blocks of weight 2, in the type B case. It turns out that (with certain assumptions on the parameters Q_1, Q_2, which eliminate 'trivial' cases) these occur in two different types. We are able to prove suitable analogues of Richards's theorem for these types. Our method is also to use the Jantzen–Schaper formula, or rather its cyclotomic version, proved by James and Mathas in [13]. Fortunately, there are considerably fewer cases for us to check than in Richards's work. On the other hand, we have some work to do in describing the Kleshchev bipartitions in these blocks.

Much of the background theory we shall use (for example, the parameterisation of simple modules by Kleshchev bipartitions and the author's own definition of the weight of a bipartition) holds in the more general context of Ariki–Koike algebras, and some of our results will generalise easily to that context. But we concentrate on the type B case in this paper without paying much attention to generalisation; this is partly to avoid over-burdening the reader with notation, and partly because even with generalisations of our results, much more work would be needed to obtain a full picture for the Ariki–Koike algebras – there are other 'types' of weight 2 block in general.

For the rest of this introduction, we summarise the background theory we shall use. In Section 2, we give a rough characterisation of weight 2 blocks. We find that these fall into two distinct types, which we call Types I and II. We describe a prototypical example for each type. In Section 3, we analyse blocks of Type I, which seem to be the most interesting. We develop the combinatorics of Type I blocks by means of a certain partial order on the set $\mathbb{Z}/e\mathbb{Z}$, and we describe the Kleshchev bipartitions and the dominance order in these blocks. Finally, we find the decomposition numbers for these blocks, proving an analogue of Richards's theorem above. In Section 4, we look at blocks of Type II. Here the combinatorics are rather different, and the blocks are easier to analyse. In particular, any two blocks of Type II have essentially the same decomposition matrix, and we may study these blocks in much the same way as we studied weight 1 blocks of Ariki–Koike algebras in [11]. We prove an analogue of Richards's theorem for Type II blocks also.

1.3 Background theory and notation

In an attempt to regulate the length of this paper, we assume familiarity with much of the background material we use, concerning the representation theory of Iwahori–Hecke algebras of types A and B. The standard reference for Iwahori–Hecke algebras of type A is Mathas's book [15], and for the Ariki–Koike algebra the reader should consult Mathas's survey article [16]; concentrating on the special case $r = 2$ will yield the theory of the Iwahori–Hecke algebra of type B. The relevant background information is also summarised in the author's paper [11], in which the definition of the weight of a bipartition (which is vital to this paper) is introduced.

Henceforth, we let $\mathcal{H}_n$ be the Iwahori–Hecke of type B presented at the start of this introduction, with q, Q_1, Q_2 elements of the field $\mathbb{F}$. We assume that q does not equal 0 or 1, and that neither of the Q_i equals 0. We let e denote the multiplicative order of q in $\mathbb{F}$; our assumptions on q mean that $e \in \{2, 3, \ldots, \infty\}$.

Note that the isomorphism type of $\mathcal{H}_n$ is unaffected if we interchange Q_1 and Q_2. However, the isomorphism types of some of the modules we use are affected under this transposition, and so it is important that we regard (Q_1, Q_2) as an ordered pair.

We assume that the reader is familiar with with following combinatorial concepts: partitions, bipartitions, Young diagrams, addable, removable, normal and good nodes and their residues, rim e-hooks and their leg lengths and foot nodes, the (e-)weight of a partition, and (e-)cores. As usual, we may abuse notation by not distinguishing a partition or a bipartition from its Young diagram. We also use the notion of Kleshchev bipartitions (as defined in [5]); although the definition of these depends on the parameters q, Q_1, Q_2, we shall simply say 'Kleshchev' without fear of confusion. We use the recursive definition of Kleshchev bipartitions, but we remark that Ariki, Kreiman and Tsuchioka [4] have recently found a non-recursive characterisation.

Let S^λ denote the Specht module indexed by a bipartition $\lambda = (\lambda^{(1)}, \lambda^{(2)})$, and let D^μ be the simple module indexed by a Kleshchev bipartition μ. The fact that the modules D^μ are precisely the simple $\mathcal{H}_n$-modules was proved by Ariki [2, Theorem 4.2].

Our main concern in this paper is the calculation of the *decomposition numbers* $[S^\lambda : D^\mu]$. The following fundamental result follows from the fact that $\mathcal{H}_n$ is cellular, with the Specht modules being a set of cell modules.

Theorem 1.2. *Suppose λ and μ are bipartitions of n with μ Kleshchev.*

1. *If $\mu = \lambda$, then $[S^\lambda : D^\mu] = 1$.*
2. *If $[S^\lambda : D^\mu] > 0$, then $\lambda \trianglerighteq \mu$.*

We shall also need the following lemma, which will aid us in determining which bipartitions are Kleshchev.

Proposition 1.3.

1. *Suppose λ is a bipartition of n and that x is a good node of λ, and let μ be the bipartition of $n-1$ with $\mu = \lambda \setminus \{x\}$. Then λ is Kleshchev if and only if μ is.*
2. *Suppose λ is a bipartition with at least k normal nodes of residue f. Let μ be the bipartition obtained by removing the highest k normal nodes of residue f from λ. Then λ is Kleshchev if and only if μ is.*

Proof.

1. This follows from [5, Theorem 2.9 & Corollary 2.11], in which it is shown that the crystal graph of a certain highest weight module for $U_v(\widehat{\mathfrak{sl}_e})$ (or for $U_v(\mathfrak{sl}_\infty)$, if $e = \infty$) has vertices indexed by Kleshchev bipartitions and edges corresponding to removal of good nodes.
2. This is proved by induction on k, with the case $k = 1$ being part (1) of the present theorem. If $k > 1$, let λ^- be the bipartition obtained by removing the good node (that is, the highest normal node) x of residue f from λ. By (1), λ is Kleshchev if and only if λ^- is Kleshchev. The f-signature of λ^- is obtained from the f-signature for λ by replacing the $-$ corresponding to x with a $+$. Hence the reduced f-signature of λ^- is obtained from the reduced f-signature of λ by replacing the $-$ corresponding to x with a $+$. So the normal nodes of λ^- of residue f are precisely the normal nodes of λ of residue f other than x. So μ is obtained from λ^- by removing the $k-1$ highest normal nodes of λ^- of residue f, and so by induction λ^- is Kleshchev if and only if μ is. □

1.3.1 The blocks of $\mathcal{H}_n$ and the weight of a bipartition

The block structure of $\mathcal{H}_n$ (and for the Ariki–Koike algebras in general) was conjectured, and proved in one direction, by Graham and Lehrer [12]. The proof has recently been completed by Lyle and Mathas [14]. Given a bipartition λ and an element f of $\mathbb{F}$, let $c_f(\lambda)$ denote the number of nodes of λ of residue f.

Theorem 1.4. [12, Proposition 5.9(ii)], [14, Theorem 2.11] *Suppose λ and μ are two bipartitions of n. Then S^λ and S^μ lie in the same block of $\mathcal{H}_n$ if and only if $c_f(\lambda) = c_f(\mu)$ for all $f \in \mathbb{F}$.*

If B is a block of $\mathcal{H}_n$, then in view of Theorem 1.4 we may define $c_f(B)$ to equal $c_f(\boldsymbol{\lambda})$ for any bipartition $\boldsymbol{\lambda}$ in B. We abuse notation by saying that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ lie in the same block of $\mathcal{H}_n$ if and only if $S^{\boldsymbol{\lambda}}$ and $D^{\boldsymbol{\mu}}$ lie in the same block.

Now we define the weight of a bipartition, as introduced by the author in [11]. Retaining the notation $c_f(\boldsymbol{\lambda})$ from above, we define the weight of $\boldsymbol{\lambda}$ to be

$$w(\boldsymbol{\lambda}) = c_{Q_1}(\boldsymbol{\lambda}) + c_{Q_2}(\boldsymbol{\lambda}) - \frac{1}{2}\sum_{f\in\mathbb{F}}(c_f(\boldsymbol{\lambda}) - c_{qf}(\boldsymbol{\lambda}))^2.$$

Note that the notion of weight depends not only on q but also on Q_1, Q_2 (or rather, on the ratio Q_1/Q_2).

It is immediate from Theorem 1.4 that w is a block invariant, and we define the weight of a block to be the weight of any bipartition in that block. Below we describe a simpler way to calculate the weight of a bipartition, but first we need to make certain assumptions on the parameters Q_1, Q_2.

1.3.2 q-connected cyclotomic parameters

The representation theory of $\mathcal{H}_n$ depends crucially on the parameters Q_1, Q_2. It is clear that the isomorphism type of $\mathcal{H}_n$ is unaffected if these parameters are simultaneously multiplied by a non-zero scalar. In [7], Dipper and James showed that, as far as representation theory is concerned, we may assume that the set $\{Q_1, Q_2\}$ is *q-connected*, that is, $Q_2 = q^s Q_1$ for some integer s. In fact, they showed that if $\{Q_1, Q_2\}$ is not q-connected, then [7, Theorem 4.17] $\mathcal{H}_n$ is Morita equivalent to the direct sum

$$\bigoplus_{i=0}^{n} H_i \otimes H_{n-i},$$

where H_i is the Iwahori–Hecke algebra of type A. In this situation, it is easy to analyse blocks of $\mathcal{H}_n$ of weight 2; such a block B is Morita equivalent to the tensor product $B_1 \otimes B_2$, where B_1 is a block of H_i and B_2 is a block of H_{n-i}, and by [11, §3.1] the weights of B_1 and B_2 sum to 2. Furthermore, under this Morita equivalence a Specht module $S^{(\lambda^{(1)},\lambda^{(2)})}$ maps to the tensor product $S^{\lambda^{(1)}} \otimes S^{\lambda^{(2)}}$, and so it is easy to calculate the decomposition numbers for B from the decomposition numbers for B_1 and B_2, and prove an analogue of Richards's theorem.

In view of this, we shall assume for the rest of this paper that the set $\{Q_1, Q_2\}$ is q-connected. In fact, given the above remark about simultaneous re-scaling of $\{Q_1, Q_2\}$, we assume henceforth that each Q_i is a power of q. Note that the residue of any node of a Young diagram is then also a power of q; we shall use the term 'i-node' to mean 'node of residue q^i'. We now define integers $\delta_i(\boldsymbol{\lambda})$ for any bipartition $\boldsymbol{\lambda}$ and for $i \in \mathbb{Z}/e\mathbb{Z}$: $\delta_i(\boldsymbol{\lambda})$ is simply the number of removable i-nodes of $\boldsymbol{\lambda}$ minus the number of addable i-nodes. The importance of the δ_i lies in the following result.

Proposition 1.5. [11, Proposition 3.2] *Suppose $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are bipartitions of n. Then $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ lie in the same block of $\mathcal{H}_n$ if and only if $\delta_i(\boldsymbol{\lambda}) = \delta_i(\boldsymbol{\mu})$ for all $i \in \mathbb{Z}/e\mathbb{Z}$.*

Proposition 1.5 allows us to define the integers $\delta_i(B)$ for any block B of $\mathcal{H}_n$: we set $\delta_i(B) = \delta_i(\boldsymbol{\lambda})$ for any bipartition $\boldsymbol{\lambda}$ in B.

1.3.3 Dual Specht modules and conjugate Kleshchev bipartitions

In this section we summarise some of the results from Mathas's paper [17] which we shall need. If λ is a partition, let λ' denote the conjugate partition. If $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ is a bipartition, the conjugate bipartition is defined to be

$$\boldsymbol{\lambda}' = (\lambda^{(2)\prime}, \lambda^{(1)\prime}).$$

Using Theorem 1.4, it is easy to see that two bipartitions λ, μ of n lie in the same block of $\mathcal{H}_n$ if and only if λ' and μ' lie in the same block; for there is a bijection between the nodes of λ and the nodes of λ' given by

$$(i,j,k) \longmapsto (j,i,3-k),$$

and satisfying

$$\mathrm{res}((j,i,3-k)) = Q_1Q_2\,\mathrm{res}((i,j,k))^{-1}.$$

In view of this, we say that the block of $\mathcal{H}_n$ containing λ' is conjugate to the block of $\mathcal{H}_n$ containing λ, for any bipartition λ.

In [17], Mathas constructs an $\mathcal{H}_n$-module $S'(\lambda)$ for each bipartition λ of n, which he calls a *dual Specht module*. The dual Specht modules perform a similar rôle to the Specht modules: each $S'(\lambda)$ has a quotient $D'(\lambda)$ which is either zero or irreducible, and the non-zero $D'(\lambda)$ give all the irreducibles for $\mathcal{H}_n$. In fact, we can say more.

Theorem 1.6. *The module $D'(\mu)$ is non-zero if and only if μ is Kleshchev. Furthermore, for such a μ, we have*

- $[S'(\mu) : D'(\mu)] = 1$, *and*
- $[S'(\lambda) : D'(\mu)] = 0$ *if* $\lambda \not\trianglerighteq \mu$.

Proof. This follows from the discussion in [17, §4]. The algebra $\mathcal{H}_n$ has a second presentation which shows it to be the Iwahori–Hecke algebra of type B with parameters q^{-1}, Q_2, Q_1. The dual Specht module $S'(\lambda)$ is then simply the Specht module S^λ for this latter algebra, and the quotient $D'(\lambda)$ is the quotient D^λ. It is easy to see from the definition of Kleshchev bipartitions that a bipartition is Kleshchev for the parameters q^{-1}, Q_2, Q_1 if and only if it is Kleshchev for the parameters q, Q_1, Q_2, and the result then follows from the usual Specht module theory. □

The importance for us of dual Specht modules is as follows. There is an anti-automorphism of $\mathcal{H}_n$ defined by $T_i \mapsto T_i$ for $i = 0, \dots, n-1$, and this allows us to define, for each $\mathcal{H}_n$-module M, a *contragredient dual* module M°. We then have the following.

Theorem 1.7. [17, Theorem 5.7] *For each bipartition λ,*

$$S'(\lambda) \cong (S^{\lambda'})^\circ.$$

As a consequence of Theorem 1.6 and Theorem 1.7, we deduce an important result on decomposition numbers. Say that a bipartition μ is *conjugate Kleshchev* if μ' is Kleshchev.

Proposition 1.8. *There is a bijection $\mu \mapsto \mu^\diamond$ from the set of Kleshchev bipartitions of n to the set of conjugate Kleshchev bipartitions of n such that*

- $[S^{\mu^\diamond} : D^\mu] = 1$, *and*
- $[S^\lambda : D^\mu] = 0$ *if* $\lambda \not\trianglelefteq \mu^\diamond$.

Proof. Contragredient duality is an exact contravariant functor which induces a permutation of the irreducible modules, so there is a bijection g from the set of irreducibles to the set of Kleshchev bipartitions such that $[S'(g(D))^\circ : D] = 1$ and $[S'(\lambda)^\circ : D] = 0$ if $\lambda \not\trianglerighteq g(D)$ (in fact, the simple modules are contragredient self-dual, so that $g(D'(\mu)) = \mu$, but we do not need this). Conjugation of bipartitions reverses the dominance order, so Theorem 1.7 implies that $[S^{g(D)'} : D] = 1$ and $[S^\lambda : D] = 0$ if $\lambda \not\trianglelefteq g(D)'$. Now for each Kleshchev μ we put $\mu^\diamond = g(D^\mu)'$. □

We shall make frequent use of Proposition 1.8 later in this paper, and we shall use the notation $\mu^\diamond$ without further comment. We shall use the following lemma to find the bijection $\lambda \mapsto \lambda^\diamond$; this is simply an adaptation of [18, Lemma 2.12], and the proof is the same as for that result.

Lemma 1.9. *Suppose B is a block of $\mathcal{H}_n$, and we have a bijection $\boldsymbol{\lambda} \mapsto \boldsymbol{\lambda}^*$ from the set of Kleshchev bipartitions in B to the set of conjugate Kleshchev bipartitions in B such that $[S^{\boldsymbol{\lambda}^*} : D^{\boldsymbol{\lambda}}] > 0$ for all Kleshchev $\boldsymbol{\lambda}$. Then $\boldsymbol{\lambda}^* = \boldsymbol{\lambda}^\diamond$ for all $\boldsymbol{\lambda}$.*

1.3.4 Beta-numbers and the abacus

When working with partitions and bipartitions, it is often useful to define *beta-numbers*. Suppose λ is a partition, and choose an integer a. For $i \geqslant 1$ define

$$\beta_i = \lambda_i + a - i.$$

Then the integers $\beta_1, \beta_2, \dots$ are distinct, and the set $B(\lambda) = \{\beta_1, \beta_2, \dots\}$ is referred to as the set of *beta-numbers for λ with charge a*. An important feature of beta-numbers is that removing a rim r-hook from λ corresponds to reducing one of the beta-numbers for λ by r. In particular, λ has a rim r-hook if and only if there exists some m such that $m \in B(\lambda) \not\ni m - r$.

Given a set of beta-numbers for λ, we may construct an *abacus display*: we take an abacus with e vertical runners, labelled $0, \dots, e-1$ from left to right (or labelled with $\dots, -1, 0, 1, 2, \dots$ from left to right, if $e = \infty$). On runner i, we mark positions corresponding to the integers congruent to i modulo e, increasing from top to bottom if $e < \infty$, and such that position $i-1$ is directly to the left of position i, if $i \not\equiv 0 \pmod{e}$; if $e = \infty$, we mark only one position on each runner, and these positions lie on a horizontal line. Now given a set $B(\lambda)$ of beta-numbers for λ, we place a bead at position β_i for each i. This configuration is called the abacus display for λ with charge a.

Now suppose we have a bipartition $(\lambda^{(1)}, \lambda^{(2)})$. Since we are assuming that Q_1 and Q_2 are powers of q, we may choose integers a_1, a_2 such that $Q_j = q^{a_j}$ for $j = 1, 2$; we refer to such a pair (a_1, a_2) as a *bicharge*. For $j = 1, 2$, we construct the set $B(\lambda^{(j)}) = \{\beta_1^{(j)}, \beta_2^{(j)}, \dots\}$ of beta-numbers for $\boldsymbol{\lambda}$ with charge a_j. The *abacus display for $\boldsymbol{\lambda}$ with bicharge* (a_1, a_2) is obtained by constructing the abacus displays for $\lambda^{(1)}$ and $\lambda^{(2)}$ with charges a_1 and a_2 respectively, and placing them side by side.

Again, removing a rim r-hook from $\boldsymbol{\lambda}$ corresponds to reducing one of the beta-numbers by r. Given the condition $Q_j = q^{a_j}$, we can say more: it is easy to calculate that if we replace $\beta_i^{(j)}$ with $\beta_i^{(j)} - r$, then the residue of the foot node of the corresponding rim r-hook is $q^{\beta_i^{(j)} - r + 1}$. In the special case $r = 1$, we see that removing a removable i-node from $\boldsymbol{\lambda}$ corresponds to replacing a beta-number $\beta_i^{(j)}$ congruent to i modulo e with $\beta_i^{(j)} - 1$.

1.3.5 Calculating the weight of a bipartition

In [11], a simpler way is found to calculate $w(\boldsymbol{\lambda})$ in the case when the cyclotomic parameters are q-connected; we give a full account of this here, since for this paper (where we use only bipartitions, rather than multipartitions) we can use slightly simpler notation than is needed for multipartitions in [11].

We refer to a bipartition $(\lambda^{(1)}, \lambda^{(2)})$ in which $\lambda^{(1)}$ and $\lambda^{(2)}$ are both cores as a *bicore*. We may reduce the calculation of the weight of a bipartition to that of a bicore, by removing rim e-hooks from $\boldsymbol{\lambda}$.

Proposition 1.10. [11, Corollary 3.4] *Suppose $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ is a bipartition of n, and that $\boldsymbol{\lambda}^-$ is a bipartition of $n - e$ obtained by removing a rim e-hook from $\lambda^{(1)}$ or $\lambda^{(2)}$. Then $w(\boldsymbol{\lambda}) = w(\boldsymbol{\lambda}^-) + 2$.*

By induction, this lemma enables us to restrict attention to bicores. To calculate the weight of a bicore $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$, we choose a bicharge (a_1, a_2) and calculate the beta-numbers defined above. Now for $i \in \mathbb{Z}/e\mathbb{Z}$ define

$$\gamma_i(\boldsymbol{\lambda}) = \left|\Big(B(\lambda^{(1)}) \setminus B(\lambda^{(2)})\Big) \cap \{j \in \mathbb{Z} \mid q^j = q^i\}\right| - \left|\Big(B(\lambda^{(2)}) \setminus B(\lambda^{(1)})\Big) \cap \{j \in \mathbb{Z} \mid q^j = q^i\}\right|.$$

So $\gamma_i(\lambda)$ is the number of beta-numbers of $\lambda^{(1)}$ congruent to i modulo e which are not beta-numbers of $\lambda^{(2)}$, minus the number of beta-numbers of $\lambda^{(2)}$ congruent to i modulo e which are not beta-numbers of $\lambda^{(1)}$.

We can use the integers $\gamma_i(\lambda)$ to 'modify' λ in such a way as to reduce its weight. Suppose we are given integers i, j which are incongruent modulo e, and if $e = \infty$ suppose also that

$$i \in B(\lambda^{(1)}) \setminus B(\lambda^{(2)}), \qquad j \in B(\lambda^{(2)}) \setminus B(\lambda^{(1)}).$$

Let l_1 be the largest element of $B(\lambda^{(1)})$ which is congruent to i modulo e, and let l_2 be the largest element of $B(\lambda^{(2)})$ which is congruent to j modulo e. Let m_1 be the smallest integer not in $B(\lambda^{(1)})$ which is congruent to j modulo e, and let m_2 be the smallest integer not in $B(\lambda^{(2)})$ which is congruent to i modulo e (note that the extra assumption on i, j in the case where $e = \infty$ guarantees that l_1, l_2, m_1, m_2 are defined). Now define $s_{ij}(\lambda)$ to be the bipartition obtained from λ by replacing l_k with m_k in $B(\lambda^{(k)})$, for $k = 1, 2$.

If $e < \infty$, then replacing λ with $s_{ij}(\lambda)$ may be visualised as follows: in the abacus for $\lambda^{(1)}$, we slide all the beads on runner i up one space and all the beads on runner j down one space; in the abacus for $\lambda^{(2)}$, we do exactly the opposite; see the example below. Replacing λ with $s_{ij}(\lambda)$ helps us to calculate weight recursively, using the following result.

Proposition 1.11. [11, Lemma 3.7(ii–iii)]

1.

$$\gamma_l(s_{ij}(\lambda)) = \begin{cases} \gamma_l(\lambda) - 2 & (l = i) \\ \gamma_l(\lambda) + 2 & (l = j) \\ \gamma_l(\lambda) & (\textit{otherwise}). \end{cases}$$

2.

$$w(s_{ij}(\lambda)) = w(\lambda) - 2(\gamma_i(\lambda) - \gamma_j(\lambda) - 2).$$

This proposition is used as follows: if we have $\gamma_i(\lambda) - \gamma_j(\lambda) \geqslant 3$ for some i, j, then we find that $w(s_{ij}(\lambda)) < w(\lambda)$, and so we may replace λ with $s_{ij}(\lambda)$. Continuing in this way, we will certainly reach a bicore λ for which $\gamma_i(\lambda) - \gamma_j(\lambda) \leqslant 2$ for all i, j. (Note that when $e = \infty$, any bicore (i.e. any bipartition) will have $\gamma_i(\lambda) - \gamma_j(\lambda) \leqslant 2$ for all i, j.) The weight of this bicore is then given by the following result.

Proposition 1.12. [11, Proposition 3.8] *Suppose λ is a bicore, and that the integers $\gamma_i(\lambda)$ defined above satisfy*

$$\gamma_i(\lambda) - \gamma_j(\lambda) \leqslant 2$$

for all i, j. Define

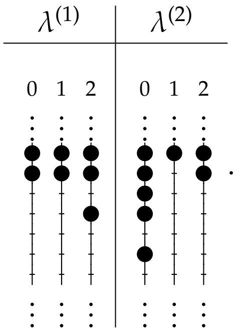
$$I = \{i \mid \gamma_i(\lambda) - \gamma_j(\lambda) = 2 \textit{ for some } j\}, \qquad J = \{j \mid \gamma_i(\lambda) - \gamma_j(\lambda) = 2 \textit{ for some } i\}.$$

Then $w(\lambda) = \min\{|I|, |J|\}$.

Armed with Propositions 1.10–1.12, we can easily calculate the weight of a bipartition, or indeed classify bipartitions of a given weight. We also note the following, which is a consequence of Proposition 1.11 and [11, Lemma 3.7(i)].

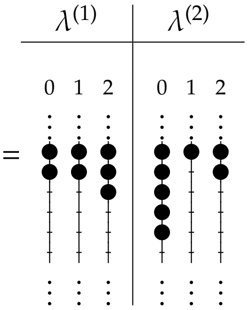
Lemma 1.13. *Suppose $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ is a bipartition of n which is a bicore, and that $\gamma_i(\lambda) - \gamma_j(\lambda) = 2$ for some i, j. Then $s_{ij}(\lambda)$ is a bipartition of n, and lies in the same block of $\mathcal{H}_n$ as λ.*

Example. Suppose $e = 3$ and $(Q_1, Q_2) = (q, q^2)$. Let λ be the bipartition $((5),(8,3,1^2))$. This has the following abacus display (with bicharge $(1,2)$):



We begin by removing rim e-hooks, which corresponds to sliding beads up their runners. We get $w(\lambda) = w(\lambda^-) + 4$, where

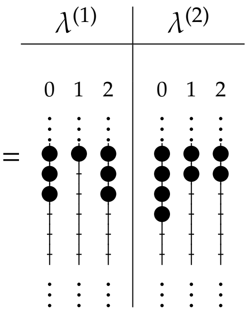
$\lambda^- =$



$= ((2),(5,3,1^2)).$

We have $\gamma_1(\lambda^-) - \gamma_0(\lambda^-) = 4$, and so we have $w(\lambda^-) = w(s_{10}(\lambda^-)) + 4$, where

$s_{10}(\lambda^-) =$



$= ((2,1^2),(2)).$

We have

$$\gamma_2(s_{10}(\lambda^-)) - \gamma_0(s_{10}(\lambda^-)) = \gamma_2(s_{10}(\lambda^-)) - \gamma_1(s_{10}(\lambda^-)) = 2,$$

and so $w(s_{10}(\lambda^-)) = 1$, and hence $w(\lambda) = 9$.

1.3.6 The cyclotomic Jantzen–Schaper formula

The Jantzen–Schaper formula is a very valuable tool for calculating and estimating decomposition numbers. The version for Ariki–Koike algebras was proved by James and Mathas in [13]. We state a weak version here which will be adequate for our purposes, specialising to the case $r = 2$.

Suppose R is a principal ideal domain and $\hat{q}, \hat{Q}_1, \hat{Q}_2$ are elements of R, with $\hat{q}$ invertible. Suppose also that $\mathfrak{p}$ is a prime ideal in R such that $R/\mathfrak{p}R \cong \mathbb{F}$, and that the images of $\hat{q}, \hat{Q}_1, \hat{Q}_2$ under this quotient map are q, Q_1, Q_2 respectively. Let $\mathbb{K}$ denote the field of fractions of R, and for $(i,j,k) \in \mathbb{N} \times \mathbb{N} \times \{1,2\}$ define

$$\widehat{\mathrm{res}}((i,j,k)) = \hat{q}^{j-i}\hat{Q}_k \in \mathbb{K}.$$

Now suppose $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ are bipartitions of n. If x is a node of $\boldsymbol{\lambda}$, let r_x denote the corresponding rim hook in $\boldsymbol{\lambda}$, let $l(r_x)$ denote the leg length of r_x, and let f_x be the foot node of r_x. Make similar definitions for a node y of $\boldsymbol{\nu}$, and then define $G(\boldsymbol{\lambda},\boldsymbol{\nu})$ to be the set of all pairs (x,y) such that

- x is a node of $\boldsymbol{\lambda}$ and y a node of $\boldsymbol{\nu}$,
- $\boldsymbol{\lambda}\setminus r_x = \boldsymbol{\nu}\setminus r_y$, and
- $\operatorname{res}(f_x) = \operatorname{res}(f_y)$.

Given $(x,y)\in G(\boldsymbol{\lambda},\boldsymbol{\nu})$, define $\epsilon_{xy} = (-1)^{l(r_x)-l(r_y)}$, and let

$$g_{\boldsymbol{\lambda}\boldsymbol{\nu}} = \prod_{(x,y)\in G(\boldsymbol{\lambda},\boldsymbol{\nu})} \left(\widehat{\operatorname{res}}(f_x) - \widehat{\operatorname{res}}(f_y)\right)^{\epsilon_{xy}}.$$

Now for any pair of bipartitions $(\boldsymbol{\lambda},\boldsymbol{\mu})$ with $\boldsymbol{\mu}$ Kleshchev, we define

$$\delta_{\boldsymbol{\lambda}\boldsymbol{\mu}} = \sum_{\boldsymbol{\nu}\lhd\boldsymbol{\lambda}} \nu_{\mathfrak{p}}(g_{\boldsymbol{\lambda}\boldsymbol{\nu}})[S^{\boldsymbol{\nu}} : D^{\boldsymbol{\mu}}].$$

We also need to define the *Poincaré polynomial*

$$P(\hat{q};\hat{Q}_1,\hat{Q}_2) = \prod_{i=1}^{n}(1+\hat{q}+\dots+\hat{q}^{i-1}) \prod_{-n<d<n} (\hat{q}^d\hat{Q}_1 - \hat{Q}_2) \in \mathbb{K}.$$

The Jantzen–Schaper formula in type B may now be stated as follows.

Theorem 1.14. [13, Theorem 4.6] *Suppose $R,\hat{q},\hat{Q}_1,\hat{Q}_2$ are such that $P(\hat{q};\hat{Q}_1,\hat{Q}_2)\neq 0_{\mathbb{K}}$. Suppose $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are bipartitions of n with $\boldsymbol{\mu}$ Kleshchev. Then the decomposition number $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]$ is at most $\delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}$, and is non-zero if and only if $\delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ is non-zero.*

Given this theorem, we may refine Theorem 1.2 and Proposition 1.8, by using a coarser order than the dominance order. Specifically, suppose we have two bipartitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ with $\boldsymbol{\lambda}\trianglerighteq\boldsymbol{\mu}$, and suppose that there is a node x of $\boldsymbol{\lambda}$ and a node y of $\boldsymbol{\mu}$ such that $\boldsymbol{\lambda}\setminus r_x = \boldsymbol{\mu}\setminus r_y$ and $\operatorname{res}(f_x)=\operatorname{res}(f_y)$. Then we write $\boldsymbol{\lambda}\to\boldsymbol{\mu}$. We extend $\to$ transitively to form a partial order, which we call the *Jantzen–Schaper dominance* order; this order depends on the parameters q,Q_1,Q_2, but these parameters will always be implicit when we use this dominance order, so there should be no danger of confusion. It is easy to see that the usual dominance order is a refinement of the Jantzen–Schaper dominance order, and that conjugation of bipartitions reverses the Jantzen–Schaper dominance order. It is clear from Theorem 1.14 that Theorem 1.2 and Proposition 1.8 remain true when the usual dominance order is replaced by the Jantzen–Schaper dominance order. Given these facts, we use Jantzen–Schaper dominance exclusively from now on, and the symbol $\trianglerighteq$ will henceforth denote this order.

Remark. In fact, when we state our decomposition number theorems as analogues of Richards's theorem, it will be crucial that $\trianglerighteq$ is understood as the Jantzen–Schaper dominance order. The fact that Richards does not need (a type A analogue of) this order is simply because for a block of weight 2 of the Iwahori–Hecke algebra in type A, the Jantzen–Schaper order is essentially identical to the usual dominance order.

We now derive a corollary of the Jantzen–Schaper formula which we shall use repeatedly later.

Corollary 1.15. *Suppose $\boldsymbol{\lambda}=(\lambda^{(1)},\lambda^{(2)})$, $\boldsymbol{\mu}=(\mu^{(1)},\mu^{(2)})$ and $\boldsymbol{\nu}$ are bipartitions of n, with $\boldsymbol{\nu}$ Kleshchev. Suppose that one of the following holds:*

1. *e is finite, $\lambda^{(1)}$ and $\mu^{(1)}$ are distinct partitions of weight 1 with the same e-core, and $\lambda^{(2)} = \mu^{(2)}$;*
2. *e is finite, $\lambda^{(2)}$ and $\mu^{(2)}$ are distinct partitions of weight 1 with the same e-core, and $\lambda^{(1)} = \mu^{(1)}$;*
3. *there is a node x of $\lambda^{(1)}$ and a node y of $\mu^{(2)}$ such that $\boldsymbol{\lambda} \setminus r_x = \boldsymbol{\mu} \setminus r_y$ and $\mathrm{res}(f_x) = \mathrm{res}(f_y)$.*

Suppose also that $[S^{\boldsymbol{\mu}} : D^{\boldsymbol{\nu}}] = 1$, and that $\boldsymbol{\mu}$ is the unique bipartition such that $\boldsymbol{\lambda} \to \boldsymbol{\mu}$ and $[S^{\boldsymbol{\mu}} : D^{\boldsymbol{\nu}}] > 0$. Then $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\nu}}] = 1$.

Proof. We define $R = \mathbb{F}[\hat{q}, \hat{q}^{-1}]$ with $\hat{q}$ an indeterminate, and set $\hat{Q}_1 = Q_1$, $\hat{Q}_2 = Q_2 + a(\hat{q} - q)$, $\mathfrak{p} = (\hat{q} - q)$, for some non-zero $a \in \mathbb{F}$ to be chosen later. Then certainly $P_R(\hat{q}; \hat{Q}_1, \hat{Q}_2) \neq 0$, and it suffices to show that in each of cases (1–3) we have $\nu_{\mathfrak{p}}(g_{\boldsymbol{\lambda\mu}}) = \pm 1$; for then the Jantzen–Schaper formula will imply that $\nu_{\mathfrak{p}}(g_{\boldsymbol{\lambda\mu}}) = 1$ and $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\nu}}] = 1$.

1. We claim that there is a unique way to choose a node x of $\lambda^{(1)}$ and a node y of $\mu^{(1)}$ such that $\boldsymbol{\lambda} \setminus r_x = \boldsymbol{\mu} \setminus r_y$ and $\mathrm{res}(f_x) = \mathrm{res}(f_y)$. Let ξ be the core of $\lambda^{(1)}$ and $\mu^{(1)}$, and construct the sets $B(\xi)$, $B(\lambda^{(1)})$, $B(\mu^{(1)})$ of beta-numbers for ξ, $\lambda^{(1)}$, $\mu^{(1)}$ with charge a_1. Then we have
$$B(\lambda^{(1)}) = B(\xi) \setminus \{l\} \cup \{l + e\},$$
$$B(\mu^{(1)}) = B(\xi) \setminus \{m\} \cup \{m + e\}$$
for some $l, m \in B(\xi)$. The fact that $\boldsymbol{\lambda} \rhd \boldsymbol{\mu}$ means that $l > m$. We need to find all ways of reducing one of the beta-numbers for $\lambda^{(1)}$ by r and reducing one of the beta-numbers for $\mu^{(1)}$ by r in such a way that the resulting sets of integers are equal, and the reduced beta-numbers are congruent modulo e. The only way to do this is to replace $l + e$ with $m + e$ in $B(\lambda^{(1)})$, and to replace l with m in $B(\mu^{(1)})$. So x and y are unique. Moreover, if we have $f_x = (x_1, x_2)$ and $f_y = (y_1, y_2)$, then we see that $x_1 - x_2 - y_1 + y_2 = e$, so that
$$g_{\boldsymbol{\lambda\mu}} = (\hat{Q}_1\hat{q}^s - \hat{Q}_1\hat{q}^{s+e})^{\pm 1}$$
for some s. We have
$$\nu_{\mathfrak{p}}(g_{\boldsymbol{\lambda\mu}}) = \pm\nu_{\mathfrak{p}}(f(\hat{q})),$$
where $f(\hat{q})$ is the Laurent polynomial $Q_1\hat{q}^s(1 - \hat{q}^e)$. Since $f(q) = 0$ and
$$\frac{df}{d\hat{q}}(q) = -eQ_1q^{s-1} \neq 0,$$
we have $\nu_{\mathfrak{p}}(f) = 1$, and hence $\nu_{\mathfrak{p}}(g_{\boldsymbol{\lambda\mu}}) = \pm 1$.
2. This is done in a very similar way to (1).
3. Clearly x and y are unique, and so we have
$$g_{\boldsymbol{\lambda\mu}} = (\hat{Q}_1\hat{q}^s - \hat{Q}_2\hat{q}^t)^{\pm 1}$$
for some s, t; the fact that $\mathrm{res}(f_x) = \mathrm{res}(f_y)$ means that $Q_1q^s = Q_2q^t$. We have $\nu_{\mathfrak{p}}(g_{\boldsymbol{\lambda\mu}}) = \pm\nu_{\mathfrak{p}}(f(\hat{q}))$, where
$$f(\hat{q}) = Q_1\hat{q}^s - (Q_2 + a(\hat{q} - q))\hat{q}^t.$$
We calculate $f(q) = 0$ and we may choose a so that
$$\frac{df}{d\hat{q}}(q) = (s - 1)Q_1q^{s-1} - aq^t \neq 0,$$
so that $\nu_{\mathfrak{p}}(f) = 1$, and $\nu_{\mathfrak{p}}(g_{\boldsymbol{\lambda\mu}}) = \pm 1$. □

2 Rough classification of weight two blocks of $\mathcal{H}_n$

In this section, we gain an understanding of what blocks of $\mathcal{H}_n$ of weight 2 'look like'. We find that there are essentially two different 'types', according to the types of bipartition that occur. We describe a prototypical block of each type. In the remaining sections of the paper, we examine blocks of the two types in more detail.

2.1 Types of weight 2 bipartition

As mentioned above, we may safely assume that Q_1, Q_2 are powers of q. Using Propositions 1.10–1.12 which describe the weight of a bipartition, we can easily characterise and categorise bipartitions of weight 2.

Proposition 2.1. *Suppose $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ is a bipartition. Then λ has weight 2 if and only if one of the following occurs.*

Type Ia *e is finite, λ has a rim e-hook, and removing this rim e-hook leaves a bipartition of weight 0.*

Type Ib *e is finite, λ is a bicore and there exist i and j such that $\gamma_i(\lambda) - \gamma_j(\lambda) = 3$ and $\gamma_j(\lambda) < \gamma_k(\lambda) < \gamma_i(\lambda)$ for every $k \notin \{i, j\}$.*

Type II *λ is a bicore and $\gamma_i(\lambda) - \gamma_j(\lambda) \leqslant 2$ for all i, j. Furthermore, if*

$$I = \{i \mid \gamma_i(\lambda) - \gamma_j(\lambda) = 2 \text{ for some } j\}, \qquad J = \{j \mid \gamma_i(\lambda) - \gamma_j(\lambda) = 2 \text{ for some } i\},$$

then $\min\{|I|, |J|\} = 2$.

Proof. If λ is not a bicore, then λ has a rim e-hook. Proposition 1.10 then implies that λ is of Type Ia. So suppose that λ is a bicore. Examining the integers $\gamma_i(\lambda)$ and appealing to Proposition 1.11, we find that $\gamma_i(\lambda) - \gamma_j(\lambda) \leqslant 3$ for all i, j. If there do not exist i, j with $\gamma_i(\lambda) - \gamma_j(\lambda) = 3$, then λ is of Type II, by Proposition 1.12. If there do exist such i, j, then certainly e is finite, and Proposition 1.11 implies that the conditions of Type Ib are satisfied. □

It is clear that the types mentioned in Proposition 2.1 are mutually exclusive. Given a block of $\mathcal{H}_n$ of weight 2, we say that it is of Type I if it contains bipartitions of Type Ia or Ib, or Type II if it contains bipartitions of Type II. It is not at all clear yet that a block cannot be of more than one different type, but this will emerge later.

For the rest of this section, we describe 'prototype' blocks of the two types, and then we consider maps between blocks analogous to the Scopes bijections for blocks of Iwahori–Hecke algebras.

2.2 Some prototypical blocks

2.2.1 Type I

For our first prototypical block, we assume that e is finite. We let B_{I} be the block of $\mathcal{H}_e$ with

$$c_f(B_{\mathrm{I}}) = \begin{cases} 1 & (f \in \{1, q, \dots, q^{e-1}\}) \\ 0 & (\text{otherwise}). \end{cases}$$

It is easy to describe the bipartitions in B_{I}.

Proposition 2.2. *B_{I} has weight 2, and the bipartitions in B_{I} are precisely the following:*

1. *all bipartitions of the form* $((w+1, 1^{e-w-1}), \varnothing)$ *or* $(\varnothing, (w+1, 1^{e-w-1}))$ *for* $0 \leqslant w \leqslant e-1$;
2. *all bipartitions of the form* $((w+1, 1^x), (y+1, 1^z))$, *where* w, x, y, z *are non-negative integers satisfying*

$$w + x + y + z = e - 2$$

and

$$Q_2 = q^{w+z+1} Q_1.$$

Proof. The fact that B_{I} has weight 2 is immediate from the definition of weight. If $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ lies in B_{I}, then we must have $\lambda^{(1)}_2 \leqslant 1$ and $\lambda^{(2)}_2 \leqslant 1$ (i.e. $\lambda^{(1)}$ and $\lambda^{(2)}$ are 'hook partitions'), since the nodes of $\boldsymbol{\lambda}$ have distinct residues. It is easy to see which pairs of hook partitions will give exactly one node of each residue. □

Note that if $Q_1 = Q_2$, then the integers w, x, y, z cannot exist, and so the second type of bipartition does not occur.

Proposition 2.3. *Every bipartition in* B_{I} *is of Type Ia or Type Ib.*

Proof. The bipartitions $((w+1, 1^{e-w-1}), \varnothing)$ and $(\varnothing, (w+1, 1^{e-w-1}))$ are clearly of Type Ia. For the others, we choose a bicharge (a_1, a_2), and calculate the corresponding beta-numbers and the integers $\gamma_i(\boldsymbol{\lambda})$. For $\boldsymbol{\lambda} = ((w+1, 1^x), (y+1, 1^z))$, we can see that

$$\begin{aligned} B(\lambda^{(1)}) &= \{m \in \mathbb{Z} \mid m \leqslant a_1 - 1\} \setminus \{a_1 - x - 1\} \cup \{a_1 + w\}, \\ B(\lambda^{(2)}) &= \{m \in \mathbb{Z} \mid m \leqslant a_2 - 1\} \setminus \{a_2 - z - 1\} \cup \{a_2 + y\}. \end{aligned}$$

Hence (recalling that $a_2 \equiv a_1 + w + z + 1 \equiv a_1 - (x + y - 1) \pmod{e}$) we have

$$\gamma_k(\boldsymbol{\lambda}) = \begin{cases} \frac{1}{e}(a_1 - e - (a_2 - w - z - 1)) & (k \equiv a_1, \dots, a_1 + w - 1) \\ \frac{1}{e}(a_1 - e - (a_2 - w - z - 1)) + 2 & (k \equiv a_1 + w) \\ \frac{1}{e}(a_1 - e - (a_2 - w - z - 1)) & (k \equiv a_1 + w + 1, \dots a_1 + w + z) \\ \frac{1}{e}(a_1 - e - (a_2 - w - z - 1)) + 1 & (k \equiv a_1 - x - y - 1, \dots, a_1 - x - 2) \\ \frac{1}{e}(a_1 - e - (a_2 - w - z - 1)) - 1 & (k \equiv a_1 - x - 1) \\ \frac{1}{e}(a_1 - e - (a_2 - w - z - 1)) + 1 & (k \equiv a_1 - x, \dots, a_1 - 1), \end{cases}$$

where all congruences are modulo e. Thus we find that if we put $i \equiv a_1 + w$ and $j \equiv a_1 - x - 1$, then we have the conditions for Type Ib. □

2.2.2 Type II

For our next prototype, we assume $4 \leqslant e \leqslant \infty$. We also assume that

$$Q_2 = q^p Q_1$$

for some $0 \leqslant p \leqslant e - 4$. Then we define B_{II} to be the block of $\mathcal{H}_{2p+4}$ with

$$c_f(B_{\mathrm{II}}) = \begin{cases} 2 & (f \in \{Q_1, qQ_1, \dots, q^p Q_1\}) \\ 1 & (f \in \{q^{-1}Q_1, q^{p+1}Q_1\}) \\ 0 & (\text{otherwise}). \end{cases}$$

Proposition 2.4. *B_{II} has weight 2, and the bipartitions in B_{II} are precisely the bipartitions*

$$\boldsymbol{\lambda}_{c,d} = ((c,d),(2^{p+2-c},1^{c-d}))$$

for $0 \leqslant d \leqslant c \leqslant p+2$.

Proof. The weight is immediate from the values of $c_f(B_{\mathrm{II}})$. Now assume $\boldsymbol{\lambda}$ is in B. We have $c_{q^{-2}Q_1}(\boldsymbol{\lambda}) = c_{q^{p+2}Q_1}(\boldsymbol{\lambda}) = 0$, so the partition $\lambda^{(1)}$ is contained in $(p+2,p+2)$. Similarly, the partition $\lambda^{(2)}$ is contained in (2^{p+2}). Now it is easily checked that the only possibilities are those given. □

Proposition 2.5. *Every bipartition in B_{II} is of Type II.*

Proof. It is clear that $\boldsymbol{\lambda}_{c,d}$ does not have a rim e-hook, so is of Type Ib or Type II; we find which by examining the integers $\gamma_i(\boldsymbol{\lambda}_{c,d})$, for a given bicharge (a_1,a_2). For $\lambda^{(1)}_{c,d} = (c,d)$ we have

$$B(\lambda^{(1)}_{c,d}) = \{m \in \mathbb{Z} \mid m \leqslant a_1 - 3\} \cup \{a_1 + d - 2, a_1 + c - 1\},$$

while for $\lambda^{(2)}_{c,d} = (2^{p+2-c},1^{c-d})$ we get

$$B(\lambda^{(2)}_{c,d}) = \{m \in \mathbb{Z} \mid m \leqslant a_2 + 1\} \setminus \{a_2 - p + d - 2, a_2 - p + c - 1\}.$$

Now (recalling that $a_2 \equiv a_1 + p \pmod{e}$) we find that if $e < \infty$ then

$$\gamma_i(\boldsymbol{\lambda}_{c,d}) = \begin{cases} \frac{1}{e}(a_2 - a_1 - p) - 1 & (i \equiv a_1 - 2, \dots, a_1 + d - 3) \\ \frac{1}{e}(a_2 - a_1 - p) + 1 & (i \equiv a_1 + d - 2) \\ \frac{1}{e}(a_2 - a_1 - p) - 1 & (i \equiv a_1 + d - 1, \dots, a_1 + c - 2) \\ \frac{1}{e}(a_2 - a_1 - p) + 1 & (i \equiv a_1 + c - 1) \\ \frac{1}{e}(a_2 - a_1 - p) - 1 & (i \equiv a_1 + c, \dots, a_1 + p + 1) \\ \frac{1}{e}(a_2 - a_1 - p) & (i \equiv a_1 + p + 2, \dots, a_1 + e - 3) \end{cases}$$

with congruences modulo e; a corresponding statement holds for the case $e = \infty$. So we have $\gamma_i(\boldsymbol{\lambda}_{c,d}) - \gamma_j(\boldsymbol{\lambda}_{c,d}) \leqslant 2$ for all i,j. So $\boldsymbol{\lambda}_{c,d}$ must be of Type II. □

If $Q_1 = q^pQ_2$ for some $0 \leqslant p \leqslant e-4$, then we introduce another prototype B^*_{II}, with

$$c_f(B^*_{\mathrm{II}}) = \begin{cases} 2 & (f \in \{Q_1, qQ_1, \dots, q^pQ_1\}) \\ 1 & (f \in \{q^{-1}Q_1, q^{p+1}Q_1\}) \\ 0 & (\text{otherwise}). \end{cases}$$

All the results we shall need concerning the block B^*_{II} are analogous to corresponding results for B_{II}, and are proved in exactly the same way. For example, the bipartitions in B^*_{II} are the bipartitions

$$\boldsymbol{\lambda}_{c,d} = ((2^{p+2-c},1^{c-d}),(c,d))$$

for $0 \leqslant d \leqslant c \leqslant p+2$, and these bipartitions are all of Type II.

2.3 Scopes-type bijections

In order to prove some of our results concerning weight 2 blocks, we take an inductive approach, beginning with the prototype blocks defined above. In order to do this, we need to introduce maps between blocks of type B Iwahori–Hecke algebras analogous to the 'Scopes isometries' in type A [19].

For $i \in \mathbb{Z}/e\mathbb{Z}$, define the function $\phi_i : \mathbb{Z} \to \mathbb{Z}$ by

$$\phi_i(j) = \begin{cases} j-1 & (j \equiv i \pmod e) \\ j+1 & (j \equiv i-1 \pmod e) \\ j & \text{(otherwise).} \end{cases}$$

It is clear that ϕ_i reduces to a function $\overline{\phi}_i : \mathbb{Z}/e\mathbb{Z} \to \mathbb{Z}/e\mathbb{Z}$.

Now, given a bipartition λ and given $i \in \mathbb{Z}/e\mathbb{Z}$, we calculate the beta-numbers of λ, and then we define the bipartition $\Phi_i(\lambda)$ by replacing each beta-number $\beta^{(j)}_k$ with $\phi_i(\beta^{(j)}_k)$. Informally, $\Phi_i(\lambda)$ is obtained from λ by simultaneously removing all removable i-nodes and adding all addable i-nodes, or by swapping the $(i-1)$th and ith runners of each abacus in the abacus display for λ.

Proposition 2.6.

1. *If λ is a bipartition of n, then $\Phi_i(\lambda)$ is a bipartition of $n - \delta_i(\lambda)$.*
2. $w(\Phi_i(\lambda)) = w(\lambda)$.
3. *If λ and μ are bipartitions of n, then λ and μ lie in the same block of $\mathcal{H}_n$ if and only if $\delta_i(\lambda) = \delta_i(\mu)$ and $\Phi_i(\lambda)$ and $\Phi_i(\mu)$ lie in the same block of $\mathcal{H}_{n-\delta_i(\lambda)}$.*
4. *If λ has weight 2, then $\Phi_i(\lambda)$ is of the same type (as defined in Proposition 2.1) as λ.*

Proof. (1–3) were proved in [11, Proposition 4.6]. For (4), suppose first that λ is of Type Ia, and write $\Phi_i(\lambda)$ as $(\mu^{(1)}, \mu^{(2)})$. Then λ has a rim e-hook, so there exist j, k such that $j \in B(\lambda^{(k)})$ but $j - e \notin B(\lambda^{(k)})$. Hence we have $\phi_i(j) \in B(\mu^{(k)})$ but $\phi_i(j) - e = \phi_i(j-e) \notin B(\mu^{(k)})$, and so $\Phi_i(\lambda)$ has a rim e-hook. Next, suppose λ is a bicore. For each j, k, l, it is clear that we have $\gamma_j(\Phi_i(\lambda)) = \gamma_{\overline{\phi}_i(j)}(\lambda)$, and so the type of λ is preserved. □

Suppose B is a block of $\mathcal{H}_n$, containing a bipartition λ, and that C is the block of $\mathcal{H}_{n-\delta_i(\lambda)}$ containing $\Phi_i(\lambda)$. In view of Proposition 2.6(3), we may write $\Phi_i(B) = C$ unambiguously, so that Φ_i is also defined on blocks. Proposition 2.6 has the following corollary.

Corollary 2.7. *Suppose B is a block of $\mathcal{H}_n$ of weight 2. Then B is of Type I or Type II if and only if $\Phi_i(B)$ is of Type I or Type II, respectively.*

3 Blocks of Type I

3.1 Type I blocks and bipartitions of weight 0

We now examine blocks of Type I in detail. For each of these blocks, we find a convenient description of all the bipartitions in the block, and we find which of these are Kleshchev and which are conjugate Kleshchev. We also describe the dominance order on the bipartitions, and then finally we apply the cyclotomic Jantzen–Schaper formula to find the decomposition numbers, and hence prove an analogue of Richards's theorem.

For many of these results, we use an inductive approach, starting with the prototype block B_{I}. So we begin by showing that given an arbitrary Type I block, we can apply the functions Φ_i repeatedly to reach B_{I}. We begin by examining bipartitions of weight 0; we shall see that we can naturally associate a bipartition of weight 0 to each Type I block, in much the same way as we associate an e-core to a block in type A.

Lemma 3.1. *Suppose $\boldsymbol{\lambda}$ is a bipartition with both an addable i-node and a removable i-node. Then $w(\boldsymbol{\lambda}) > 0$.*

Proof. Write $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$. If some $\lambda^{(k)}$ has both addable and removable i-nodes, then by [11, Lemma 2.2] $\lambda^{(k)}$ is not an e-core, and so $\boldsymbol{\lambda}$ certainly has positive weight. On the other hand, if $\lambda^{(k)}$ has a removable i-node and $\lambda^{(3-k)}$ has an addable i-node for $k = 1$ or 2, then (assuming $\boldsymbol{\lambda}$ is a bicore) we have $|\gamma_i(\boldsymbol{\lambda}) - \gamma_{i-1}(\boldsymbol{\lambda})| \geqslant 2$, and so $\boldsymbol{\lambda}$ has positive weight by Proposition 1.12. □

Corollary 3.2. *Suppose $\boldsymbol{\lambda}$ is a bipartition of weight 0, and that $\delta_i(\boldsymbol{\lambda}) \leqslant 0$ for all i. Then $\boldsymbol{\lambda} = \varnothing$.*

Proof. If $\boldsymbol{\lambda} \neq \varnothing$, then $\boldsymbol{\lambda}$ has a removable i-node for some i. Now $\delta_i(\boldsymbol{\lambda})$ equals the number of removable i-nodes of $\boldsymbol{\lambda}$ minus the number of addable i-nodes of $\boldsymbol{\lambda}$, so $\boldsymbol{\lambda}$ has at least one addable i-node as well. But then $w(\boldsymbol{\lambda}) > 0$ by Lemma 3.1. □

Now if $e < \infty$ and $\boldsymbol{\lambda}$ is a bipartition of Type Ia or Ib, we define a bipartition $\overline{\boldsymbol{\lambda}}$ as follows:

- if $\boldsymbol{\lambda}$ is of Type Ia, define $\overline{\boldsymbol{\lambda}}$ by removing the rim e-hook from $\boldsymbol{\lambda}$;
- if $\boldsymbol{\lambda}$ is of Type Ib, with $\gamma_i(\boldsymbol{\lambda}) - \gamma_j(\boldsymbol{\lambda}) = 3$, define $\overline{\boldsymbol{\lambda}}$ to be $s_{ij}(\boldsymbol{\lambda})$.

Lemma 3.3. *$\overline{\boldsymbol{\lambda}}$ is a bipartition of weight 0. If $\boldsymbol{\lambda}$ lies in the block B of $\mathcal{H}_n$, then $\overline{\boldsymbol{\lambda}}$ lies in the block $\overline{B}$ of $\mathcal{H}_{n-e}$ with*

$$c_f(\overline{B}) = c_f(B) - 1$$

for $f \in \{1, q, \dots, q^{e-1}\}$. If $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are bipartitions of Type Ia or Type Ib lying in the same block of $\mathcal{H}_n$, then $\overline{\boldsymbol{\lambda}} = \overline{\boldsymbol{\mu}}$.

Proof. The fact that $w(\overline{\boldsymbol{\lambda}}) = 0$ follows from Proposition 1.10 (for Type Ia) or from Proposition 1.11(2) (for Type Ib). It is clear that $\delta_i(\overline{\boldsymbol{\lambda}}) = \delta_i(\boldsymbol{\lambda})$ for each i, and as noted in the proof of [11, Proposition 3.2] this implies that $c_{q^j}(\overline{\boldsymbol{\lambda}}) = c_{q^j}(\boldsymbol{\lambda}) - C$, for some constant C; by [11, Lemma 3.3] we have $|\overline{\boldsymbol{\lambda}}| = |\boldsymbol{\lambda}| - e$, and so $C = 1$.

The final statement now follows, since the block containing $\overline{\boldsymbol{\lambda}}$ depends only on the block containing $\boldsymbol{\lambda}$, and a block of weight 0 contains only one bipartition [11, Theorem 4.1]. □

So weight 2 blocks of $\mathcal{H}_n$ of Type I are in bijection with blocks of $\mathcal{H}_{n-e}$ of weight 0. Now we can show that from any Type I block we can reach the prototype block B_{I} by applying the functions Φ_i repeatedly.

Proposition 3.4. *Suppose that $e < \infty$ and B is a weight 2 block of $\mathcal{H}_n$ containing a bipartition $\boldsymbol{\lambda}$ of Type Ia or Type Ib. Then there is a sequence $n = n_0 > \cdots > n_m = e$ of positive integers, a sequence $B = B_0, \dots, B_m$, where B_j is a block of $\mathcal{H}_{n_j}$ for each j, and a sequence $i_1, \dots, i_m$ of elements of $\mathbb{Z}/e\mathbb{Z}$, such that*

$$B_j = \Phi_{i_j}(B_{j-1})$$

for $j = 1, \dots, m$, and B_m is the block B_{I} from §2.2.1.

Hence B contains only bipartitions of Types Ia and Ib.

Proof. We proceed by induction on n; if we can find some i such that $\delta_i(B) > 0$, then by Corollary 2.7 we can replace B with $\Phi_i(B)$, and appeal to the inductive hypothesis. So all we need to prove is that if $\delta_i(B) \leqslant 0$ for all i, then B is the block B_{I}. But if $\delta_i(B) \leqslant 0$ for all i, then we also have $\delta_i(\overline{\boldsymbol{\lambda}}) \leqslant 0$ for all i. Hence $\overline{\boldsymbol{\lambda}} = \varnothing$, by Corollary 3.2, i.e. $c_f(\overline{\boldsymbol{\lambda}}) = 0$ for all $f \in \mathbb{F}$. But then $c_f(B) = 1$ for $f \in \{1, q, \dots, q^{e-1}\}$ by Lemma 3.3, so $B = B_{\mathrm{I}}$.

The last statement now follows from Propositions 2.3 and 2.6(4). □

Now we are able to examine Type I blocks in much greater detail. We have seen that to each Type I block B we can associate a bipartition of weight 0, namely $\overline{\boldsymbol{\lambda}}$ for any $\boldsymbol{\lambda}$ in B. We call this bipartition the *root* of B.

3.2 The bipartitions in a Type I block

Our next task is to find all the bipartitions in a Type I block. Suppose B is a Type I block with root $\boldsymbol{\nu}$, and construct an abacus display for $\boldsymbol{\nu}$. The fact that $\boldsymbol{\nu}$ has weight 0 means that $\gamma_i(\boldsymbol{\nu}) - \gamma_k(\boldsymbol{\nu}) \leqslant 1$ for all i, k, by Proposition 1.12. We partition $\mathbb{Z}/e\mathbb{Z}$ into two sets I, K such that $\gamma_i(\boldsymbol{\nu}) - \gamma_k(\boldsymbol{\nu}) = 1$ whenever $i \in I, k \in K$. This defines I, K uniquely except in the case where $\gamma_i(\boldsymbol{\nu}) = \gamma_k(\boldsymbol{\nu})$ for all i, k, in which case we choose either

$$I = \mathbb{Z}/e\mathbb{Z}, \qquad K = \varnothing$$

or

$$I = \varnothing, \qquad K = \mathbb{Z}/e\mathbb{Z}$$

as we wish. Now we impose a partial order on $\mathbb{Z}/e\mathbb{Z}$. For each $i \in \mathbb{Z}/e\mathbb{Z}$ and for $a = 1, 2$, we define $b_i^{(a)}$ to be the largest beta-number of $\nu^{(a)}$ congruent to i modulo e, and then for $i, k \in \mathbb{Z}/e\mathbb{Z}$, we define $i \succcurlyeq k$ if both $b_i^{(1)} \geqslant b_k^{(1)}$ and $b_i^{(2)} \geqslant b_k^{(2)}$. Then $\succcurlyeq$ is a partial order, which restricts to a total order on each of I and K. When there are several Type I blocks under consideration and we wish to emphasise B, we may write $I_B, K_B, \succcurlyeq_B$. We use the symbol $\not\gtrless$ to indicate incomparability under the partial order $\succcurlyeq$.

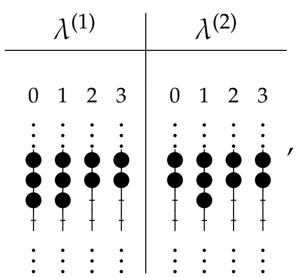
Now we can describe all the bipartitions in B. Given $h \in \mathbb{Z}/e\mathbb{Z}$ and $a = 1$ or 2, define $[h]^a$ to be the bipartition obtained from $\boldsymbol{\nu}$ by moving the lowest bead on runner h of abacus a down one space. Given $i \in I$ and $k \in K$, define $[ik]$ to be the bipartition $s_{ik}(\boldsymbol{\nu})$. Again, if we wish to emphasise B, we may write $[h]^a_B$ or $[ik]_B$.

Lemma 3.5. *The bipartitions $[h]^a$ for $h \in \mathbb{Z}/e\mathbb{Z}$ and $[ik]$ for $i \in I, k \in K$ are all the bipartitions in B.*

Proof. It is easy to verify that $[h]^a$ is a weight 2 bipartition of Type Ia with root $\boldsymbol{\nu}$, while $[ik]$ is a Type Ib bipartition with root $\boldsymbol{\nu}$ and with $\gamma_k(\boldsymbol{\lambda}) - \gamma_i(\boldsymbol{\lambda}) = 3$. On the other hand, it is easy to check that a Type I weight 2 bipartition $\boldsymbol{\lambda}$ with $\overline{\boldsymbol{\lambda}} = \boldsymbol{\nu}$ must be of one of these forms. For example, if $\boldsymbol{\lambda}$ is of Type Ib with $\gamma_k(\boldsymbol{\lambda}) - \gamma_i(\boldsymbol{\lambda}) = 3$, then $\overline{\boldsymbol{\lambda}} = s_{ki}(\boldsymbol{\lambda})$, and Proposition 1.11 implies that $\gamma_i(\overline{\boldsymbol{\lambda}}) - \gamma_k(\overline{\boldsymbol{\lambda}}) = 1$, so that $i \in I$ and $k \in K$, and $\boldsymbol{\lambda} = [ik]$. □

Examples.

1. Suppose $e = 4$, $Q_1 = q^2$ and $Q_2 = q$, and take $\boldsymbol{\nu} = (\varnothing, (1))$. Then $\boldsymbol{\nu}$ has an abacus display

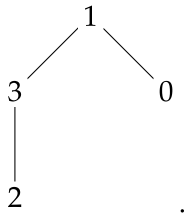


,

and we find

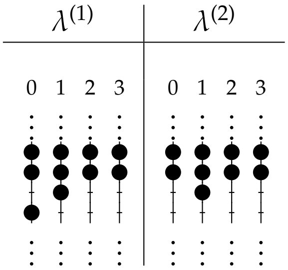
$$I = \{0\}, \qquad K = \{1, 2, 3\},$$

and that the partial order $\succcurlyeq$ is given by the following Hasse diagram:

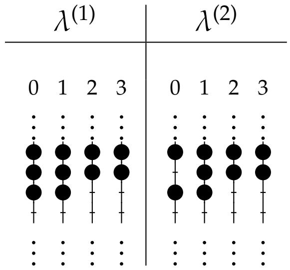


.

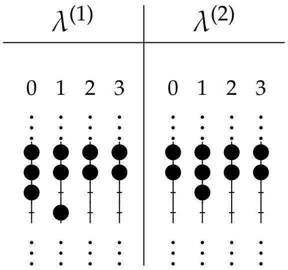
The bipartitions in B are as follows:

$[0]^1 =$ 

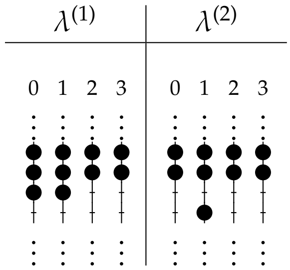
$= ((3, 1), (1)),$

$[0]^2 =$ 

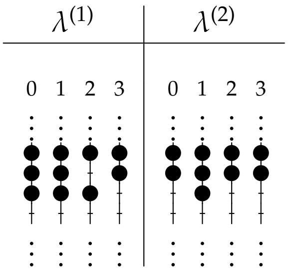
$= (\varnothing, (1^5)),$

$[1]^1 =$ 

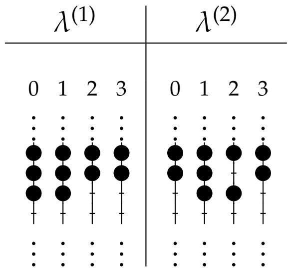
$= ((4), (1)),$

$[1]^2 =$ 

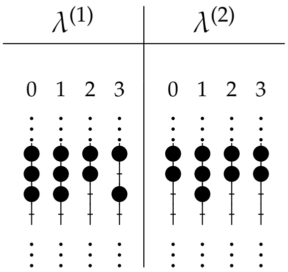
$= (\varnothing, (5)),$

$[2]^1 =$ 

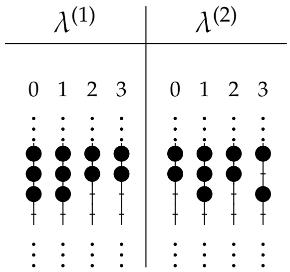
$= ((1^4), (1)),$

$[2]^2 =$ 

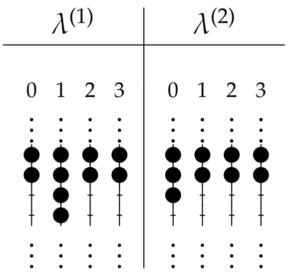
$= (\varnothing, (2^2, 1)),$

$[3]^1 =$ 

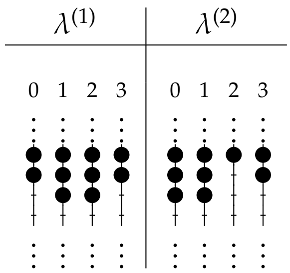
$= ((2, 1^2), (1)),$

$[3]^2 =$ 

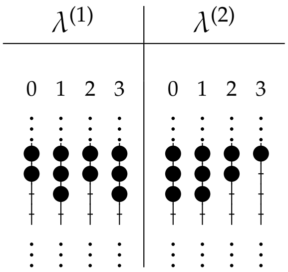
$= (\varnothing, (3, 2)),$

$[01] =$ 

$= ((4, 1), (\varnothing)),$

$[02] =$ 

$= ((1^2), (1^3)),$

$[03] =$  $= ((2,1),(1^2)).$

2. Let B be the prototype block B_{I}, and suppose without loss of generality that $Q_1 = q^a, Q_2 = 1$. Then we have $I = \{0, \dots, a-1\}$, $K = \{a, \dots, e-1\}$ and the order $\succcurlyeq$ is given by

$$0 \prec \cdots \prec a-1, \qquad a \prec \cdots \prec e-1$$

and $i \not\gtrless k$ for $i \in I, k \in K$. Furthermore, we have

$$\begin{aligned}
[i]^1 &= ((e-a+1+i, 1^{a-1-i}), \varnothing) && (0 \leqslant i < a),\\
[k]^1 &= ((k-a+1, 1^{e+a-1-k}), \varnothing) && (a \leqslant k < e),\\
[h]^2 &= (\varnothing, (h+1, 1^{e-1-h})) && (0 \leqslant h < e),\\
[ik] &= ((k-a+1, 1^{a-1-i}), (i+1, 1^{e-1-k})) && (0 \leqslant i < a \leqslant k < e).
\end{aligned}$$

3.3 The dominance order in a Type I block

Armed with our description of the bipartitions in a Type I block, we now describe the dominance order, which will be very useful later for calculating decomposition numbers. Recall that by the 'dominance order', we mean the Jantzen–Schaper dominance order described in §1.3.6. Recall also the relation $\rightarrow$ from that section.

Proposition 3.6. *Suppose B is a Type I block, and use the notation described above for bipartitions in B. Then the relation $\rightarrow$ on bipartitions in B is given as follows:*

$$\begin{aligned}
&[i]^a \to [j]^a && (a \in \{1,2\}, i,j \in I, i \succcurlyeq j), && [i]^1 \to [k]^1 && (i \in I, k \in K, i \not\preccurlyeq k),\\
&[i]^1 \to [i]^2 && (i \in I), && [i]^1 \to [ik] && (i \in I, k \in K, i \not\preccurlyeq k),\\
&[k]^1 \to [i]^1 && (i \in I, k \in K, k \succcurlyeq i), && [k]^a \to [l]^a && (a \in \{1,2\}, k,l \in K, k \succcurlyeq l),\\
&[k]^1 \to [k]^2 && (k \in K), && [k]^1 \to [ik] && (i \in I, k \in K, i \not\preccurlyeq k),\\
&[i]^2 \to [k]^2 && (i \in I, k \in K, i \succcurlyeq k), && [i]^2 \to [ik] && (i \in I, k \in K, i \succcurlyeq k),\\
&[k]^2 \to [i]^2 && (i \in I, k \in K, k \not\preccurlyeq i), && [k]^2 \to [ik] && (i \in I, k \in K, k \preccurlyeq i),\\
&[ik] \to [i]^1 && (i \in I, k \in K, i \preccurlyeq k), && [ik] \to [k]^1 && (i \in I, k \in K, i \preccurlyeq k),\\
&[ik] \to [i]^2 && (i \in I, k \in K, i \not\succcurlyeq k), && [ik] \to [k]^2 && (i \in I, k \in K, i \not\succcurlyeq k),\\
&[ik] \to [jk] && (i,j \in I, k \in K, i \preccurlyeq j), && [ik] \to [il] && (i \in I, k,l \in K, k \succcurlyeq l).
\end{aligned}$$

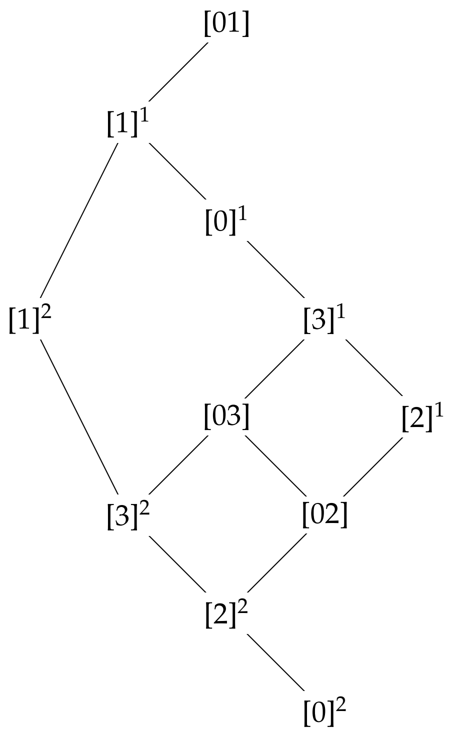
Proof. This is easily checked, by considering all possible ways of removing a rim hook from a bipartition and then adding a rim hook of the same length with foot node of the same residue. Recall that removing a rim l-hook corresponds to reducing a beta-number by l, and that the residue of the foot node of this rim hook is q^{a+1}, where a is the reduced beta-number. □

Corollary 3.7. *The Jantzen–Schaper dominance order in a Type I block is given as follows:*

$$
\begin{array}{llll}
[i]^1 \trianglerighteq [j]^1 & (i,j \in I, i \succcurlyeq j), & [i]^1 \trianglerighteq [k]^1 & (i \in I, k \in K, i \not\preccurlyeq k),\\
[i]^1 \trianglerighteq [j]^2 & (i,j \in I, i \succcurlyeq j \text{ or } i \not\preccurlyeq k \not\preccurlyeq j, \text{ some } k \in K), & [i]^1 \trianglerighteq [k]^2 & (i \in I, k \in K, i \not\preccurlyeq k),\\
[i]^1 \trianglerighteq [jk] & (i,j \in I, k \in K, i \not\preccurlyeq k \not\succcurlyeq j), & & \\
[k]^1 \trianglerighteq [i]^1 & (i \in I, k \in K, k \succcurlyeq i), & [k]^1 \trianglerighteq [l]^1 & (k,l \in K, k \succcurlyeq l),\\
[k]^1 \trianglerighteq [i]^2 & (i \in I, k \in K, k \not\preccurlyeq i), & [k]^1 \trianglerighteq [l]^2 & (k,l \in K, k \succcurlyeq l),\\
[k]^1 \trianglerighteq [il] & (i \in I, k,l \in K, k \succcurlyeq l \not\succcurlyeq i), & & \\
[i]^2 \trianglerighteq [j]^2 & (i,j \in I, i \succcurlyeq j), & [i]^2 \trianglerighteq [k]^2 & (i \in I, k \in K, i \succcurlyeq k),\\
[i]^2 \trianglerighteq [jk] & (i,j \in I, k \in K, i \succcurlyeq k \preccurlyeq j), & & \\
[k]^2 \trianglerighteq [i]^2 & (i \in I, k \in K, k \not\preccurlyeq i), & [k]^2 \trianglerighteq [l]^2 & (k,l \in K, k \succcurlyeq l),\\
[k]^2 \trianglerighteq [il] & (i \in I, k,l \in K, k \succcurlyeq l \preccurlyeq i), & & \\
[ik] \trianglerighteq [j]^1 & (i,j \in I, k \in K, i \preccurlyeq k \succcurlyeq j), & [ik] \trianglerighteq [l]^1 & (i \in I, k,l \in K, i \preccurlyeq k \succcurlyeq l),\\
[ik] \trianglerighteq [j]^2 & (i,j \in I, k \in K, i \not\succcurlyeq k \not\preccurlyeq j), & [ik] \trianglerighteq [l]^2 & (i \in I, k,l \in K, i \not\succcurlyeq k \succcurlyeq l),\\
[ik] \trianglerighteq [jl] & \multicolumn{3}{l}{(i,j \in I, k,l \in K, (i \preccurlyeq j, k \succcurlyeq l) \text{ or } (i \preccurlyeq k \succcurlyeq l \not\succcurlyeq j) \text{ or } (i \not\succcurlyeq k \succcurlyeq l \preccurlyeq j)).}
\end{array}
$$

Proof. This is simply a matter of extending $\rightarrow$ transitively. It can be checked that for any λ, μ for which we claim $\lambda \trianglerighteq \mu$, there are ν and ξ such that $\lambda \rightarrow \nu \rightarrow \xi \rightarrow \mu$. On the other hand, it can be checked that if $\lambda \trianglerighteq \mu$ appears in our list and $\mu \rightarrow \nu$, then $\lambda \trianglerighteq \nu$ appears in our list. □

Example. For the Type I block considered in the last example, the Hasse diagram of the Jantzen–Schaper dominance order is



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3.4 Kleshchev bipartitions in Type I blocks

Next, we want to determine which of the bipartitions in a Type I block are Kleshchev. First we examine how the partial order $\succcurlyeq$ changes when we apply the function Φ_h for some $h \in \mathbb{Z}/e\mathbb{Z}$. Recall the function $\overline{\phi}_h : \mathbb{Z}/e\mathbb{Z} \to \mathbb{Z}/e\mathbb{Z}$ defined above. The following result is easy to check from the definitions.

Proposition 3.8. *Suppose B is a Type I block of weight 2, and that $\delta_h(B) > 0$ for some $h \in \mathbb{Z}/e\mathbb{Z}$. Let C denote the block $\Phi_h(B)$. Then we have*

- $I_C = \overline{\phi}_h(I_B)$, $K_C = \overline{\phi}_h(K_B)$,
- $\Phi_h([g]^a_B) = [\overline{\phi}_h(g)]^a_C$ *for all* $g \in \mathbb{Z}/e\mathbb{Z}$ *and* $a \in \{1,2\}$,
- $\Phi_h([ik]_B) = [\overline{\phi}_h(i)\overline{\phi}_h(k)]_C$ *for all* $i \in I_B, k \in K_B$, *and*
- *for all* $j, l \in \mathbb{Z}/e\mathbb{Z}$ *we have* $j \succcurlyeq_B l$ *if and only if* $\overline{\phi}_h(j) \succcurlyeq_C \overline{\phi}_h(l)$, *except when* $\delta_h(B) = 1$ *and* $\{j,l\} = \{h-1,h\}$, *in which case we have*
$$h-1 \preccurlyeq_B h, \qquad h-1 \not\preccurlyeq_C h.$$

Now we examine normal nodes. If B and C are as above and $\boldsymbol{\lambda}$ is a bipartition in B, then $\boldsymbol{\lambda}$ has at least $\delta_h(B)$ normal h-nodes. We write $\Psi_h(\boldsymbol{\lambda})$ for the bipartition in C obtained by removing the $\delta_h(B)$ highest normal h-nodes from $\boldsymbol{\lambda}$. It is easy to see that Ψ_h is a bijection between the set of bipartitions in B and the set of bipartitions in C. Moreover, by Proposition 1.3(2) Ψ_h maps the set of Kleshchev bipartitions in B to the set of Kleshchev bipartitions in C. We need to describe the action of Ψ_h.

Proposition 3.9. *Suppose B and C are as in Proposition 3.8, and $\boldsymbol{\lambda}$ is a bipartition in B. If $\delta_h(B) \geqslant 2$, then we have $\Psi_h(\boldsymbol{\lambda}) = \Phi_h(\boldsymbol{\lambda})$ for all bipartitions $\boldsymbol{\lambda}$ in B. If $\delta_h(B) = 1$, then exactly one of $h-1$ and h lies in I_B, and the action of Ψ_h is as follows.*

- *If $h \in I_B$, then $\Psi_h(\boldsymbol{\lambda}) = \Phi_h(\boldsymbol{\lambda})$ for all bipartitions $\boldsymbol{\lambda}$ in B other than the 'exceptional' bipartitions $[h-1]^2_B, [h]^2_B, [h(h-1)]_B$, for which we have*
$$\begin{aligned}\Psi_h([h-1]^2_B) &= [(h-1)h]_C,\\ \Psi_h([h]^2_B) &= [h]^2_C,\\ \Psi_h([h(h-1)]_B) &= [h-1]^2_C.\end{aligned}$$
- *If $h-1 \in I_B$, then $\Psi_h(\boldsymbol{\lambda}) = \Phi_h(\boldsymbol{\lambda})$ for all bipartitions $\boldsymbol{\lambda}$ in B other than the 'exceptional' bipartitions $[h-1]^1_B, [h]^1_B, [(h-1)h]_B$, for which we have*
$$\begin{aligned}\Psi_h([h-1]^1_B) &= [h(h-1)]_C,\\ \Psi_h([h]^1_B) &= [h]^1_C,\\ \Psi_h([(h-1)h]_B) &= [h-1]^1_C.\end{aligned}$$

Proof. Suppose first that either $\delta_h(B) \geqslant 2$ or $\boldsymbol{\lambda}$ is not an exceptional bipartition. By examining the runners labelled $h-1$ and h in the abacus displays for the bipartitions in B, we find that every bipartition $\boldsymbol{\lambda}$ has exactly $\delta_h(B)$ removable h-nodes and no addable h-nodes, so $\Psi_h(\boldsymbol{\lambda})$ is obtained simply by removing the removable h-nodes. If $\delta_h(B) = 1$ and $\boldsymbol{\lambda}$ is an exceptional bipartition, we may calculate $\Psi_h(\boldsymbol{\lambda})$ using the abacus display for $\boldsymbol{\lambda}$. As above, let $\boldsymbol{\nu}$ be the root of B and let $b^{(a)}_i$ be the largest beta-number of $\nu^{(a)}$ congruent to i modulo e; then there are integers $t^{(1)}, t^{(2)}$ such that $b^{(a)}_h = b^{(a)}_{h-1} + 1 + et^{(a)}$. We have
$$|t^{(1)} - t^{(2)}| \leqslant 1$$

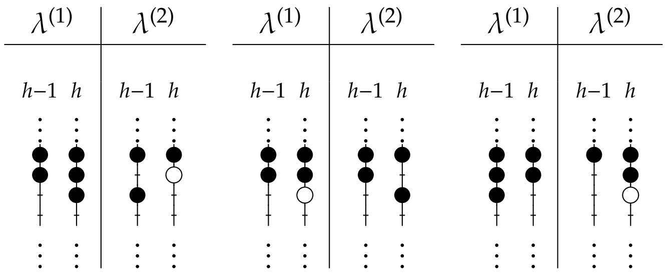
(since $\boldsymbol{\nu}$ has weight 0), and we also have

$$t^{(1)} + t^{(2)} = \delta_h(B) = 1.$$

Hence either $t^{(1)} = 1$ and $t^{(2)} = 0$ (in which case $h \in I_B$ and $h-1 \in K_B$), or $t^{(1)} = 0$ and $t^{(2)} = 1$ (in which case $h \in K_B$ and $h-1 \in I_B$). We now illustrate the abacus displays for each of the exceptional partitions in these two cases; in each case the good h-node corresponds to the white bead.

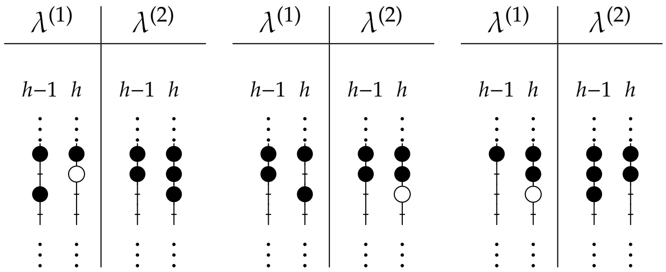
$[h \in I_B]$

$\boldsymbol{\lambda} = [h-1]^2_B$ $\quad$ $\boldsymbol{\lambda} = [h]^2_B$ $\quad$ $\boldsymbol{\lambda} = [h(h-1)]$



$[h-1 \in I_B]$

$\boldsymbol{\lambda} = [h-1]^1_B$ $\quad$ $\boldsymbol{\lambda} = [h]^1_B$ $\quad$ $\boldsymbol{\lambda} = [(h-1)h]$



□

We are almost ready to state which bipartitions in a Type I block are Kleshchev. First, we do this for the prototype block B_{I}.

Proposition 3.10. *The Kleshchev bipartitions in B_{I} are*

- *the bipartitions $((w+1,1^x),(y+1,1^z))$ (where $w+x+y+z = e-2$ and $Q_2 = q^{w+z+1}Q_1$) and*
- *the bipartitions $(\varnothing,(w+1,1^{e-w-1}))$ with $1 \leqslant w+1 \leqslant e-1$.*

Proof. We begin by showing that the bipartitions listed are Kleshchev. For the bipartitions $\boldsymbol{\lambda} = ((w+1,1^x),(y+1,1^z))$, we may get from $\boldsymbol{\lambda}$ to $\varnothing$ by repeatedly removing the lowest removable node:

$$\begin{aligned}
((w+1,1^x),(y+1,1^z)) &\to ((w+1,1^x),(y+1,1^{z-1})) &&\to \cdots \to ((w+1,1^x),(y+1))\\
&\to ((w+1,1^x),(y)) &&\to \cdots \to ((w+1,1^x),\varnothing)\\
&\to ((w+1,1^{x-1}),\varnothing) &&\to \cdots \to ((w+1),\varnothing)\\
&\to ((w),\varnothing) &&\to \cdots \to (\varnothing,\varnothing).
\end{aligned}$$

It is easily checked that the removed node at each stage is normal, and hence (since there is at most one node of any residue) good.

Now we consider the bipartitions of the form $\boldsymbol{\lambda} = (\varnothing, (w+1, 1^{e-w-1}))$, with $w+1 \leqslant e-1$. Again, we repeatedly remove the lowest removable node:

$$\begin{aligned}(\varnothing, (w+1, 1^{e-w-1}) &\to (\varnothing, (w+1, 1^{e-w-2}) \to \cdots \to (\varnothing, (w+1))\\ &\to (\varnothing, (w)) \qquad\quad \to \cdots \to (\varnothing, \varnothing).\end{aligned}$$

The removed node at each stage is normal (the condition $w+1 < e$ guarantees this for the first removed node) and hence good.

Now we show that the remaining bipartitions are not Kleshchev. The bipartition $(\varnothing, (e))$, is easy to deal with, since it has no normal nodes. For a partition of the form $((w+1, 1^{e-w-1}), \varnothing)$, suppose we can remove good nodes one by one to reach the empty bipartition. At some point, we must remove the node of residue Q_2; but the Q_2-signature at this stage must be $-+$, and so the node of residue Q_2 is not normal; contradiction. □

Now we can state which Type I bipartitions are Kleshchev.

Proposition 3.11. *Suppose B is a Type I block, with the sets I, K as above.*

- *For $i \in I$, the bipartition $[i]^1$ is Kleshchev if and only if there is some $k \in K$ with $i \preccurlyeq k$.*
- *For $i \in I$, the bipartition $[i]^2$ is Kleshchev if and only if there is some $m \in \mathbb{Z}/e\mathbb{Z}$ with $i \not\succcurlyeq m$.*
- *For $k \in K$, the bipartition $[k]^1$ is Kleshchev if and only if there are some $i \in I, l \in K$ with $l \succ k \succcurlyeq i$.*
- *For $k \in K$, the bipartition $[k]^2$ is Kleshchev if and only if there is some $m \in \mathbb{Z}/e\mathbb{Z}$ with $k \prec m$.*
- *For $i \in I$ and $k \in K$, the bipartition $[ik]$ is Kleshchev if and only if either $k \not\succcurlyeq i$ or there exist $j \in I, l \in K$ with $i \succ j$ and $k \prec l$.*

Proof. We use induction on n, with the initial case being the prototype block B_{I}. The proposition holds here by Proposition 3.10 – using that result and recalling the notation from Example 2 in §3.2, we see that the Kleshchev bipartitions in B_{I} are the bipartitions $[h]^2$ for $0 \leqslant h \leqslant e-2$, together with all the bipartitions $[ik]$.

Now suppose B is some Type I block other than B_{I}. Then we have $\delta_h(B) > 0$ for some $h \in \mathbb{Z}/e\mathbb{Z}$, and we let $C = \Phi_h(B)$ and assume that the proposition holds for C. Let $\mathcal{L}_B$ denote the set of bipartitions in B which the proposition claims to be Kleshchev. We must show that for $\boldsymbol{\lambda}$ a bipartition in B, $\Psi_h(\boldsymbol{\lambda})$ is Kleshchev if and only if $\boldsymbol{\lambda} \in \mathcal{L}_B$.

If $\delta_h(B) \geqslant 2$, the proposition holds by Proposition 3.9: the set $\mathcal{L}_B$ depends only on the sets I_B and K_B and the order $\succcurlyeq_B$, and these are obtained from those for C by applying the function $\overline{\phi_h}$. On the other hand, the correspondence $\boldsymbol{\lambda} \leftrightarrow \Psi_h(\boldsymbol{\lambda})$ is also obtained by applying $\overline{\phi_h}$.

So suppose $\delta_h(B) = 1$, and that $\boldsymbol{\lambda}$ is a bipartition in B. If $\boldsymbol{\lambda}$ is not one of the three exceptional bipartitions, then we may apply essentially the same argument as in the case where $\delta_h(B) \geqslant 2$; it is easily checked that if $\boldsymbol{\lambda}$ is non-exceptional, then the conditions for $\boldsymbol{\lambda}$ to lie in $\mathcal{L}_B$ do not depend upon whether $h \succcurlyeq h-1$, and this is (up to relabelling using $\overline{\phi_h}$) the only place where $\succcurlyeq_B$ and $\succcurlyeq_C$ differ.

Finally we check the three exceptional partitions. We begin by noting the following.

Claim. If $m \in \mathbb{Z}/e\mathbb{Z}$, then $m \prec_B h$ if and only if $m \not\succ_B h-1$.

Proof. This is a matter of considering the possible abacus configurations; if m were a counterexample to the claim, then m would have to lie strictly between $h-1$ and h, which is absurd.

Now there are two cases, according to which of $h-1$ and h lies in I_B.

$[h \in I_B,\ h-1 \in K_B]$

- $[h-1]^2_B$ lies in $\mathcal{L}_B$ because $h-1 \prec_B h$. On the other hand, $[(h-1)h]_C$ is Kleshchev because $h \nsucccurlyeq_C h-1$.
- $[h]^2_B$ lies in $\mathcal{L}_B$ if and only if there is some $m \in \mathbb{Z}/e\mathbb{Z}$ with $m \npreccurlyeq_B h$. On the other hand, $[h]^2_C$ is Kleshchev if and only there is some $m \in \mathbb{Z}/e\mathbb{Z}$ with $m \succ_C h$. Given any m, we have

$$\begin{aligned} m \npreccurlyeq_B h &\Leftrightarrow m \nprec_B h, m \neq h \\ &\Leftrightarrow m \succ_B h-1, m \neq h \\ &\Leftrightarrow m \succ_C h, \end{aligned}$$

  and so $[h]^2_B \in \mathcal{L}_B$ if and only if $[h]^2_C$ is Kleshchev.
- $[h(h-1)]_B$ lies in $\mathcal{L}_B$ because $h-1 \nsucccurlyeq_B h$. On the other hand, $[h-1]^2_C$ is Kleshchev because $h-1 \nsucccurlyeq_C h$.

$[h-1 \in I_B,\ h \in K_B]$

- $[h-1]^1_B$ lies in $\mathcal{L}_B$ since $h \succcurlyeq_B h-1$. On the other hand, $[h(h-1)]_C$ is Kleshchev because $h-1 \nsucccurlyeq_C h$.
- Since $h \succcurlyeq_B h-1$, we find that $[h]^1_B$ lies in $\mathcal{L}_B$ if and only if there is some $l \in K_B$ with $l \succ_B h$. On the other hand, $[h]^1_C$ is Kleshchev if and only if there is some $l \in K_C$ such that $l \succcurlyeq_C h$. For $l \in \mathbb{Z}/e\mathbb{Z}$, we have

$$\begin{aligned} l \in K_B, l \succ_B h &\Leftrightarrow l \in K_B, l \nprec_B h, l \neq h \\ &\Leftrightarrow l \in K_B, l \succ_B h-1, l \neq h \\ &\Leftrightarrow l \in K_C, l \succ_C h, \end{aligned}$$

  and so $[h]^1_B \in \mathcal{L}_B$ if and only if $[h]^2_C$ is Kleshchev.
- $[(h-1)h]_B$ lies in $\mathcal{L}_B$ if and only if there exist $j \in I_B$, $l \in K_B$ with $j \prec_B h-1$, $l \succ_B h$. On the other hand, $[h-1]^1_C$ is Kleshchev if and only if there are $j \in I_C$, $l \in K_C$ with $l \succ_C h-1 \succ_C j$. Given any j, we have

$$\begin{aligned} j \in I_B, j \prec_B h-1 &\Leftrightarrow j \in I_B, j \nsucc_B h-1, j \neq h-1 \\ &\Leftrightarrow j \in I_B, j \prec_B h, j \neq h-1 \\ &\Leftrightarrow j \in I_C, j \prec_C h-1, \end{aligned}$$

  while for any l we have

$$l \in K_B, l \succ_B h \Leftrightarrow l \in K_C, l \succ_C h-1,$$

  and so the two conditions are equivalent. □

We wish to describe the set of conjugate Kleshchev bipartitions also. To do this, we describe the conjugation action on bipartitions in Type I blocks. If B is a Type I block of $\mathcal{H}_n$ with root $\boldsymbol{\nu}$, let B' be the Type I block of $\mathcal{H}_n$ with root $\boldsymbol{\nu}'$. Then a bipartition $\boldsymbol{\lambda}$ lies in B if and only if $\boldsymbol{\lambda}'$ lies in B', so B' is the block conjugate to B. The relationship between B and B' in terms of our notation for Type I blocks is as follows.

Lemma 3.12. *Define the bijection* $\check{}: \mathbb{Z}/e\mathbb{Z} \to \mathbb{Z}/e\mathbb{Z}$ *by* $\check{h} = a_1 + a_2 - 1 - h$. *Then we have*

$$\begin{aligned} I_{B'} &= \{\check{\imath} \mid i \in I_B\}, \\ K_{B'} &= \{\check{k} \mid k \in K_B\}, \\ (g \succcurlyeq_{B'} h) &\Leftrightarrow (\check{g} \preccurlyeq_B \check{h}), \\ ([h]^a_B)' &= [\check{h}]^{3-a}_{B'}, \\ ([ik]_B)' &= [\check{\imath}\check{k}]_{B'}. \end{aligned}$$

Proof. It is easily checked that if ν is a partition and $B(\nu)$ is the set of beta-numbers for ν with charge a, then $\mathbb{Z} \setminus \{b - \beta \mid \beta \in B(\nu)\}$ is the set of beta-numbers for ν' with charge $b + 1 - a$. Hence if $\boldsymbol{\nu} = (\nu^{(1)}, \nu^{(2)})$ and we calculate $B(\nu^{(1)})$ and $B(\nu^{(2)})$ using the bicharge (a_1, a_2), then

$$\mathbb{Z} \setminus \{\check{\beta} \mid \beta \in B(\nu^{(2)})\}$$

and

$$\mathbb{Z} \setminus \{\check{\beta} \mid \beta \in B(\nu^{(1)})\}$$

give the beta-numbers for $\boldsymbol{\nu} = (\nu^{(2)\prime}, \nu^{(1)\prime})$ with bicharge (a_1, a_2). From this we can calculate the integers $\gamma_i(\boldsymbol{\nu}')$ in terms of the integers $\gamma_i(\boldsymbol{\nu})$, and hence calculate $I_{B'}$, $K_{B'}$ and $\succcurlyeq_{B'}$. The conjugates of the various bipartitions are calculated in the same way. □

We can immediately read off the set of conjugate Kleshchev bipartitions.

Corollary 3.13. *Suppose B is a Type I block, with bipartitions* $[h]^a$ *and* $[ik]$ *as defined above.*

- *For* $i \in I$, *the bipartition* $[i]^1$ *is conjugate Kleshchev if and only if there is some* $m \in \mathbb{Z}/e\mathbb{Z}$ *with* $i \not\preccurlyeq m$.
- *For* $i \in I$, *the bipartition* $[i]^2$ *is conjugate Kleshchev if and only if there is some* $k \in K$ *with* $i \succcurlyeq k$.
- *For* $k \in K$, *the bipartition* $[k]^1$ *is conjugate Kleshchev if and only if there is some* $m \in \mathbb{Z}/e\mathbb{Z}$ *with* $k \succ m$.
- *For* $k \in K$, *the bipartition* $[k]^2$ *is conjugate Kleshchev if and only if there are some* $i \in I, l \in K$ *with* $i \succcurlyeq k \succ l$.
- *For* $i \in I$ *and* $k \in K$, *the bipartition* $[ik]$ *is conjugate Kleshchev if and only if either* $k \not\preccurlyeq i$ *or there exist* $j \in I, l \in K$ *with* $i \prec j$ *and* $k \succ l$.

3.5 Decomposition numbers for Type I blocks

In this section, we calculate the decomposition numbers for a Type I block. This is done in the same way as the corresponding calculation by Richards for weight 2 blocks of Iwahori–Hecke algebras, using the Jantzen–Schaper formula and analysing several cases. Fortunately, we do not have quite as many cases to contend with.

Of course, the decomposition numbers $[S^\lambda : D^\mu]$ are easier to calculate if we know what the bipartition $\boldsymbol{\mu}^\diamond$ is. But this will emerge from our calculations, using Lemma 1.9. The logic of our argument is as follows: our main theorem will be a statement of the decomposition numbers for Type I bipartitions. This will be split into several cases, and will inherently specify a map $\boldsymbol{\mu} \mapsto \boldsymbol{\mu}^*$, which will be a bijection from the set of Kleshchev bipartitions in B to the set of conjugate Kleshchev bipartitions in B. For each case, we attempt to calculate the decomposition numbers $[S^\lambda : D^\mu]$ for those λ with $\mu \trianglelefteq \lambda \trianglelefteq \mu^*$. We are able to find these exactly except for $[S^{\mu^*} : D^\mu]$, where we find simply that $[S^{\mu^*} : D^\mu] > 0$ in each case. By Lemma 1.9 we shall have $\mu^* = \mu^\diamond$ for all μ, and we shall know all the decomposition numbers by Proposition 1.8.

We introduce further notation: given h in I_B (or in K_B, respectively), we let h^+ be the least element (with respect to the order $\succcurlyeq$) of I_B (respectively K_B) such that $h^+ \succ h$, if there is any such element. And we define h^- to be the greatest element of I_B (or K_B, respectively) such that $h^- \prec h$, if there is such an element.

Table 1 is split into thirteen cases, according to the possible pairs μ, μ^*. Each case is then split into sub-cases, according to the possible λ such that $[S^\lambda : D^\mu] = 1$.

Theorem 3.14. *Suppose $r = 2$, $e < \infty$ and B is a Type I weight 2 block of $\mathcal{H}_n$. Let $I, K, \succcurlyeq$ be as defined above for B. If λ and μ are bipartitions in B with μ Kleshchev, then the decomposition number $[S^\lambda : D^\mu]$ is either 0 or 1. For each μ, the bipartitions λ with $[S^\lambda : D^\mu] = 1$ are listed in Table 1. In each case, conditions involving i^+, i^-, k^+, k^- should be ignored if these elements do not exist.*

Before examining each case of Theorem 3.14 separately, we prove a useful lemma which uses the Jantzen–Schaper formula to calculate decomposition numbers for Type I blocks. Recall the relation $\lambda \to \mu$ and the function $\epsilon_{\lambda\mu}$ from Section 1.3.6.

Lemma 3.15. *Suppose B is a Type I weight 2 block of $\mathcal{H}_n$, and that μ, ν, ξ are bipartitions in B such that:*

- *μ is Kleshchev;*
- *$[S^\nu : D^\mu] = 1$;*
- *$\xi \to \mu$ and $\xi \to \nu$, and $\epsilon_{\xi\mu} \neq \epsilon_{\xi\nu}$;*
- *μ and ν are the only bipartitions π in B for which $\xi \to \pi$ and $[S^\pi : D^\mu] > 0$;*
- *(μ, ν, ξ) takes one of the following forms:*
  1. *$([hk], [ik], [jk])$ $(h, i, j \in I, k \in K)$;*
  2. *$([hk], [ik], [k]^a)$ $(h, i \in I, k \in K, a \in \{1, 2\})$;*
  3. *$([k]^2, [ik], [jk])$ $(i, j \in I, k \in K)$;*
  4. *$([k]^2, [ik], [k]^1)$ $(i \in I, k \in K)$;*
  5. *$([f]^a, [g]^a, [h]^a)$ $(f, g, h \in \mathbb{Z}/e\mathbb{Z}, a \in \{1, 2\})$.*

Then $[S^\xi : D^\mu] = 0$.

Proof. For cases (1–4), we show that $g_{\xi\mu} g_{\xi\nu} = 1$ for any appropriate $R, \hat{q}, \hat{Q}_1, \hat{Q}_2$; then the Jantzen–Schaper formula gives the result. In each of these cases we find, by checking the abacus displays, that:

- μ is obtained from ξ by adding a rim hook h_1 to the first component, and removing a rim hook h_2 from the second component;
- ν is obtained from ξ by adding a rim hook l_1 to the first component, and removing a rim hook l_2 from the second component;
- h_i and l_i have *the same* foot node, for $i = 1, 2$.

Case	μ	conditions	μ^*	additional conditions	λ for which $\mu \lhd \lambda \lhd \mu^*$ and $[S^\lambda : D^\mu] = 1$
A	$[i]^1$	$i \in I$ $(\exists\, k \in K)(k \succcurlyeq i \not\preccurlyeq k^-)$	$[ik]$	$(\exists\, i^+, k \succcurlyeq i^+)$	$[i^+k], [i^+]^1$
				$(k \not\succcurlyeq i^+)$	$[k]^1$
A′	$[ik]$	$k \preccurlyeq i \not\succcurlyeq k^+$	$[i]^2$	$(\exists\, i^-, k \preccurlyeq i^-)$	$[i^-]^2, [i^-k]$
				$(k \not\preccurlyeq i^-)$	$[k]^2$
B	$[i]^2$	$i \in I, \exists\, i^+$ $(\forall\, k \in K)(k \succcurlyeq i^+ \text{ or } i \succcurlyeq k)$	$[i^+]^1$	—	$[i]^1, [i^+]^2$
C	$[i]^2$	$i \in I$ $(\exists\, k \in K)(i^+ \not\preccurlyeq k \succcurlyeq i \succcurlyeq k^-)$	$[k]^1$	$(\exists\, i^+, k \not\preccurlyeq i^+)$	$[i]^1, [i^+k], [i^+]^2$
				$(k \preccurlyeq i^+)$	$[i]^1, [k]^2$
C′	$[k]^2$	$k \in K$ $(\exists\, i \in I)(k^+ \succcurlyeq i \succcurlyeq k \not\preccurlyeq i^-)$	$[i]^1$	$(\exists\, i^-, k \not\succcurlyeq i^-)$	$[i]^2, [i^-k], [i^-]^1$
				$(k \succcurlyeq i^-)$	$[i]^2, [k]^1$
D	$[i]^2$	$i \in I$ $(\exists\, k \in K)(k \not\gtrless i \succcurlyeq k^-)$	$[ik]$	$(\exists\, i^+, k \not\preccurlyeq i^+)$	$[i^+k], [i^+]^2$
				$(k \preccurlyeq i^+)$	$[k]^2$
D′	$[ik]$	$k^+ \succcurlyeq i \not\gtrless k$	$[i]^1$	$(\exists\, i^-, k \not\succcurlyeq i^-)$	$[i^-]^1, [i^-k]$
				$(k \succcurlyeq i^-)$	$[k]^1$
E	$[k]^1$	$k \in K, \exists\, k^+$ $(\exists\, i \in I)(i^+ \not\preccurlyeq k \succcurlyeq i)$	$[ik^+]$	$(\exists\, i^+, k^+ \succcurlyeq i^+)$	$[ik], [i^+k^+], [i^+]^1$
				$(k^+ \not\succcurlyeq i^+)$	$[ik], [k^+]^1$
E′	$[ik]$	$\exists\, k^+$ $i \succcurlyeq k^+ \not\preccurlyeq i^-$	$[k^+]^2$	$(\exists\, i^-, k \preccurlyeq i^-)$	$[ik^+], [i^-]^2, [i^-k]$
				$(k \not\preccurlyeq i^-)$	$[ik^+], [k]^2$
F	$[k]^2$	$k \in K, \exists\, k^+$ $(\forall\, i \in I)(k^+ \not\succcurlyeq i \succcurlyeq k$ $\text{or } k^+ \succcurlyeq i \not\succcurlyeq k)$	$[k^+]^1$	$(\exists\, j \in I)(j \not\gtrless k \preccurlyeq j^+)$ $(\exists\, i \in I)(i \not\gtrless k^+ \succcurlyeq i^-)$	$[ik^+], [i]^2, [j]^1, [jk]$
				$(\nexists\, j \in I)(j \not\gtrless k)$ $(\exists\, i \in I)(i \not\gtrless k^+ \succcurlyeq i^-)$	$[ik^+], [i]^2, [k]^1$
				$(\exists\, j \in I)(j \not\gtrless k \preccurlyeq j^+)$ $(\nexists\, i \in I)(i \not\gtrless k^+)$	$[k^+]^2, [j]^1, [jk]$
				$(\nexists\, j \in I)(j \not\gtrless k)$ $(\nexists\, i \in I)(i \not\gtrless k^+)$	$[k^+]^2, [k]^1$
G	$[k]^2$	$k \in K, \exists\, k^+$ $(\exists\, i \in I)(i^+ \succcurlyeq k \not\gtrless i \not\gtrless k^+)$	$[ik^+]$	$(\exists\, i^+, k^+ \not\preccurlyeq i^+)$	$[ik], [i^+k^+], [i^+]^2$
				$(k^+ \preccurlyeq i^+)$	$[ik], [k^+]^2$
G′	$[ik]$	$\exists\, k^+$ $k \not\gtrless i \not\gtrless k^+ \succcurlyeq i^-$	$[k^+]^1$	$(\exists\, i^-, k \not\succcurlyeq i^-)$	$[ik^+], [i^-]^1, [i^-k]$
				$(k \succcurlyeq i^-)$	$[ik^+], [k]^1$
H	$[ik]$	$\exists\, i^-, k^+$ $(k \succcurlyeq i \text{ or } i^- \succcurlyeq k^+$ $\text{or } (k \not\gtrless i, i^- \not\gtrless k^+)$	$[i^-k^+]$	—	$[ik^+], [i^-k]$

Table 1

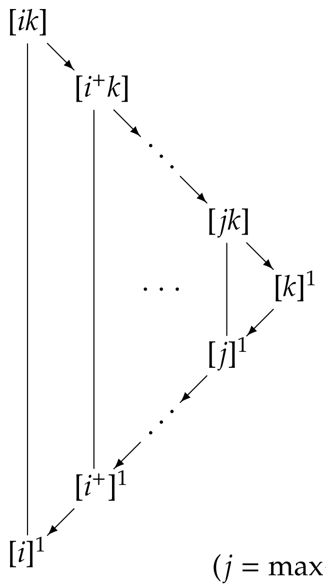
Now the condition $\epsilon_{\xi\mu} = -\epsilon_{\xi\nu}$ implies that $g_{\xi\mu}g_{\xi\nu} = 1$.

A similar argument deals with case 5. □

Now we prove Theorem 3.14. In each case, we find all the bipartitions $\boldsymbol{\lambda}$ such that $\boldsymbol{\mu} \trianglelefteq \boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}^*$, using Corollary 3.7. We then find the restriction of the relation $\rightarrow$ to this set of bipartitions, using Proposition 3.6. We indicate this relation in a diagram; our convention in the diagrams below is that whenever there are parallel arrows $\boldsymbol{\nu} \rightarrow \boldsymbol{\xi}$ and $\boldsymbol{\xi} \rightarrow \boldsymbol{\pi}$, there is an implicit arrow $\boldsymbol{\nu} \rightarrow \boldsymbol{\pi}$ parallel to these. Now we can find the decomposition numbers $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]$ for $\boldsymbol{\mu} \trianglelefteq \boldsymbol{\lambda} \triangleleft \boldsymbol{\mu}^*$: for each $\boldsymbol{\lambda}$, we either apply Corollary 1.15 to get $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}] = 1$ or Lemma 3.15 to get $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}] = 0$. By ad hoc use of the Jantzen–Schaper formula, we can easily find $[S^{\boldsymbol{\mu}^*} : D^{\boldsymbol{\mu}}] > 0$ in each case also.

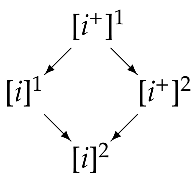
We indicate the diagrams for cases A–H; the diagrams for cases A′, C′, D′, E′ and G′ may be found by inverting the diagrams for cases A, C, D, E, G and conjugating all the bipartitions. Theorem 3.14 may now be verified, case by case.

Case A

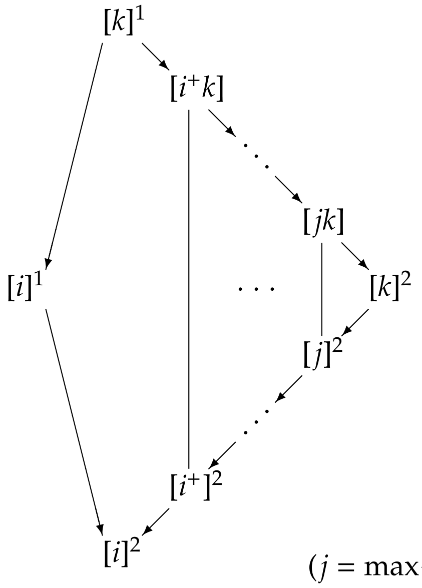


$(j = \max\{j \in I \mid j \preccurlyeq k\})$

Case B

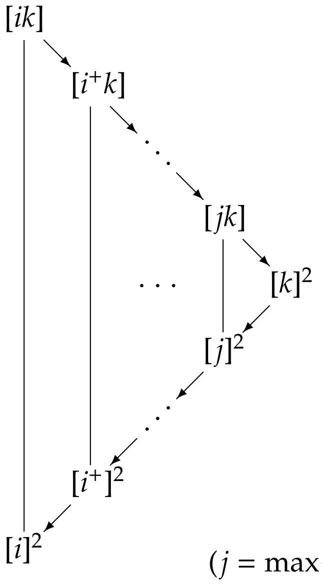


Case C



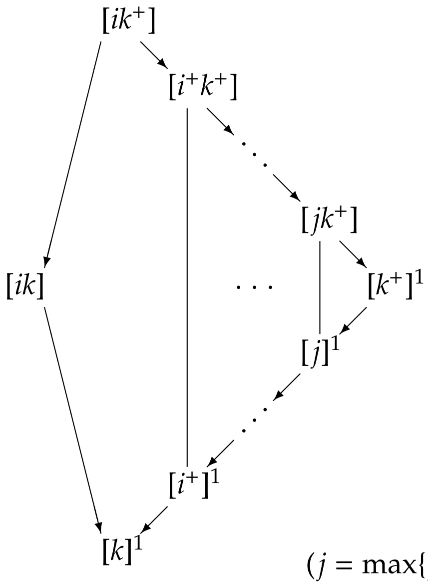
($j = \max\{j \in I \mid j \not\succcurlyeq k\}$)

Case D



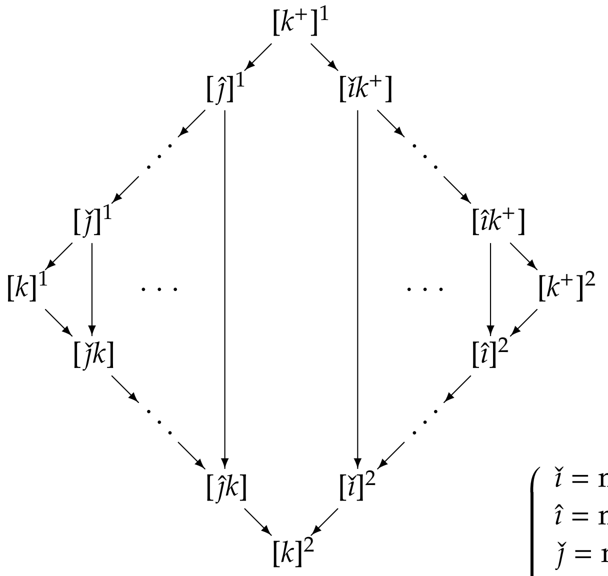
($j = \max\{j \in I \mid j \not\succcurlyeq k\}$)

Case E



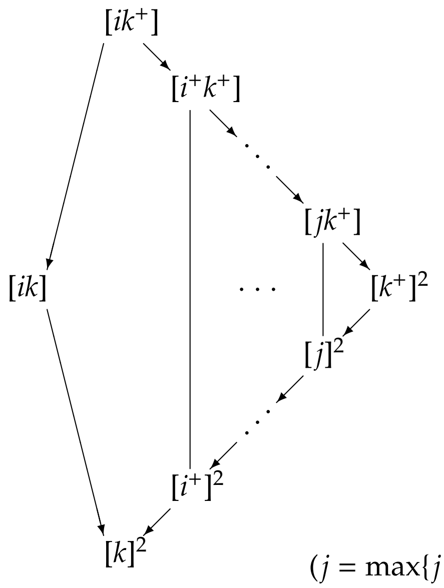
($j = \max\{j \in I \mid j \preccurlyeq k^+\}$)

Case F



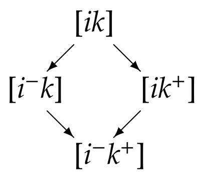
$$\begin{pmatrix} \check{\imath} = \min\{i \in I \mid i \not\gtrless k^+\} \\ \hat{\imath} = \max\{i \in I \mid i \not\gtrless k^+\} \\ \check{\jmath} = \min\{j \in I \mid j \not\gtrless k\} \\ \hat{\jmath} = \max\{j \in I \mid j \not\gtrless k\} \end{pmatrix}$$

Case G



$(j = \max\{j \in I \mid j \not\geq k^+\})$

Case H



3.6 A Richards-type theorem for Type I blocks

We now give a simpler description of the decomposition numbers for a Type I block, analogous to Richards's description for weight two blocks of Iwahori–Hecke algebras.

Suppose $\boldsymbol{\lambda}$ is a bipartition of Type I, with root $\boldsymbol{\nu}$. Then $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ are related in one of three ways:

- $\boldsymbol{\lambda}$ is obtained from $\boldsymbol{\nu}$ by adding a rim e-hook to one component (if $\boldsymbol{\lambda}$ is a bipartition of the form $[h]^a$);

- λ is obtained from ν by adding a rim d-hook to the first component and a rim $(e-d)$-hook to the second component (if λ is of the form $[ik]$ with $i \mathrel{\not\gtreqless} k$);
- λ is obtained from ν by adding a rim $(d+e)$-hook to one component and removing a rim d-hook from the other (if λ is of the form $[ik]$ with $i \succcurlyeq k$ or $k \succcurlyeq i$).

We define an integer $\partial\lambda$ as follows.

- If λ is obtained by adding a rim e-hook to the first component of ν, we define $\partial\lambda$ to be the leg length of this hook plus 1.
- If λ is obtained by adding a rim e-hook to the second component of ν, we define $\partial\lambda$ to be the leg length of this hook.
- If λ is obtained by adding a rim d-hook to one component of ν and a rim $(e-d)$-hook to the other component, we define $\partial\lambda$ to be the sum of the leg lengths of the two hooks plus 1.
- If λ is obtained by adding a rim $(d+e)$-hook to the first component of ν and removing a rim d-hook from the second, we define $\partial\lambda$ to be the leg length of the added hook minus the leg length of the removed hook plus 1.
- If λ is obtained by adding a rim $(d+e)$-hook to the second component of ν and removing a rim d-hook from the first, we define $\partial\lambda$ to be the leg length of the added hook minus the leg length of the removed hook minus 1.

We re-interpret this definition in terms of $I_B, K_B, \succcurlyeq_B$.

Proposition 3.16. *Suppose B is a Type I block. If $i \in I_B$ and $k \in K_B$ then*

$$\begin{aligned}
\partial[i]^1 &= |\{h \in \mathbb{Z}/e\mathbb{Z} \mid h \succcurlyeq i\}|,\\
\partial[k]^1 &= |\{h \in \mathbb{Z}/e\mathbb{Z} \mid h \nprec k\}|,\\
\partial[i]^2 &= |\{h \in \mathbb{Z}/e\mathbb{Z} \mid h \not\preccurlyeq i\}|,\\
\partial[k]^2 &= |\{h \in \mathbb{Z}/e\mathbb{Z} \mid h \succ k\}|,\\
\partial[ik] &= \begin{cases} |\{j \in I_B \mid j \succ i\}| + |\{l \in K_B \mid l \succ k\}| & (i \succcurlyeq k)\\ |\{j \in I_B \mid j \succ i\}| + |\{l \in K_B \mid l \succ k\}| + 1 & (i \mathrel{\not\gtreqless} k)\\ |\{j \in I_B \mid j \succ i\}| + |\{l \in K_B \mid l \succ k\}| + 2 & (i \preccurlyeq k). \end{cases}
\end{aligned}$$

Proof. Recall that adding a rim hook to a partition corresponds to increasing one of the beta-numbers for that partition. If this beta-number is increased from b to c, then the leg length of the rim hook equals the number of beta-numbers lying in $\{b+1, \dots, c-1\}$. Given this, it is easy to check the various cases. □

Lemma 3.17. *Suppose B is a Type I block, and $d \in \mathbb{Z}$. Then the bipartitions λ in B with $\partial\lambda = d$ are totally ordered by $\trianglerighteq$.*

Proof. Suppose λ and μ are bipartitions in B with $\lambda \ntrianglerighteq \mu \ntrianglerighteq \lambda$. Using Corollary 3.7, we can find the various possibilities for λ and μ, and show that $\partial\lambda \neq \partial\mu$ in each case. For example, if $\lambda = [k]^1$ and $\mu = [i]^2$ with $i \in I_B$, $k \in K_B$, then the condition $\lambda \ntrianglerighteq \mu$ implies that $i \succcurlyeq k$. We have

$$\partial\lambda = |\{h \in \mathbb{Z}/e\mathbb{Z} \mid h \nprec k\}|, \qquad \partial\mu = |\{h \in \mathbb{Z}/e\mathbb{Z} \mid h \not\preccurlyeq i\}|;$$

the condition $i \succcurlyeq k$ implies that

$$\{h \in \mathbb{Z}/e\mathbb{Z} \mid h \not\succ k\} \supseteq \{h \in \mathbb{Z}/e\mathbb{Z} \mid h \not\succcurlyeq i\},$$

and this inclusion is strict, since the first set contains i and k while the second does not. So we have $\partial\lambda > \partial\mu$. The other possibilities may be checked just as easily. □

Now we can state our Richards-type theorem.

Theorem 3.18. *Suppose B is a Type I block, and μ is a bipartition in B. Then μ is Kleshchev if and only if there is some ν in B with $\nu \rhd \mu$ and $\partial\nu = \partial\mu$. In this case, $\mu^\diamond$ is the least dominant such ν, and for any bipartition λ in B we have*

$$[S^\lambda : D^\nu] = \begin{cases} 1 & (\mu \trianglelefteqslant \lambda \trianglelefteqslant \mu^\diamond, |\partial\lambda - \partial\mu| \leqslant 1) \\ 0 & (\textit{otherwise}). \end{cases}$$

Proof. For each of cases A–H above, we calculate $\partial\lambda - \partial\mu$ for each λ with $\mu \trianglelefteqslant \lambda \trianglelefteqslant \mu^\diamond$, using Proposition 3.16. The diagrams of these cases are arranged so that two bipartitions in the same column have the same ∂-value, and these values decrease from left to right. We find that in each case:

- $\partial\mu^\diamond = \partial\mu$;
- there is no $\mu \lhd \lambda \lhd \mu^\diamond$ with $\partial\lambda = \partial\mu$;
- the bipartitions λ with $[S^\lambda : D^\mu] = 1$ are precisely those with $|\partial\lambda - \partial\mu| \leqslant 1$.

For example, suppose we are in Case C with $i^+ \not\succcurlyeq k$. We calculate the ∂-value of each bipartition in the diagram, from left to right:

$$\begin{aligned}
\partial[i]^1 &= |\{h \mid h \succcurlyeq i\}| &&= |\{i, i^+, i^{++}, \dots\} \cup \{k, k^+, k^{++}, \dots\}|;\\
\partial[k]^1 &= |\{h \mid h \not\succ k\}| &&= |\{i^+, i^{++}, \dots\} \cup \{k, k^+, k^{++}, \dots\}|;\\
\partial[i]^2 &= |\{h \mid h \not\succcurlyeq i\}| &&= |\{i^+, i^{++}, \dots\} \cup \{k, k^+, k^{++}, \dots\}|;\\
\partial[i^+k] &= |\{h \in I \mid h \succ i^+\}| + |\{h \in K \mid h \succ k\}| + 1 &&= |\{i^{++}, i^{+++}, \dots\} \cup \{k, k^+, k^{++}, \dots\}|;\\
\partial[i^+]^2 &= |\{h \mid h \not\succcurlyeq i^+\}| &&= |\{i^{++}, i^{+++}, \dots\} \cup \{k, k^+, k^{++}, \dots\}|;\\
&\vdots\\
\partial[jk] &= |\{h \in I \mid h \succ j\}| + |\{h \in K \mid h \succ k\}| + 1 &&= |\{j^+, j^{++}, \dots\} \cup \{k, k^+, k^{++}, \dots\}|;\\
\partial[j]^2 &= |\{h \mid h \not\succcurlyeq j\}| &&= |\{j^+, j^{++}, \dots\} \cup \{k, k^+, k^{++}, \dots\}|;\\
\partial[k]^2 &= \{h \mid h \succ k\} &&= |\{j^+, j^{++}, \dots\} \cup \{k^+, k^{++}, \dots\}|.
\end{aligned}$$

The bipartitions λ with $[S^\lambda : D^{[i]^2}] = 1$ are precisely those in the first three columns of the diagram.

The result follows for cases A–H. Cases A′, C′, D′, E′, G′ follow from these and the easily-verified formula

$$\partial\lambda' = e - \partial\lambda.$$ □

4 Blocks of Type II

We now turn to blocks of Type II. We undertake the same tasks as for Type I blocks: describing the bipartitions and the Kleshchev bipartitions in a Type II block, finding the dominance order for these bipartitions, and calculating the decomposition numbers.

Blocks of Type II behave differently from blocks of Type I. They do not have a naturally associated bipartition of weight 0, which makes the partitions in them slightly awkward to describe. On the other hand, we shall see that if B is any block of Type II, then B and $\Phi_i(B)$ have the same decomposition matrix. This makes it easy to prove a Richards-type theorem for these blocks inductively.

As with Type I blocks, we begin by showing that we may get from a Type II block to one of the prototype blocks B_{II} or B^*_{II} by a sequence of the maps Φ_i; this will facilitate an inductive approach to many of our results.

Proposition 4.1. *Suppose B is a weight 2 block of $\mathcal{H}_n$ containing a bipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ of Type II. Then there is a sequence $n = n_0 > \cdots > n_m$ of positive integers, a sequence $B = B_0, \dots, B_m$, where B_j is a block of $\mathcal{H}_{n_j}$ for each j, and a sequence $i_1, \dots, i_m$ of elements of $\mathbb{Z}/e\mathbb{Z}$, such that*

$$B_j = \Phi_{i_j}(B_{j-1})$$

for $j = 1, \dots, m$, and B_m is either the block B_{II} or the block B^*_{II} from §2.2.1.

In particular, B contains only bipartitions of Type II.

Proof. We proceed by induction on n. As in the proof of Proposition 3.4, it suffices to prove that if $\delta_i(B) \leqslant 0$ for all i, then B is either the block B_{II} or the block B^*_{II}.

Suppose that $\delta_i(B) \leqslant 0$ for all i. Suppose $\lambda^{(1)} \neq \varnothing$, so that $\lambda^{(1)}$ has a removable i-node for some i. Either $\lambda^{(1)}$ or $\lambda^{(2)}$ must have an addable i-node, but $\lambda^{(1)}$ can't by Lemma 3.1, since $\lambda^{(1)}$ is a core. So $\lambda^{(2)}$ has at least one addable i-node. Now if $\lambda^{(1)}$ has x removable i-nodes and $\lambda^{(2)}$ has y addable i-nodes, then

$$\gamma_i(\boldsymbol{\lambda}) - \gamma_{i-1}(\boldsymbol{\lambda}) = x + y,$$

so the condition for $\boldsymbol{\lambda}$ to be Type II means that $x = y = 1$. If we define $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)})$ by removing the removable i-node from $\lambda^{(1)}$ and adding the addable i-node of $\lambda^{(2)}$, then $\boldsymbol{\mu} = s_{i(i-1)}(\boldsymbol{\lambda})$ lies in B, and we have $|\mu^{(1)}| = |\lambda^{(1)}| - 1$.

So by induction on $|\lambda^{(1)}|$ we may assume that $\lambda^{(1)} = \varnothing$. Certainly $\lambda^{(2)} \neq \varnothing$, since otherwise we should have $w(\boldsymbol{\lambda}) = 0$. So $\lambda^{(2)}$ has a removable i-node, for some i. By a similar argument to that used above, we find that $\lambda^{(2)}$ has exactly one removable i-node, and $\lambda^{(1)}$ has exactly one addable i-node. Hence $q^i = Q_1$, and so $\lambda^{(2)}$ has only one removable node. So $\lambda^{(2)}$ is a rectangular partition, say $\lambda^{(2)} = (c^d)$. We have $c + d \leqslant e$ since $\lambda^{(2)}$ is a core, and so by examining the residues of the nodes we find that

$$c_{Q_2}(\boldsymbol{\lambda}) = c_{Q_1}(\boldsymbol{\lambda}) = \min\{c, d\},$$

and that $(c_f(\boldsymbol{\lambda}) - c_{qf}(\boldsymbol{\lambda}))^2$ equals 1 for exactly $2\min\{c, d\}$ values of f, and 0 for all other values. Hence

$$2 = w(\boldsymbol{\lambda}) = \min\{c, d\}.$$

If $c = 2 \leqslant d$, we get $Q_2 = q^{d-2}Q_1$, and we find that $\boldsymbol{\lambda}$ lies in the block B_{II}. If $c \geqslant 2 = d$, then $Q_1 = q^{c-2}Q_2$, and so $\boldsymbol{\lambda}$ lies in the block B^*_{II}. □

4.1 The dominance order, Kleshchev bipartitions and decomposition numbers in B_{II} and B^*_{II}

In order to work out the decomposition numbers for Type II blocks, we begin by looking at the prototype blocks B_{II} and B^*_{II}. First we must describe the Jantzen–Schaper dominance order and find the Kleshchev bipartitions in these blocks. For the dominance order, recall the relation $\rightarrow$ which generates $\trianglerighteq$.

To begin with, we look at the block B_{II}. We shall state corresponding results for B^*_{II}, which are proved in exactly the same way, at the end of this section.

Proposition 4.2. *Suppose B is the weight 2 block B_{II}, with the integer p and the bipartitions $\boldsymbol{\lambda}_{c,d}$ as defined in §2.2.2. Then we have $\boldsymbol{\lambda}_{c,d} \rightarrow \boldsymbol{\lambda}_{a,b}$ if and only if*

- *$c = a$ and $d \geqslant b$, or*
- *$c \geqslant a$ and $d = b$, or*
- *$d = a + 1$.*

Hence we have $\boldsymbol{\lambda}_{c,d} \trianglerighteq \boldsymbol{\lambda}_{a,b}$ if and only if $c \geqslant a$ and $d \geqslant b$.

Proof. This is easy to check. □

Now we find which bipartitions in B_{II} are Kleshchev.

Proposition 4.3. *Suppose B is the weight 2 block B_{II}. Then the bipartition $\boldsymbol{\lambda}_{c,d}$ in B is Kleshchev if and only if $c < p + 2$ and $d < p + 1$.*

Proof. Suppose first that $c < p+2$ and $d < p+1$. Then we may remove nodes from $\boldsymbol{\lambda}_{c,d}$ in the following order to get to the empty bipartition; it is easy to check that at each stage the removed node is good:

$$\begin{aligned}
\boldsymbol{\lambda}_{c,d} &= ((c,d),(2^{p+2-c},1^{c-d}))\\
&\rightarrow ((c,d),(2^{p+1-c},1^{c-d+1}))\rightarrow \cdots \rightarrow ((c,d),(1^{p+2-d}))\\
&\rightarrow ((c-1,d),(1^{p+2-d})) \quad \rightarrow \cdots \rightarrow ((d,d),(1^{p+2-d}))\\
&\rightarrow ((d,d),(1^{p+1-d})) \quad \rightarrow \cdots \rightarrow ((d,d),\varnothing)\\
&\rightarrow ((d,d-1),\varnothing) \quad \rightarrow \cdots \rightarrow ((d),\varnothing)\\
&\rightarrow ((d-1),\varnothing) \quad \rightarrow \cdots \rightarrow (\varnothing,\varnothing).
\end{aligned}$$

Now we look at the other bipartitions. $\boldsymbol{\lambda}_{p+2,p+2}$ has no normal nodes, so cannot be Kleshchev. From the bipartition $\boldsymbol{\lambda}_{p+2,d}$ with $d < p + 2$, we remove good nodes as follows:

$$\begin{aligned}
\boldsymbol{\lambda}_{p+2,d} &= ((p+2,d),(1^{p+2-d}))\\
&\rightarrow ((p+2,d),(1^{p+1-d}))\rightarrow \cdots \rightarrow ((p+2,d),\varnothing)\\
&\rightarrow ((p+2,d-1),\varnothing) \quad \rightarrow \cdots \rightarrow ((p+2),\varnothing)\\
&\rightarrow ((p+1),\varnothing).
\end{aligned}$$

This last bipartition has no normal nodes, so is not Kleshchev, and so by Proposition 1.3 $\boldsymbol{\lambda}_{p+2,d}$ is not Kleshchev.

Finally, consider $\lambda_{p+1,p+1}$. We remove good nodes as follows:

$$\begin{aligned}\lambda_{p+1,p+1} &= ((p+1,p+1),(2))\\ &\to ((p+1,p+1),(1)) \to ((p+1,p+1),\varnothing) \to ((p+1,p),\varnothing)\\ &\to ((p+1,p-1),\varnothing) \ \to \cdots \qquad\qquad \to ((p+1),\varnothing).\end{aligned}$$

So $\lambda_{p+1,p+1}$ is not Kleshchev either. □

Corollary 4.4. *Suppose B is the block B_{II}. Then $\lambda_{c,d}$ is conjugate Kleshchev if and only if $c > 1$ and $d > 0$.*

Proof. By examining residues, we find that B_{II} is self-conjugate. The conjugation map is given by

$$\lambda_{c,d} \longmapsto \lambda_{p+2-d,p+2-c},$$

and the result follows. □

Now we can describe the decomposition numbers for B_{II}.

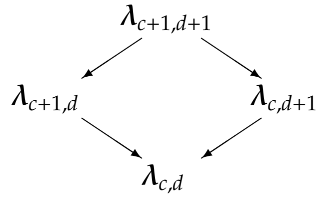
Theorem 4.5. *Suppose B is the block B_{II}. If λ, μ are bipartitions in B with μ Kleshchev, then the decomposition number $[S^\lambda : D^\mu]$ equals 0 or 1. For each μ, the bipartitions λ with $[S^\lambda : D^\mu] = 1$ are described in Table 2.*

Case	μ	conditions	μ^*	λ for which $\mu \lhd \lambda \lhd \mu^*$ and $[S^\lambda : D^\mu] = 1$
A	$\lambda_{c,d}$	$0 \leqslant d < c \leqslant p+1$	$\lambda_{c+1,d+1}$	$\lambda_{c+1,d}, \lambda_{c,d+1}$
B	$\lambda_{d,d}$	$0 \leqslant d \leqslant p$	$\lambda_{d+2,d+2}$	$\lambda_{d+1,d}, \lambda_{d+2,d+1}$

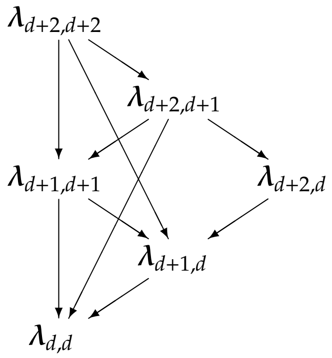
Table 2

Our approach to proving Theorem 4.5 is much the same as our approach to Theorem 3.14, although the details are much simpler. As for Theorem 3.14, we draw diagrams of the relation $\to$ on the set of bipartitions λ with $\mu \trianglelefteq \lambda \trianglelefteq \mu^*$. It then remains to use the Jantzen–Schaper formula; since this is rather easier than for Type I, we omit the details.

Case A



Case B



Now we give the corresponding results for the block B^*_{II}.

Theorem 4.6. *Suppose B is the weight 2 block B^*_{II}, with the integer p and the bipartitions $\boldsymbol{\lambda}_{c,d}$ defined as above.*

1. *We have $\boldsymbol{\lambda}_{c,d} \trianglerighteqslant \boldsymbol{\lambda}_{a,b}$ if and only if $c \leqslant a$ and $d \leqslant b$.*
2. *$\boldsymbol{\lambda}_{c,d}$ is Kleshchev if and only if $c > 1$ and $d > 0$, and is conjugate Kleshchev if and only if $c < p+2$ and $d < p+1$.*
3. *Given $\boldsymbol{\lambda}, \boldsymbol{\mu}$ in B with $\boldsymbol{\mu}$ Kleshchev, the decomposition number $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]$ equals 0 or 1. For each $\boldsymbol{\mu}$, the bipartitions $\boldsymbol{\lambda}$ for which $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}] = 1$ are listed in Table 3.*

Case	$\boldsymbol{\mu}$	conditions	$\boldsymbol{\mu}^*$	$\boldsymbol{\lambda}$ for which $\boldsymbol{\mu} \lhd \boldsymbol{\lambda} \lhd \boldsymbol{\mu}^*$ and $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}] = 1$
A	$\boldsymbol{\lambda}_{c,d}$	$1 \leqslant d < c \leqslant p+2$	$\boldsymbol{\lambda}_{c-1,d-1}$	$\boldsymbol{\lambda}_{c-1,d}, \boldsymbol{\lambda}_{c,d-1}$
B	$\boldsymbol{\lambda}_{d,d}$	$2 \leqslant d \leqslant p+2$	$\boldsymbol{\lambda}_{d-2,d-2}$	$\boldsymbol{\lambda}_{d-1,d}, \boldsymbol{\lambda}_{d-2,d-1}$

Table 3

4.2 A Richards-type theorem for B_{II}

We may re-state Theorem 4.5 to describe the decomposition numbers for B_{II} in a way analogous to Richards's description of decomposition numbers for weight 2 blocks in type *A*.

Let *B* be the block B_{II} or B^*_{II}. We define a function ∂ on the set of bipartitions in *B* by $\partial\boldsymbol{\lambda}_{c,d} = c - d$. Furthermore, we say that $\boldsymbol{\lambda}_{d,d}$ is *black* if *d* is even, and *white* otherwise. Then we may re-state Theorem 4.5 and Theorem 4.6(3) as follows.

Theorem 4.7. *Let B be the block B_{II} or B^*_{II}.*

1. *The bipartitions $\boldsymbol{\lambda}$ in B with a given value of $\partial\boldsymbol{\lambda}$ are totally ordered by (Jantzen–Schaper) dominance.*
2. *A bipartition $\boldsymbol{\mu}$ in B is Kleshchev if and only if there is a bipartition $\boldsymbol{\nu}$ in B such that $\boldsymbol{\nu} \rhd \boldsymbol{\mu}$, $\partial\boldsymbol{\nu} = \partial\boldsymbol{\mu}$, and (if $\partial\boldsymbol{\mu} = 0$) $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ have the same colour. In this case, $\boldsymbol{\mu}^\diamond$ is the least dominant such $\boldsymbol{\nu}$.*
3. *The decomposition numbers for B are given by*

$$[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}] = \begin{cases} 1 & (\boldsymbol{\lambda} = \boldsymbol{\mu}, \boldsymbol{\lambda} = \boldsymbol{\mu}^\diamond \text{ or } (\boldsymbol{\mu} \trianglelefteqslant \boldsymbol{\lambda} \trianglelefteqslant \boldsymbol{\mu}^\diamond \text{ and } |\partial\boldsymbol{\mu} - \partial\boldsymbol{\lambda}| = 1)) \\ 0 & (\text{otherwise}). \end{cases}$$

4.3 The bipartitions in a Type II block

Now we consider Type II blocks in general. First, we need to describe the bipartitions in a Type II block; we use similar arguments to those used in the discussion of weight 1 blocks in [11].

Suppose $\boldsymbol{\lambda}$ is a Type II bipartition. There are well-defined sets $V_{\boldsymbol{\lambda}}, W_{\boldsymbol{\lambda}} \subset \mathbb{Z}/e\mathbb{Z}$ such that

$$\gamma_i(\boldsymbol{\lambda}) - \gamma_j(\boldsymbol{\lambda}) \begin{cases} = 2 & (i \in V_{\boldsymbol{\lambda}}, j \in W_{\boldsymbol{\lambda}}) \\ \leqslant 1 & (\text{otherwise}), \end{cases}$$

and we have either $|V_{\boldsymbol{\lambda}}| = 2 \leqslant |W_{\boldsymbol{\lambda}}|$ or $|V_{\boldsymbol{\lambda}}| \geqslant 2 = |W_{\boldsymbol{\lambda}}|$.

For $v \in V_{\boldsymbol{\lambda}}, w \in W_{\boldsymbol{\lambda}}$, we define $\boldsymbol{\lambda}_{vw}$ to be the bipartition $s_{vw}(\boldsymbol{\lambda})$ as defined in Section 1.3.5. For $u, v \in V_{\boldsymbol{\lambda}}$ and $w, x \in W_{\boldsymbol{\lambda}}$ with $u \neq v, w \neq x$, we define $\boldsymbol{\lambda}_{(uv)(wx)}$ to be the bipartition $s_{uw}(s_{vx}(\boldsymbol{\lambda}))$. Note that we have

$$\boldsymbol{\lambda}_{(uv)(wx)} = \boldsymbol{\lambda}_{(uv)(xw)} = \boldsymbol{\lambda}_{(vu)(wx)}.$$

$\boldsymbol{\lambda}_{vw}$ and $\boldsymbol{\lambda}_{(uv)(wx)}$ lie in the same block as $\boldsymbol{\lambda}$, by Lemma 1.13.

Now we can describe the bipartitions in a Type II block.

Proposition 4.8. *Suppose $\boldsymbol{\lambda}$ is a Type II bipartition lying in a block B of $\mathcal{H}_n$, and define $\boldsymbol{\lambda}_{vw}$ for $v \in V_{\boldsymbol{\lambda}}, w \in W_{\boldsymbol{\lambda}}$ as above. Then the set of bipartitions in B is precisely*

$$\{\boldsymbol{\lambda}\} \cup \{\boldsymbol{\lambda}_{vw} \mid v \in V_{\boldsymbol{\lambda}}, w \in W_{\boldsymbol{\lambda}}\} \cup \{\boldsymbol{\lambda}_{(uv)(vw)} \mid u \neq v \in V_{\boldsymbol{\lambda}}, w \neq x \in W_{\boldsymbol{\lambda}}\}.$$

We shall prove Proposition 4.8 by reducing to the case where $B = B_{\mathrm{II}}$ or B^*_{II}.

Lemma 4.9. *Suppose $\boldsymbol{\lambda}$ is a Type II bipartition and $i \in \mathbb{Z}/e\mathbb{Z}$. Then Proposition 4.8 holds for $\boldsymbol{\lambda}$ if and only if it holds for $\Phi_i(\boldsymbol{\lambda})$.*

Proof. Recall the bijections $\phi_i : \mathbb{Z} \to \mathbb{Z}$ and $\overline{\phi}_i : \mathbb{Z}/e\mathbb{Z} \to \mathbb{Z}/e\mathbb{Z}$. We know from Proposition 2.6(4) that $\Phi_i(\boldsymbol{\lambda})$ is of Type II, and we examine the sets $V_{\Phi_i(\boldsymbol{\lambda})}, W_{\Phi_i(\boldsymbol{\lambda})}$. Since the beta-numbers for $\Phi_i(\boldsymbol{\lambda})$ are obtained from those for $\boldsymbol{\lambda}$ by applying the function ϕ_i, we obtain

$$V_{\Phi_i(\boldsymbol{\lambda})} = \overline{\phi}_i(V_{\boldsymbol{\lambda}}), \quad W_{\Phi_i(\boldsymbol{\lambda})} = \overline{\phi}_i(W_{\boldsymbol{\lambda}}).$$

We also get

$$\Phi_i(\boldsymbol{\lambda}_{vw}) = (\Phi_i(\boldsymbol{\lambda}))_{\overline{\phi}_i(v)\overline{\phi}_i(w)}$$

and

$$\Phi_i(\boldsymbol{\lambda}_{(uv)(wx)}) = (\Phi_i(\boldsymbol{\lambda}))_{(\overline{\phi}_i(u)\overline{\phi}_i(v))(\overline{\phi}_i(w)\overline{\phi}_i(x))}$$

for $u \neq v \in V_{\boldsymbol{\lambda}}, w \neq x \in W_{\boldsymbol{\lambda}}$, and so Φ_i gives a bijection between

$$\{\boldsymbol{\lambda}\} \cup \{\boldsymbol{\lambda}_{vw} \mid v \in V_{\boldsymbol{\lambda}}, w \in W_{\boldsymbol{\lambda}}\} \cup \{\boldsymbol{\lambda}_{(uv)(vw)} \mid u \neq v \in V_{\boldsymbol{\lambda}}, w \neq x \in W_{\boldsymbol{\lambda}}\}$$

and

$$\{\Phi_i(\boldsymbol{\lambda})\} \cup \{(\Phi_i(\boldsymbol{\lambda}))_{vw} \mid v \in V_{\Phi_i(\boldsymbol{\lambda})}, w \in W_{\Phi_i(\boldsymbol{\lambda})}\} \cup \{\boldsymbol{\lambda}_{(uv)(vw)} \mid u \neq v \in V_{\Phi_i(\boldsymbol{\lambda})}, w \neq x \in W_{\Phi_i(\boldsymbol{\lambda})}\}. \qquad \square$$

Proof of Proposition 4.8. By Proposition 4.1 and Lemma 4.9, we may assume that B is the block B_{II} or B^*_{II}. In fact, we assume that B is B_{II}; the other case is similar.

We show also that we may also reduce to the case where $\lambda^{(1)} = \varnothing$: if $\lambda^{(1)}$ has a removable i-node for some i, then, since $\delta_i(\boldsymbol{\lambda}) \leqslant 0$, $\lambda^{(2)}$ has an addable i-node. So we have $\gamma_i(\boldsymbol{\lambda}) - \gamma_{i-1}(\boldsymbol{\lambda}) \geqslant 2$, so there is exactly one removable i-node and exactly one addable i-node. Applying the function Φ_i is equivalent to removing all removable i-nodes and adding all addable i-nodes, and so we replace $\boldsymbol{\lambda}$ with $\Phi_i(\boldsymbol{\lambda})$, and appeal to Lemma 4.9. We repeat this until we have removed all nodes from $\lambda^{(1)}$.

So we have $\boldsymbol{\lambda} = (\varnothing, (2^{p+2}))$, and we wish to calculate $V_{\boldsymbol{\lambda}}$ and $W_{\boldsymbol{\lambda}}$. We choose an integer a such that $Q_1 = q^a$, $Q_2 = q^{a+p}$, and as in the proof of Proposition 2.5, we find that

$$B(\lambda^{(1)}) = \{m \in \mathbb{Z} \mid m \leqslant a - 1\},$$

while

$$B(\lambda^{(2)}) = \{m \in \mathbb{Z} \mid m \leqslant a + p + 1\} \setminus \{a - 2, a - 1\}.$$

Hence, writing $\overline{m}$ for the residue of an integer m modulo e, we have

$$V_{\boldsymbol{\lambda}} = \{\overline{a-2}, \overline{a-1}\}, \qquad W_{\boldsymbol{\lambda}} = \{\overline{a}, \overline{a+1}, \ldots, \overline{a+p+1}\}.$$

We find

$$\lambda_{(a-2)w} = ((w-a+1,1),(2^{p+1+a-w},1^{w-a})) \qquad (a \leqslant w \leqslant a+p+1),$$
$$\lambda_{(a-1)w} = ((w-a+1),(2^{p+1+a-w},1^{w-a+1})) \qquad (a \leqslant w \leqslant a+p+1),$$
$$\lambda_{((a-2)(a-1))(wx)} = ((x-a+1,w-a+2),(2^{p+1+a-x},1^{x-w-1})) \qquad (a \leqslant w < x \leqslant a+p+1),$$

and so by Proposition 2.4 the result follows. □

Now, as we did for weight 1 blocks of Ariki–Koike algebras in [11], we wish to give a description of the set of bipartitions in a Type II block which is independent of the choice of $\boldsymbol{\lambda}$. Suppose $\boldsymbol{\lambda}$ is of Type II and lies in a block B, and let V_λ, W_λ be as above. Using Proposition 4.8, we may easily find that $|V_\lambda|$, $|W_\lambda|$ and $V_\lambda \cup W_\lambda$ are independent of the choice of $\boldsymbol{\lambda}$ in B, and so we write these as v_B, w_B, X_B respectively. Our description of the bipartitions in B will depend upon which of v_B and w_B equals 2; if both equal 2, then both descriptions apply.

- If $v_B = 2$, then we write $V_\lambda = \{u, v\}$, and for distinct $y, z \in X_B$ we define

$$\boldsymbol{\nu}_{y,z} = \begin{cases} \boldsymbol{\lambda} & (\{y,z\} = \{u,v\}) \\ \boldsymbol{\lambda}_{vz} & (y = u, z \in W_\lambda) \\ \boldsymbol{\lambda}_{uz} & (y = v, z \in W_\lambda) \\ \boldsymbol{\lambda}_{vy} & (z = u, y \in W_\lambda) \\ \boldsymbol{\lambda}_{uy} & (z = v, y \in W_\lambda) \\ \boldsymbol{\lambda}_{(uv)(yz)} & (y, z \in W_\lambda). \end{cases}$$

  Then $\boldsymbol{\nu}_{y,z} = \boldsymbol{\nu}_{z,y}$ is independent of the choice of $\boldsymbol{\lambda}$, and by Proposition 4.8 the bipartitions $\boldsymbol{\nu}_{y,z}$ are precisely the bipartitions in B. We shall write $\boldsymbol{\nu}_{y,z}$ as $\boldsymbol{\nu}_{y,z}(B)$ when there is a danger of ambiguity.

- If $w_B = 2$, then we write $W_\lambda = \{w, x\}$, and for distinct $y, z \in X_B$ we define

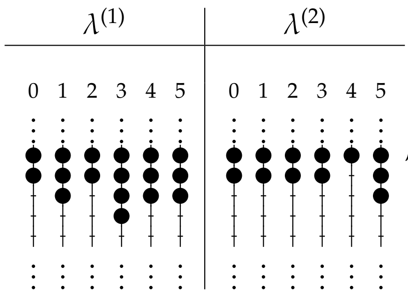
$$\boldsymbol{\xi}_{y,z} = \begin{cases} \boldsymbol{\lambda} & (\{y,z\} = \{w,x\}) \\ \boldsymbol{\lambda}_{zx} & (y = w, z \in V_\lambda) \\ \boldsymbol{\lambda}_{zw} & (y = x, z \in V_\lambda) \\ \boldsymbol{\lambda}_{yx} & (z = w, y \in V_\lambda) \\ \boldsymbol{\lambda}_{yw} & (z = x, y \in V_\lambda) \\ \boldsymbol{\lambda}_{(yz)(wx)} & (y, z \in V_\lambda). \end{cases}$$

  Again, $\boldsymbol{\xi}_{y,z} = \boldsymbol{\xi}_{z,y}$ is independent of the choice of $\boldsymbol{\lambda}$; we may write $\boldsymbol{\xi}_{y,z}$ as $\boldsymbol{\xi}_{y,z}(B)$.

If $v_B = w_B = 2$, with $X_B = \{a, b, c, d\}$, say, then we have $\boldsymbol{\nu}_{a,b} = \boldsymbol{\xi}_{c,d}$, $\boldsymbol{\nu}_{a,c} = \boldsymbol{\xi}_{b,d}$ et cetera.

Example.

1. Suppose $e = 6$, $Q_1 = q^5$, $Q_2 = 1$, and $\boldsymbol{\lambda} = ((5, 2^3, 1), (6, 1))$. This has an abacus display



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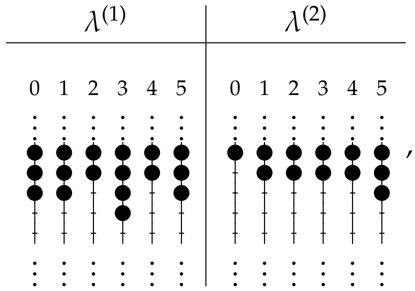
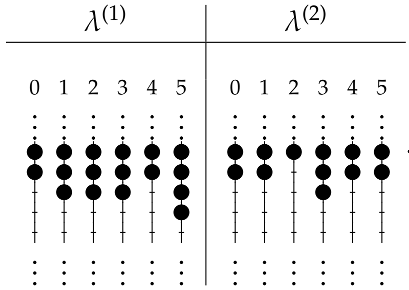
and we find $V_\lambda = \{3, 4\}$, $W_\lambda = \{0, 2, 5\}$. The bipartitions in the same block include $\boldsymbol{\nu}_{3,4} = \boldsymbol{\lambda}$, and

$\boldsymbol{\nu}_{0,3} = ((5, 2, 1), (6, 1^5)) =$



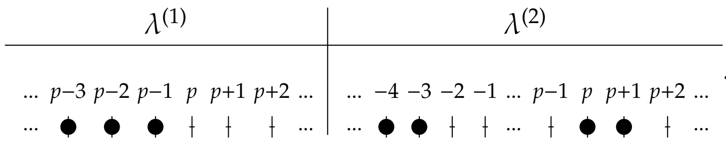
,

$\boldsymbol{\nu}_{2,5} = ((7, 2, 1^3), (4, 1^3)) =$



.

2. Suppose $e = \infty$ and B is the block B^*_{II}. Suppose $Q_1 = q^p$ and $Q_2 = 1$. B contains the bipartition $\boldsymbol{\lambda} = (\varnothing, (p + 2, p + 2))$, which has an abacus display



.

We have $V_\lambda = \{-2, -1, \ldots, p - 1\}$ and $W_\lambda = \{p, p + 1\}$. For $-2 \leqslant y < z \leqslant p + 1$, we have

$$\boldsymbol{\xi}_{y,z} = ((2^{p+1-z}, 1^{z-y-1}), (z + 1, y + 2)).$$

4.4 The dominance order in a Type II block

Next we work out the dominance order for the bipartitions in a Type II block.
If B is a Type II block, let v_B, w_B, X_B be as above. We define a total order on X_B.

- If $v_B = 2$, then for $x \in X_B$ define $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)})$ to be the bipartition $\boldsymbol{\nu}_{x,y}$, for any $y \in X_B$ distinct from x, and let $\beta^B(x)$ be the largest beta-number of $\mu^{(1)}$ congruent to x modulo e; it is easy to see that $\beta^B(x)$ does not depend on the choice of y. The integers $\beta^B(x)$ for $x \in X_B$ are distinct, and we totally order X_B according to the usual order of these integers: $x \succcurlyeq y$ if and only if $\beta^B(x) \geqslant \beta^B(y)$.
- If $w_B = 2$, then for $x \in X_B$ define $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)})$ to be the bipartition $\boldsymbol{\nu}_{y,z}$ for any $y, z \in X_B$ distinct from x, and let $\beta^B(x)$ be the largest beta-number of $\mu^{(1)}$ congruent to x modulo e. $\beta^B(x)$ does not depend on the choice of y, z, and we totally order X_B according to the order of the integers $\beta^B(x)$.

Now we can describe the dominance order in a Type II block.

Proposition 4.10. *Suppose B is a Type II block of $\mathcal{H}_n$, with $v_B, w_B, X_B, \succcurlyeq$ and the bipartitions $\boldsymbol{\nu}_{y,z}$ or $\boldsymbol{\xi}_{y,z}$ as defined above.*

- *If $v_B = 2$, then we have $\boldsymbol{\nu}_{y,z} \to \boldsymbol{\nu}_{y',z'}$ if and only if one of*

- $y = y'$ *and* $z \succcurlyeq z'$
- $y = z'$ *and* $z \succcurlyeq y'$
- $z = y'$ *and* $y \succcurlyeq z'$
- $z = z'$ *and* $y \succcurlyeq y'$

occurs. Hence we have $\boldsymbol{\nu}_{y,z} \trianglerighteq \boldsymbol{\nu}_{y',z'}$ *if and only if one of*

- $y \succcurlyeq y'$ *and* $z \succcurlyeq z'$
- $y \succcurlyeq z'$ *and* $z \succcurlyeq y'$

occurs.

- *If* $w_B = 2$*, then we have* $\xi_{y,z} \to \xi_{y',z'}$ *if and only if one of*

- $y = y'$ *and* $z \preccurlyeq z'$
- $y = z'$ *and* $z \preccurlyeq y'$
- $z = y'$ *and* $y \preccurlyeq z'$
- $z = z'$ *and* $y \preccurlyeq y'$

occurs. Hence we have $\boldsymbol{\nu}_{y,z} \trianglerighteq \boldsymbol{\nu}_{y',z'}$ *if and only if one of*

- $y \preccurlyeq y'$ *and* $z \preccurlyeq z'$
- $y \preccurlyeq z'$ *and* $z \preccurlyeq y'$

occurs.

4.5 Kleshchev bipartitions and decomposition numbers for Type II blocks

The decomposition numbers for Type II blocks are easy to calculate, given our work on B_{II}. Again, we mimic the argument for weight 1 blocks of Ariki–Koike algebras in [11], and show that the decomposition numbers are preserved under the Scopes bijections Φ_i. In the following proposition, we use the notation $M \sim dN$ to mean that the module M has the same composition factors as the module $N^{\oplus d}$.

Proposition 4.11. *Suppose B is a Type II block, and* $i \in \mathbb{Z}/e\mathbb{Z}$ *is such that* $\delta_i(B) > 0$*; let* $C = \Phi_i(B)$*.*

1. *If* $\boldsymbol{\lambda}$ *is a bipartition in B, then* $\boldsymbol{\lambda}$ *has exactly* $\delta_i(B)$ *removable nodes and no addable nodes.*
2. $\Phi_i(\boldsymbol{\lambda})$ *is obtained by removing all the removable i-nodes from* $\boldsymbol{\lambda}$*, and is Kleshchev if and only if* $\boldsymbol{\lambda}$ *is.*
3. *There is a bijection* σ *between the set of Kleshchev bipartitions in B and the set of Kleshchev bipartitions in C, such that*
$$S^{\boldsymbol{\lambda}}{\downarrow}_C^B \sim \delta_i(B)!S^{\Phi_i(\boldsymbol{\lambda})}, \qquad S^{\Phi_i(\boldsymbol{\lambda})}{\uparrow}_C^B \sim \delta_i(B)!S^{\boldsymbol{\lambda}},$$
$$D^{\boldsymbol{\mu}}{\downarrow}_C^B \sim \delta_i(B)!D^{\sigma(\boldsymbol{\mu})}, \qquad D^{\sigma(\boldsymbol{\mu})}{\uparrow}_C^B \sim \delta_i(B)!D^{\boldsymbol{\mu}}$$
and
$$[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}] = [S^{\Phi_i(\boldsymbol{\lambda})} : D^{\sigma(\boldsymbol{\mu})}]$$
for any bipartition $\boldsymbol{\lambda}$ *and any Kleshchev bipartition* $\boldsymbol{\mu}$ *in B.*
4. Φ_i *preserves the dominance order of bipartitions in B.*

Proof. This is proved using the branching rule for Specht modules [1, Lemma 2.1] in exactly the same way as [11, Proposition 4.11], but citing Proposition 4.10 of the present paper rather than Lemma 4.8 of [11]. □

In fact, we know what the bijection in (3) is.

Lemma 4.12. *The bijection described in Proposition 4.11(3) is the restriction of Φ_i to the set of Kleshchev bipartitions.*

Proof. Suppose $\boldsymbol{\mu}$ is a Kleshchev bipartition in B, and that $\sigma(\boldsymbol{\pi}) = \Phi_i(\boldsymbol{\pi})$ for all Kleshchev bipartitions $\boldsymbol{\pi}$ in B for which $\sigma(\boldsymbol{\pi}) \rhd \sigma(\boldsymbol{\mu})$. We have

$$1 = [S^{\boldsymbol{\mu}} : D^{\boldsymbol{\mu}}] = [S^{\Phi_i(\boldsymbol{\mu})} : D^{\sigma(\boldsymbol{\mu})}],$$

so that $\Phi_i(\boldsymbol{\mu}) \trianglerighteq \sigma(\boldsymbol{\mu})$ by Theorem 1.2. If $\Phi_i(\boldsymbol{\mu}) \rhd \sigma(\boldsymbol{\mu})$, then (since Φ_i and σ are bijections on the set of Kleshchev bipartitions) we have $\Phi_i(\boldsymbol{\mu}) = \sigma(\boldsymbol{\pi})$ for some Kleshchev bipartition $\boldsymbol{\pi}$, and (by assumption) $\sigma(\boldsymbol{\pi}) = \Phi_i(\boldsymbol{\pi})$. But then we get $\boldsymbol{\pi} = \boldsymbol{\mu}$, so $\sigma(\boldsymbol{\mu}) = \Phi_i(\boldsymbol{\mu})$. □

Now we can state our Richards-type theorem.

Theorem 4.13. *Suppose B is a Type II block of $\mathcal{H}_n$. Then there is a function ∂ from the set of bipartitions in B to the non-negative integers, and a function from the set $\{\boldsymbol{\lambda} \text{ in } B \mid \partial\boldsymbol{\lambda} = 0\}$ to the set $\{black, white\}$ such that the following hold.*

1. *The bipartitions in B with a given value of ∂ are totally ordered by dominance;*
2. *Given a bipartition $\boldsymbol{\mu}$ in B, $\boldsymbol{\mu}$ is Kleshchev if there is a bipartition $\boldsymbol{\nu}$ in B such that $\boldsymbol{\nu} \rhd \boldsymbol{\mu}$, $\partial\boldsymbol{\nu} = \partial\boldsymbol{\mu}$, and $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ have the same colour if $\partial\boldsymbol{\mu} = 0$. In this case, $\mu^\diamond$ is the least dominant such $\boldsymbol{\nu}$.*
3. *The decomposition numbers for B are given by*

$$[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}] = \begin{cases} 1 & (\boldsymbol{\lambda} = \boldsymbol{\mu}, \boldsymbol{\lambda} = \boldsymbol{\mu}^\diamond \text{ or } (\boldsymbol{\mu} \trianglelefteq \boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}^\diamond \text{ and } |\partial\boldsymbol{\mu} - \partial\boldsymbol{\lambda}| = 1)) \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. We prove this by induction on n, with the initial cases $B = B_{\mathrm{II}}$ and B^*_{II} already proved. If B is not B_{II} or B^*_{II}, then we have $\delta_i(B) > 0$ for some $i \in \mathbb{Z}/e\mathbb{Z}$, and we may assume that the result holds for $C = \Phi_i(B)$.

We define ∂ and the colour function on B simply by taking those for C and composing with Φ_i. Proposition 4.11 implies the result. □

Remark. It is easy to get an explicit expression for ∂ and the colour function: we find that if $y \prec z \in X_B$, then

$$\partial\boldsymbol{\nu}_{y,z} = |\{x \in X_B \mid y \prec x \prec z\}|$$

if $v_B = 2$, while

$$\partial\boldsymbol{\xi}_{y,z} = |\{x \in X_B \mid y \prec x \prec z\}|$$

if $w_B = 2$. If $v_B = 2$ and $\partial\boldsymbol{\nu}_{y,z} = 0$, then $\boldsymbol{\nu}_{y,z}$ is black if $|\{x \in X_B \mid x \succ z\}|$ is even, and white otherwise. If $w_B = 2$ and $\partial\boldsymbol{\xi}_{y,z} = 0$, then $\boldsymbol{\xi}_{y,z}$ is black if $|\{x \in X_B \mid x \prec y\}|$ is even, and white otherwise.

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