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Schur subalgebras II

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1 Introduction

Let p be a prime, and k an infinite field of characteristic p. In [2], the author and Martin re-proved a result of Henke [4] in which the Schur algebra $S(2, d)$ over k is shown to embed in the Schur algebra $S(2, r)$ for certain values of d and r, corresponding to certain self-similarity properties of the decomposition matrices for $S(2, r)$. We also constructed embeddings of $S(2, r)$ in $S(2, rp)$ for all r, reflecting further the structure of the decomposition matrices. Here, an embedding is not necessarily an injective homomorphism of algebras, but simply a linear injection preserving the multiplication rule.

In this paper we continue to study such embeddings, and examine their consequences for decomposition numbers. Essential results concerning Schur algebras can be found in the books of Green [3] and Martin [6]; further results and notation are taken from [2].

In Section 2 we construct embeddings

$$S(2, r) \hookrightarrow S(2, rp + q)$$

for all q between 0 and $p-1$; this then gives each Schur algebra $S(2, R)$ an embedded algebra isomorphic to $S(2, \lfloor \frac{R}{p} \rfloor)$. In Section 3 we examine the consequences of these embeddings for (dual) Weyl modules; we find explicitly the restrictions of the dual Weyl modules to the embedded subalgebras. In Section 4 we use these results to rediscover the decomposition matrices for $S(2, r)$, first found by Carter and Cline [1].

1.1 Notation

We use the notation from [2]; in particular, we take as a basis for the Schur algebra $S(2, r)$ the set $M(r)$ of 2 by 2 matrices with non-negative integer entries summing to r. For $A \in M(r)$, we denote by $c_i(A)$, $r_i(A)$ the ith row and column sums of A respectively. For $A, B \in M(r)$ with $c_i(A) = r_i(B)$ we define $N(A, B)$ to be the set of matrices in $M(r)$ with the same row sums as A and the same column sums as B, and we define $R(A, B)$ to be the set of 2×2 matrices D with (possibly negative) integer entries, and with $r_i(D) = a_{i1}, c_i(D) = b_{1i}$, for $i = 1, 2$.

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The multiplication $\circ$ in $S(2,r)$ is then given on basis elements by [2, Proposition 2.1]

$$A \circ B = \begin{cases} 0 & (c_1(A) \neq r_1(B)) \\ \sum_{C \in N(A,B)} (\sum_{D \in R(A,B)} \binom{C}{D}).1_k.C & (c_1(A) = r_1(B)), \end{cases}$$

where, for 2 by 2 matrices C, D, we define

$$\binom{C}{D} = \prod_{i,j=1}^{2} \binom{c_{ij}}{d_{ij}}.$$

2 Schur algebra embeddings

First we recall the embedding of $S(2,r)$ in $S(2,rp)$ [2, Theorem 4.2].

Theorem 2.1. *Let $p = \mathrm{char}(k)$. There exists an embedding $\Psi : S(2,r) \longrightarrow S(2,rp)$ defined on basis elements by*

$$\Psi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{cases} \begin{pmatrix} pa & pb \\ pc & pd \end{pmatrix} & \textit{(if b or c equals 0)} \\ \sum_{\epsilon=0}^{p-1} \begin{pmatrix} pa+\epsilon & pb-\epsilon \\ pc-\epsilon & pd+\epsilon \end{pmatrix} & \textit{(otherwise).} \end{cases}$$

Now we generalise this result and embed $S(2,r)$ in $S(2,rp+q)$ for all $0 \leqslant q \leqslant p-1$. Given $A \in M(r)$, write

$$A_{\beta,q} = \begin{pmatrix} pa_{11}+q+\beta & pa_{12}-\beta \\ pa_{21}-\beta & pa_{22}+\beta \end{pmatrix}$$

if this has non-negative entries, and $A_{\beta,q} = 0$ otherwise. We then have the following embedding. Note that the embedded algebra is neither a subalgebra with 1 of $S(2,rp+q)$ nor a subalgebra of the form $eS(2,rp+q)e$ for e an idempotent.

Theorem 2.2. *Let $p = \mathrm{char}(k)$, and let $0 \leqslant q < p$. There exists an embedding $\Psi : S(2,r) \longrightarrow S(2,rp+q)$ defined on basis elements by*

$$\Psi : A \longmapsto \sum_{\beta=0}^{p-1} \binom{\beta+q}{q} A_{\beta,q} + \sum_{\gamma=0}^{p-1} (1 - \binom{p-\gamma+q}{q}) A_{-\gamma,q}.$$

Ψ is clearly injective (consider the coefficients of matrices $A_{0,q}$); in order to prove the theorem, we must show that multiplication of basis elements is preserved. We proceed along the same lines as in [2]; we write down the product $\Psi(A) \circ \Psi(B)$ for $A, B \in M(r)$, and reduce it modulo p using Lemma 2.3 and splitting into cases. First we recall Lucas's lemma, and state an additional lemma concerning binomial coefficients, whose proof is trivial.

Lemma 2.3.

1. *Let p be a prime, and let a, b, c, d be integers, with a non-negative and $0 \leqslant c, d < p$. Then*

$$\binom{pa+c}{pb+d} \equiv \binom{a}{b}\binom{c}{d} \pmod p.$$

2. *Let a, b, c, d be integers, with a and c non-negative. Then*

$$\sum_{r\in\mathbb{Z}}\binom{r}{b}\binom{a}{r}\binom{c}{d-r}=\binom{a}{b}\binom{a+c-b}{d-b}.$$

(Note that we may safely ignore $\binom{r}{b}$ *when* $r<0$*, since then* $\binom{a}{r}=0$*.)*

Put $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $B=\begin{pmatrix}e & f\\ g & h\end{pmatrix}$; unless $a+c=e+g$, we have $A\circ B=0$ and $\Psi(A)\circ\Psi(B)=0$, so assume $a+c=e+g$, and take $C\in M(rp+q)$; the coefficient of C in $\Psi(A)\circ\Psi(B)$ is zero unless $r_1(C)=pr_1(A)+q$ and $c_1(C)=pc_1(B)+q$; so we assume the latter, and we may write C uniquely as $\begin{pmatrix}pk+q+\epsilon & pl-\epsilon\\ pm-\epsilon & po+\epsilon\end{pmatrix}$ with $0\leqslant\epsilon\leqslant p-1$. We then have

$$\begin{aligned}R(A_{\beta,q},B_{\gamma,q})&=\{\begin{pmatrix}v & pa+q-v+\beta\\ pe+q-v+\gamma & v+pc-pe-q-\beta-\gamma\end{pmatrix}\mid v\in\mathbb{Z}\}\\&=\{\begin{pmatrix}pw+\alpha & pa+q-pw-\alpha+\beta\\ pe+q-pw-\alpha+\gamma & pc-pe-q+pw+\alpha-\beta-\gamma\end{pmatrix}\mid w\in\mathbb{Z},0\leqslant\alpha\leqslant p-1\}\end{aligned}\tag{1}$$

if $A_{\beta,q},B_{\gamma,q}\neq 0$. If one of $A_{\beta,q}$,$B_{\gamma,q}$ is zero, then every matrix in the set on the right-hand side of (1) has a negative entry, and so the product will not be affected if we assume (1) even when $A_{\beta,q}$ or $B_{\gamma,q}$ is zero.

Putting $D_{w,\alpha}=\begin{pmatrix}pw+\alpha & pa+q-pw-\alpha+\beta\\ pe+q-pw-\alpha+\gamma & pc-pe-q+pw+\alpha-\beta-\gamma\end{pmatrix}$, we write the coefficient of C in $\Psi(A)\circ\Psi(B)$ as $\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4$, where

$$\begin{aligned}\Sigma_1&=\sum_{\beta=0}^{p-1}\sum_{\gamma=0}^{p-1}\binom{\beta+q}{q}\binom{\gamma+q}{q}\sum_{w\in\mathbb{Z}}\sum_{\alpha=0}^{p-1}\binom{C}{D_{w,\alpha}},\\ \Sigma_2&=\sum_{\beta=0}^{p-1}\sum_{\gamma=0}^{p-1}\binom{\beta+q}{q}(1-\binom{p-\gamma+q}{q})\sum_{w\in\mathbb{Z}}\sum_{\alpha=0}^{p-1}\binom{C}{D_{w,\alpha}},\\ \Sigma_3&=\sum_{\beta=0}^{p-1}\sum_{\gamma=0}^{p-1}(1-\binom{p-\beta+q}{q})\binom{\gamma+q}{q}\sum_{w\in\mathbb{Z}}\sum_{\alpha=0}^{p-1}\binom{C}{D_{w,\alpha}},\\ \Sigma_4&=\sum_{\beta=0}^{p-1}\sum_{\gamma=0}^{p-1}(1-\binom{p-\beta+q}{q})(1-\binom{p-\gamma+q}{q})\sum_{w\in\mathbb{Z}}\sum_{\alpha=0}^{p-1}\binom{C}{D_{w,\alpha}}.\end{aligned}$$

Proposition 2.4. *With notation as above, we have*

$$\Sigma_1=\binom{\epsilon+q}{q}\sum_{w\in\mathbb{Z}}\begin{pmatrix}\begin{pmatrix}k & l\\ m & o\end{pmatrix}\\ \begin{pmatrix}w & a-w\\ e-w & c+e-w\end{pmatrix}\end{pmatrix}+(1-\binom{p-\epsilon+q}{q})\sum_{w\in\mathbb{Z}}\begin{pmatrix}\begin{pmatrix}k+1 & l-1\\ m-1 & o\end{pmatrix}\\ \begin{pmatrix}w & a-w\\ e-w & c+e-w-1\end{pmatrix}\end{pmatrix};$$

$$\Sigma_2=\Sigma_3=-\Sigma_4=(1-\binom{p-\epsilon+q}{q})\sum_{w\in\mathbb{Z}}\begin{pmatrix}\begin{pmatrix}k+1 & l-1\\ m-1 & o\end{pmatrix}\\ \begin{pmatrix}w & a-w\\ e-w & c+e-w\end{pmatrix}\end{pmatrix}.$$

(If l or m is zero or $k = -1$, undefined terms should be treated as zero.)

Proof. In order to reduce all the binomial coefficients modulo p, we need to know the greatest multiple of p less than each of the entries of the matrices in $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$. So we must consider separately the cases $\epsilon = 0$, $0 < \epsilon < p - q$ and $\epsilon \geqslant p - q$. First we deal with the case $\epsilon = 0$. Now $\binom{pn}{r} \equiv 0 \pmod p$ unless p divides r, so for a non-zero term in any of $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$, we must have $\beta = \gamma = 0$ and $\alpha = q$. This immediately gives $\Sigma_2 = \Sigma_3 = \Sigma_4 = 0$, while for Σ_1 we use the congruence $\binom{np}{rp} \equiv \binom{n}{r} \pmod p$ to give

$$\Sigma_1 \equiv \sum_{w\in\mathbb{Z}} \begin{pmatrix} \begin{pmatrix} k & l \\ m & o \end{pmatrix} \\ \begin{pmatrix} w & a-w \\ e-w & c-e+w \end{pmatrix} \end{pmatrix}$$

as required.

Next we consider the case $\epsilon < p - q$, and evaluate Σ_1.

We write $B_{\alpha,\beta,\gamma}$ for

$$\binom{\beta+q}{q}\binom{\gamma+q}{q} \sum_{w\in\mathbb{Z}} \begin{pmatrix} \begin{pmatrix} pk+q+\epsilon & pl-\epsilon \\ pm-\epsilon & po+\epsilon \end{pmatrix} \\ \begin{pmatrix} pw+\alpha & pa+q-pw-\alpha+\beta \\ pe+q-pw-\alpha+\gamma & pc-pe-q+pw+\alpha-\beta-\gamma \end{pmatrix} \end{pmatrix};$$

note that if $\beta \geqslant p - q + \alpha$, then $p \leqslant \beta + q < p + q$, so $\binom{\beta+q}{q} \equiv 0$; similarly for γ. So we may assume that

$$pa + q - pw - \alpha + \beta < p(a - w + 1),$$
$$pe + q - pw - \alpha + \gamma < p(e - w + 1),$$

and split Σ_1 up as

$$\Sigma_1 = S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 + S_8 + S_9 + S_{10},$$

where

$$S_1 = \sum_{\substack{0\leqslant\alpha,\beta,\gamma<p \\ \beta+q<\alpha \\ \gamma+q<\alpha \\ \beta+\gamma+q\leqslant\alpha}} B_{\alpha,\beta,\gamma} \qquad S_6 = \sum_{\substack{0\leqslant\alpha,\beta,\gamma<p \\ \beta+q\geqslant\alpha \\ \alpha\leqslant\gamma+q \\ \alpha\geqslant\gamma+q+\beta}} B_{\alpha,\beta,\gamma}$$

$$S_2 = \sum_{\substack{0\leqslant\alpha,\beta,\gamma<p \\ \beta+q<\alpha \\ \gamma+q\geqslant\alpha \\ \beta+\gamma+q\leqslant\alpha}} B_{\alpha,\beta,\gamma} \qquad S_7 = \sum_{\substack{0\leqslant\alpha,\beta,\gamma<p \\ \beta+q\geqslant\alpha \\ \alpha>\gamma+q \\ \alpha<\gamma+q+\beta}} B_{\alpha,\beta,\gamma}$$

$$S_3 = \sum_{\substack{0\leqslant\alpha,\beta,\gamma<p \\ \beta+q<\alpha \\ \gamma+q<\alpha \\ \beta+\gamma+q>\alpha}} B_{\alpha,\beta,\gamma} \qquad S_8 = \sum_{\substack{0\leqslant\alpha,\beta,\gamma<p \\ \beta+q\geqslant\alpha \\ \alpha\leqslant\gamma+q \\ \alpha<\gamma+q+\beta\leqslant\alpha+p}} B_{\alpha,\beta,\gamma}$$

$$S_4 = \sum_{\substack{0\leqslant\alpha,\beta,\gamma<p \\ \beta+q<\alpha \\ \gamma+q\geqslant\alpha \\ \beta+\gamma+q>\alpha}} B_{\alpha,\beta,\gamma} \qquad S_9 = \sum_{\substack{0\leqslant\alpha,\beta,\gamma<p \\ \beta+q\geqslant\alpha \\ \alpha+p<\gamma+q+\beta\leqslant\alpha+2p}} B_{\alpha,\beta,\gamma}$$

$$S_5 = \sum_{\substack{0\leqslant\alpha,\beta,\gamma<p \\ \beta+q\geqslant\alpha \\ \gamma+q<\alpha \\ \beta+\gamma+q\leqslant\alpha}} B_{\alpha,\beta,\gamma} \qquad S_{10} = \sum_{\substack{0\leqslant\alpha,\beta,\gamma<p \\ \beta+q\geqslant\alpha \\ \alpha+2p<\gamma+q+\beta}} B_{\alpha,\beta,\gamma}.$$

Lemma 2.5.

$$S_1 \equiv \sum_{w\in\mathbb{Z}} \begin{pmatrix} \begin{pmatrix} k & l-1 \\ m-1 & o \end{pmatrix} \\ \begin{pmatrix} w & a-w-1 \\ e-w-1 & c-e+w \end{pmatrix} \end{pmatrix}, \qquad S_2 \equiv \binom{\epsilon+q}{q} \sum_{w\in\mathbb{Z}} \begin{pmatrix} \begin{pmatrix} k & l-1 \\ m-1 & o \end{pmatrix} \\ \begin{pmatrix} w & a-w-1 \\ e-w & c-e+w \end{pmatrix} \end{pmatrix},$$

$$S_5 \equiv \binom{\epsilon+q}{q} \sum_{w\in\mathbb{Z}} \begin{pmatrix} \begin{pmatrix} k & l-1 \\ m-1 & o \end{pmatrix} \\ \begin{pmatrix} w & a-w \\ e-w-1 & c-e+w \end{pmatrix} \end{pmatrix}, \qquad S_6 \equiv \binom{\epsilon+q}{q} \sum_{w\in\mathbb{Z}} \begin{pmatrix} \begin{pmatrix} k & l-1 \\ m-1 & o \end{pmatrix} \\ \begin{pmatrix} w & a-w \\ e-w & c-e+w \end{pmatrix} \end{pmatrix},$$

$$S_8 \equiv \left(1 - \binom{e+q}{q}\right) \sum_{w\in\mathbb{Z}} \begin{pmatrix} \begin{pmatrix} k & l-1 \\ m-1 & o \end{pmatrix} \\ \begin{pmatrix} w & a-w \\ e-w & c-e+w-1 \end{pmatrix} \end{pmatrix},$$

$$S_3 \equiv S_4 \equiv S_7 \equiv S_9 \equiv S_{10} \equiv 0.$$

Proof. We prove the expression for S_8; the other expressions follow similarly (or much more simply).

By Lemma 2.3 we find

$$S_8 = T_8 \times \sum_{w\in\mathbb{Z}} \begin{pmatrix} \begin{pmatrix} k & l-1 \\ m-1 & o \end{pmatrix} \\ \begin{pmatrix} w & a-w \\ e-w & c-e+w-1 \end{pmatrix} \end{pmatrix},$$

where

$$T_8 = \sum_{\substack{0\leqslant\alpha,\beta,\gamma<p \\ \beta+q\geqslant\alpha \\ \alpha\leqslant\gamma+q \\ \alpha<\gamma+q+\beta\leqslant\alpha+p}} \binom{\beta+q}{q}\binom{\gamma+q}{q}\binom{\epsilon+q}{\alpha}\binom{p-\epsilon}{q-\alpha+\beta}\binom{p-\epsilon}{q-\alpha+\gamma}\binom{\epsilon}{p+\alpha-\beta-\gamma-q}.$$

We would like to replace the range of summation for γ in T_8 with $\sum_{\gamma\in\mathbb{Z}}$, so we examine which values of γ outside the given range give non-zero values of $B_{\alpha,\beta,\gamma}$. If $\gamma<\alpha-q-\beta+1$ or $\gamma<\alpha-q$ or $\gamma<0$ or $\gamma>p-q+\alpha-\beta$, then one of the binomial coefficients is congruent to zero, so we need only consider $p-1<\gamma\leqslant p-q+\alpha-\beta$. This can only happen if $\alpha-q\geqslant\beta$; but $\binom{p-\epsilon}{q-\alpha+\beta}$ is zero unless $\beta\leqslant\alpha-q$. So the only value of γ outside the given range which can give a non-zero $B_{\alpha,\beta,\gamma}$ is $\gamma=p$ when $\beta=\alpha-q$; this gives

$$\sum_{\alpha=0}^{p-1} B_{\alpha,\alpha-q,p} \equiv \sum_{\alpha=0}^{p-1}\binom{\alpha}{q}\binom{\epsilon+q}{\alpha}\binom{p-\epsilon}{p+q-\alpha} \equiv \binom{\epsilon+q}{q}.$$

Now we sum over γ; terms involving γ give

$$\begin{aligned} &\sum_{\gamma\in\mathbb{Z}}\binom{\gamma+q}{q}\binom{p-\epsilon}{q-\alpha+\gamma}\binom{\epsilon}{p+\alpha-\beta-\gamma-q} \\ \equiv &\sum_{\gamma\in\mathbb{Z}}\sum_{\zeta\in\mathbb{Z}}\binom{\alpha}{\zeta}\binom{\gamma+q-\alpha}{q-\zeta}\binom{p-\epsilon}{q+\gamma-\alpha}\binom{\epsilon}{p+\alpha-\beta-\gamma-q} \\ \equiv &\sum_{\zeta\in\mathbb{Z}}\binom{\alpha}{\zeta}\binom{p-\epsilon}{q-\zeta}\binom{p-q+\zeta}{p-\beta-q+\zeta} \end{aligned}$$

by Lemma 2.3. Hence we have

$$T_8 \equiv \sum_{\alpha=0}^{p-1} \sum_{\beta=\max(0,\alpha-q)}^{p-1} \binom{\beta+q}{q}\binom{\epsilon+q}{\alpha}\binom{p-\epsilon}{q-\alpha+\beta} \sum_{\gamma\in\mathbb{Z}} \binom{\alpha}{\zeta}\binom{p-\epsilon}{q-\zeta}\binom{p-q+\zeta}{p-\beta-q+\zeta} - \binom{\epsilon+q}{q}.$$

Next we would like to replace the range of summation for β with $\sum_{\beta\in\mathbb{Z}}$, so we check which values of β outside the given range contribute non-trivially. If $\beta < 0$ or $\beta < \alpha - q$ then one of the binomial coefficients is congruent to zero, so we need only consider $p - 1 < \beta$. Now for $\binom{p-q+\zeta}{p-\beta-q+\zeta}\binom{p-\epsilon}{q-\zeta}$ to be non-zero, we must have $\beta \leqslant p - q + \zeta$ and $q \geqslant \zeta$. So we need only consider the case $\beta = p$, $q = \zeta$. This gives a term $\binom{\epsilon+q}{q}$ exactly as before; summing over β, we get

$$\begin{aligned}
&\sum_{\beta\in\mathbb{Z}} \binom{\beta+q}{q}\binom{p-q+\zeta}{\beta}\binom{p-\epsilon}{p-\epsilon-q+\alpha-\beta} \\
&\equiv \sum_{\beta\in\mathbb{Z}}\sum_{\eta\in\mathbb{Z}} \binom{\beta}{q-\eta}\binom{q}{\eta}\binom{p-q+\zeta}{\beta}\binom{p-\epsilon}{p-\epsilon-q+\alpha-\beta} \\
&\equiv \sum_{\eta\in\mathbb{Z}} \binom{q}{\eta}\binom{p-q+\zeta}{q-\eta}\binom{2p-2q-\epsilon+\zeta+\eta}{p-\alpha+\zeta}
\end{aligned}$$

by Lemma 2.3. Hence

$$T_8 \equiv \sum_{\alpha=0}^{p-1}\sum_{\zeta\in\mathbb{Z}}\sum_{\eta\in\mathbb{Z}} \binom{\epsilon+q}{\alpha}\binom{\alpha}{\zeta}\binom{p-\epsilon}{q-\zeta}\binom{q}{\eta}\binom{p-q+\zeta}{q-\eta}\binom{2p-2q-\epsilon+\zeta+\eta}{p-\alpha+\zeta} - 2\binom{\epsilon+q}{q}.$$

Now we sum over α; we may change the range of summation to $\sum_{\alpha\in\mathbb{Z}}$ without compunction, since $\epsilon + q < p$; Lemma 2.3 gives

$$T_8 \equiv \sum_{\eta,\zeta\in\mathbb{Z}} \binom{\epsilon+q}{\zeta}\binom{2p-q+\eta}{p}\binom{p-q+\zeta}{q-\eta}\binom{q}{\eta}\binom{p-\epsilon}{q-\zeta} - 2\binom{\epsilon+q}{q}.$$

Now if $\eta > q$ then $\binom{q}{\eta} = 0$, so we may restrict attention to the range $\eta \leqslant q$. If $\eta = q$, we have

$$\sum_{\zeta\in\mathbb{Z}} \binom{\epsilon+q}{\zeta}.2.\binom{p-\epsilon}{q-\zeta} \equiv 2.\binom{p+q}{q} \equiv 2,$$

while if $\eta < q$, the summand is

$$\sum_{\zeta\in\mathbb{Z}} \binom{\epsilon+q}{\zeta}\binom{p-q+\zeta}{q-\eta}\binom{q}{\eta}\binom{p-\epsilon}{q-\zeta}.$$

Putting these together, we get

$$\begin{aligned}
T_8 &\equiv \sum_{\zeta,\eta\in\mathbb{Z}} \binom{\epsilon+q}{\zeta}\binom{p-q+\zeta}{q-\eta}\binom{q}{\eta}\binom{p-\epsilon}{q-\zeta} + 1 - 2\binom{\epsilon+q}{q} \\
&\equiv \sum_{\zeta\in\mathbb{Z}} \binom{p+\zeta}{q}\binom{\epsilon+q}{\zeta}\binom{p-\epsilon}{q-\zeta} + 1 - 2\binom{\epsilon+q}{q};
\end{aligned}$$

if $\zeta \neq q$ the summand is congruent to zero, so we take $\zeta = q$ to get

$$T_8 \equiv \binom{\epsilon+q}{q} + 1 - 2\binom{\epsilon+q}{q}.$$

Hence

$$S_8 \equiv (1 - \tbinom{e+q}{q}) \sum_{w\in\mathbb{Z}} \begin{pmatrix} \begin{pmatrix} k & l-1 \\ m-1 & o \end{pmatrix} \\ \begin{pmatrix} w & a-w \\ e-w & c-e+w-1 \end{pmatrix} \end{pmatrix}. \qquad \square$$

Thus we obtain the expression for Σ_1 in the case $\epsilon < p - q$; the expressions for Σ_2, Σ_3 and Σ_4 follow similarly, as does the case $\epsilon \geqslant p - q$. This completes the proof of Proposition 2.4. $\square$

Proof of Theorem 2.2. Adding together the expressions for $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ in Proposition 2.4, we find that the coefficient of $\begin{pmatrix} pk+q+\epsilon & pl-\epsilon \\ pm-\epsilon & po+\epsilon \end{pmatrix}$ in $\Psi(A) \circ \Psi(B)$ is

$$\tbinom{\epsilon+q}{q} \sum_{w\in\mathbb{Z}} \begin{pmatrix} \begin{pmatrix} k & l \\ m & o \end{pmatrix} \\ \begin{pmatrix} w & a-w \\ e-w & c-e+w \end{pmatrix} \end{pmatrix} + (1 - \tbinom{\epsilon+q}{q}) \sum_{w\in\mathbb{Z}} \begin{pmatrix} \begin{pmatrix} k+1 & l-1 \\ m-1 & o+1 \end{pmatrix} \\ \begin{pmatrix} w & a-w \\ e-w & c-e+w \end{pmatrix} \end{pmatrix}.$$

But this is simply $\tbinom{\epsilon+q}{q}$ times the coefficient of $\begin{pmatrix} k & l \\ m & o \end{pmatrix}$ in $A \circ B$ plus $(1 - \tbinom{\epsilon+q}{q})$ times the coefficient of $\begin{pmatrix} k+1 & l-1 \\ m-1 & o+1 \end{pmatrix}$ in $A \circ B$, which is what we require. $\square$

Remark. Note that in the case $q = p - 1$, Ψ takes the form

$$\Psi : A \longmapsto \sum_{\beta=0}^{p-1} A_{-\beta,q};$$

this is reminiscent of the case $q = 0$, and the simpler forms of Ψ in these cases afford much simpler proofs. The author has been unable to generalise these proofs for all q.

3 Dual Weyl modules

3.1 Bideterminants

We use the definition of $\nabla(\lambda)$ given in [6] (where it is called $M(\lambda)$) and [3] (where it is called D_λ). We revert temporarily to the notation $\{\xi_{i,j} \mid i, j \in I(n,r)\}$ for the standard basis of $S(n,r)$. Here $I(n,r)$ is the set of functions from $\{1,\ldots,r\}$ to $\{1,\ldots,n\}$, usually written as multi-indices $i_1 \ldots i_r$. We use $\sim$ to indicate conjugacy under the natural actions of $\mathfrak{S}_r$ on both $I(n,r)$ and $I(n,r) \times I(n,r)$, and we identify $\xi_{i,j}$ and $\xi_{k,l}$ if $(k,l) \sim (i,j)$.

Let $A(n,r)$ be the dual vector space to $S(n,r)$, with basis element $c_{i,j}$ dual to $\xi_{i,j}$. This has a natural $S(n,r)$-module structure via

$$\xi \circ c_{i,j} = \sum_{s\in I(n,r)} c_{s,j}(\xi).c_{i,s}.$$

(Note that historically $S(n,r)$ has been constructed as the dual of $A(n,r)$, and so many authors write $\xi(c_{s,j})$ where we write $c_{s,j}(\xi)$. We use $\circ$ to denote the module action of $S(n,r)$ on $A(n,r)$ as opposed to the dual vector space action. No confusion with the multiplication $\circ$ in $S(n,r)$ need arise.) We shall construct $\nabla(\lambda)$ as a submodule of $A(n,r)$.

Given a partition λ of r, we construct the corresponding Young diagram, and then define a *basic λ-tableau* T^λ to be a bijection from the set of nodes of the Young diagram to the set $\{1,\dots,r\}$; we usually write T^λ by drawing the diagram of λ with each node replaced by its image under T^λ. For $i \in I(n,r)$ we write T^λ_i for the composite iT^λ, and similarly indicate T^λ_i by means of a diagram. We then define $C(T^\lambda) \leqslant \mathfrak{S}_r$ to be the subgroup of $\mathfrak{S}_r$ which fixes the set of values corresponding to each column of the Young diagram. Given $i, j \in I(n,r)$ we can then define the *bideterminant*

$$T^\lambda(i:j) = \sum_{\pi \in C(T^\lambda)} (-1)^\pi c_{i,\pi(j)}.$$

Let $l = l(\lambda) \in I(n,r)$ be such that if T^λ maps a node in the xth row of the Young diagram to y, then $l_y = x$. Then we define the dual Weyl module $\nabla(\lambda)$ to be the k-span of all the bideterminants $T^\lambda(l:i)$. The isomorphism type of this module does not depend on our choice of T^λ.

Given T^λ, we say that $i \in I(n,r)$ is *standard* if the entries in T^λ_i are increasing along rows and strictly increasing down columns. A basis for $\nabla(\lambda)$ is then given by

$$\{T^\lambda(l:i) \mid i \text{ is standard}\}.$$

In [6], it is noted that $T^\lambda(i:\pi j) = (-1)^\pi T^\lambda(i:j)$ for $\pi \in C(T^\lambda)$, and an explicit formula for the action of $S(n,r)$ on bideterminants is given:

$$\xi \circ T^\lambda(i:j) = \sum_{u \in I(n,r)} c_{u,j}(\xi) T^\lambda(i:u). \tag{2}$$

When we specialise to the case $n = 2$, the dual Weyl module takes a particularly simple form. We take $\lambda = (a,b)$ with $a + b = r$, and choose

$$T^\lambda = \begin{matrix} 1 & \dots & \dots & a \\ a+1 & \dots & r & \end{matrix}\ ;$$

$i \in I(2,r)$ is then standard precisely if

$$i = 1^s 2^{r-s}$$

for some $b \leqslant s \leqslant a$; write x_s for the corresponding bideterminant $T^\lambda(l:i)$. We seek a description of the module action.

Let $i = 1^s 2^{r-s}$ be standard. From (2) we see that $\xi_{u,v} \circ T^\lambda(l:i)$ is zero unless $v \sim i$. So assume this, and write (without loss)

$$(u,v) = (1^e 2^g 1^f 2^h, 1^s 2^{r-s})$$

with $e + g = s$. We then have

$$\xi_{u,v} \circ T^\lambda(l:i) = \sum_w T^\lambda(l:w),$$

the sum being over all $w \in I(n,r)$ with exactly e 1s among $w_1,\dots,w_s$ and exactly f 1s among $w_{s+1},\dots,w_r$. Take such a w, and consider the first b columns of T^λ_w; for $T^\lambda(l:w)$ to be non-zero, each of these must be of the form $\substack{1\\2}$ or $\substack{2\\1}$, and if this is the case, then $T^\lambda(l:w) = (-1)^\kappa x_{e+f}$, where κ is the number of columns of T^λ_w of the form $\substack{2\\1}$.

In order to find the number of such w corresponding to each value of κ, we must choose which of the first b columns of T^λ_w will equal $\begin{smallmatrix}2\\1\end{smallmatrix}$, and then choose how to arrange the remaining $a-b$ entries in the first row. This gives exactly

$$\binom{b}{\kappa}\binom{e+g-b}{g-\kappa}\binom{f+h-b}{f-\kappa}$$

such w. Hence, reverting to the notation using $M(r)$ for the standard basis of $S(2,r)$, we have the following.

Proposition 3.1. *Let $\lambda = (a,b)$ be a partition of r. Then $\nabla(\lambda)$ has a basis $\{x_s \mid a \geqslant s \geqslant b\}$, and the action of $S(2,r)$ on $\nabla(\lambda)$ is given by*

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix} \circ x_s = \begin{cases} \left(\sum_{\kappa\in\mathbb{Z}}(-1)^\kappa\binom{b}{\kappa}\binom{e+g-b}{g-\kappa}\binom{f+h-b}{f-\kappa}\right)x_{e+f} & (e+g=s) \\ 0 & (e+g\neq s). \end{cases}$$

Remark. To aid notation, we write

$$C_{(a,b)}(\begin{pmatrix} e & f \\ g & h \end{pmatrix}) = \sum_{\kappa\in\mathbb{Z}}(-1)^\kappa\binom{b}{\kappa}\binom{e+g-b}{g-\kappa}\binom{f+h-b}{f-\kappa}$$

and extend $C_{(a,b)}$ linearly over $S(2,r)$.

Note that $e+g$ might lie between a and b, while $e+f$ does not. No ambiguity need arise in Proposition 3.1, since in this case it is easily seen that $C_{(a,b)}(\begin{pmatrix} e & f \\ g & h \end{pmatrix}) = 0$.

3.2 Restriction of dual Weyl modules

Given an algebra A with an idempotent e, there is a natural functor between the module categories of A and eAe, given by sending an A-module M to eM. Now recall the embedding of $S(2,d)$ in $S(2,r)$ from [2, Theorem 3.2].

Theorem 3.2. *Let $p = \mathrm{char}(k)$, and let s be any non-negative integer. If $d < r$, $d < 2.p^s$ and $d \equiv r \pmod{p^s}$, put $m = r-d$. Then we may embed $S(2,d)$ in $S(2,r)$ via*

$$\Phi : \begin{pmatrix} i & j \\ k & l \end{pmatrix} \longmapsto \begin{cases} \begin{pmatrix} i+m & j \\ k & l \end{pmatrix} & \textit{(if } i+j \geqslant k+l \textit{ and } i+k \geqslant j+l\textit{)} \\ \begin{pmatrix} i & f+m \\ k & l \end{pmatrix} & \textit{(if } i+j \geqslant k+l \textit{ and } i+k < j+l\textit{)} \\ \begin{pmatrix} i & f \\ k+m & l \end{pmatrix} & \textit{(if } i+j < k+l \textit{ and } i+k \geqslant j+l\textit{)} \\ \begin{pmatrix} i & j \\ k & l+m \end{pmatrix} & \textit{(if } i+j < k+l \textit{ and } i+k < j+l\textit{).} \end{cases}$$

Moreover, if we put

$$e = \sum_{\epsilon=0}^{d} \Phi(\begin{pmatrix} \epsilon & 0 \\ 0 & d-\epsilon \end{pmatrix}),$$

then e is an idempotent in $S(2,r)$ and $\Phi(S(2,d)) = eS(2,r)e$.

Given d,r as in Theorem 3.2, let

$$G_{d,r} : \mathrm{mod}(S(2,r)) \longrightarrow \mathrm{mod}(S(2,d))$$

be the functor sending a module M to eM. We then have the following result.

Theorem 3.3. *Let d, r be as above, and let (a,b) be a partition of r. Then*

$$G_{d,r}(\nabla(a,b)) \cong \begin{cases} \nabla(a-m,b) & (\text{if } a-b \geqslant m) \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. Since $\begin{pmatrix} i & 0 \\ 0 & l \end{pmatrix} \circ x_s$ equals x_s if $i = s$ and zero otherwise, we have

$$e \circ \nabla(a,b) = \langle x_s \mid a \geqslant s \geqslant b;\, s \geqslant \frac{r+m}{2} \text{ or } s < \frac{r-m}{2} \rangle.$$

In particular, if $a - b < m$, we have $e \circ \nabla(a,b) = 0$. If $a - b \geqslant m$, we define

$$\alpha(s) = \begin{cases} s - m & (s \geqslant \frac{r+m}{2}) \\ s & (s < \frac{r-m}{2}), \end{cases}$$

and then define a linear isomorphism

$$f : e \circ \nabla(a,b) \longrightarrow \nabla(a-m,b)$$

via

$$x_s \longmapsto x_{\alpha(s)};$$

we need to show that this is an isomorphism of modules. For $A \in M(d)$, we have

$$c_1(\Phi(A)) = s \Leftrightarrow c_1(A) = \alpha(s)$$

and

$$r_1(\Phi(A)) = s \Leftrightarrow r_1(A) = \alpha(s)$$

so we need only check that

$$C_{(a,b)}(\Phi(A)) \equiv C_{(a-m,b)}(A).$$

Put $A = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$; $C_{(a,b)}(A)$ is unchanged if we swap the columns and/or the rows of A, so we may assume that $i + j \geqslant k + l$ and $i + k \geqslant j + l$, so that $\Phi(A) = \begin{pmatrix} i+m & j \\ k & l \end{pmatrix}$. Then the left-hand side equals

$$\sum_\kappa (-1)^\kappa \binom{b}{\kappa} \binom{i+m+k-b}{k-\kappa} \binom{j+l-b}{j-\kappa};$$

for the summand to be non-zero we must have $\kappa \geqslant 0$, whence

$$k - \kappa \leqslant k \leqslant \frac{d}{2} < p^s;$$

then, by [2, Lemma 3.1],

$$\binom{i+m+k-b}{k-\kappa} \equiv \binom{i+k-b}{k-\kappa}$$

and so

$$C_{(a,b)}(\begin{pmatrix} i+m & j \\ k & l \end{pmatrix}) \equiv \sum_{\kappa} (-1)^{\kappa} \binom{b}{\kappa} \binom{i+k-b}{k-\kappa} \binom{j+l-b}{j-\kappa}$$
$$= C_{(a-m,b)}(\begin{pmatrix} i & j \\ k & l \end{pmatrix}). \qquad \square$$

For the embedding $\Psi : S(2, r) \hookrightarrow S(2, rp + q)$ of Theorem 2.2, we cannot use exactly the same kind of functor, since the embedded algebra is not of the form $eS(2, rp + q)e$. But we may use a composition of such a functor with restriction: let S be the image of Ψ, and let $e = \Psi(1_{S(2,r)})$ be the identity element of S. Put $\overline{S} = eS(2, rp + q)e$. Then there is a natural functor

$$\mathrm{mod}(\overline{S}) \longrightarrow \mathrm{mod}(S)$$

given by restriction: S is a subalgebra (with 1) of $\overline{S}$, and so we simply regard an $\overline{S}$-module as an S-module. Now define

$$F_{r,q} : \mathrm{mod}(S(2, rp + q)) \longrightarrow \mathrm{mod}(S(2, r))$$

to be the composite of this functor with the functor

$$\mathrm{mod}(S(2, rp + q)) \longrightarrow \mathrm{mod}(\overline{S})$$

given by sending a module M to eM. We wish to identify $F_{r,q}(\nabla(\lambda))$.

Theorem 3.4. *Let $\lambda = (a, b)$ be a partition of $pr + q$, and put $a = pc + d$ with $0 \leqslant d < p$. Then*

$$F_{r,q}(\nabla(\lambda)) \cong \begin{cases} \nabla(c, r - c) & (\text{if } d \geqslant q \text{ and } c \geqslant \frac{r}{2}) \\ 0 & (\text{otherwise}). \end{cases}$$

To prove this, we begin by examining the coefficients $C_{(a,b)}(\Psi(A))$.

Lemma 3.5. *Take $A = \begin{pmatrix} i & j \\ k & l \end{pmatrix} \in M(r)$. Then, provided $a \geqslant p(i + k) + q \geqslant b$,*

$$C_{(a,b)}(\Psi(A)) = \begin{cases} 0 & (d < q) \\ C_{(c,r-c)}(A) & (d \geqslant q). \end{cases}$$

Proof. We have

$$C_{(a,b)}(\Psi(A)) = \sum_{\beta=0}^{p-1} \binom{\beta+q}{q} C_{(a,b)}(A_{\beta,q}) + \sum_{\gamma=0}^{p-1} (1 - \binom{p-\gamma+q}{q}) C_{(a,b)}(A_{-\gamma,q});$$

If $\gamma = 0$ or $\gamma = p$, then $1 - \binom{p-\gamma+q}{q} \equiv 0$, so we may replace γ with $p - \beta$ in the second sum and still sum over $0 \leqslant \beta \leqslant p - 1$. Rearranging, we get

$$C_{(a,b)}(\Psi(A)) \equiv \Sigma_5 + \Sigma_6,$$

where

$$\begin{aligned}\Sigma_5 &= \sum_{\beta=0}^{p-1}\binom{\beta+q}{q}(C_{(a,b)}(A_{\beta,q}) - C_{(a,b)}(A_{\beta-p,q}))\\ &= \sum_{\beta=0}^{p-1}\sum_{\kappa\in\mathbb{Z}}\binom{\beta+q}{q}(-1)^\kappa\binom{pr+q-a}{\kappa}\binom{a-pj-pl}{pk-\beta-\kappa}\binom{a-pi-pk-q}{pj-\beta-\kappa}\\ &-\sum_{\beta=0}^{p-1}\sum_{\kappa\in\mathbb{Z}}\binom{\beta+q}{q}(-1)^\kappa\binom{pr+q-a}{\kappa}\binom{a-pj-pl}{pk+p-\beta-\kappa}\binom{a-pi-pk-q}{pj+p-\beta-\kappa}\end{aligned}$$

and

$$\begin{aligned}\Sigma_6 &= \sum_{\beta=0}^{p-1}C_{(a,b)}(A_{\beta-p,q})\\ &= \sum_{\beta=0}^{p-1}\sum_{\kappa\in\mathbb{Z}}(-1)^\kappa\binom{pr+q-a}{\kappa}\binom{a-pj-pl}{pk+p-\beta-\kappa}\binom{a-pi-pk-q}{pj+p-\beta-\kappa}.\end{aligned}$$

We deal with Σ_5 first. Replacing κ with $p+\kappa$ in the second half of the sum and noting that

$$\binom{pr+q-a}{\kappa}+\binom{pr+q-a}{p+\kappa}\equiv\binom{pr+p+q-a}{p+\kappa},$$

we have

$$\Sigma_5 \equiv \sum_{\beta=0}^{p-1}\sum_{\kappa\in\mathbb{Z}}\binom{\beta+q}{q}(-1)^\kappa\binom{pr+p+q-a}{p+\kappa}\binom{a-pj-pl}{pk-\beta-\kappa}\binom{a-pi-pk-q}{pj-\beta-\kappa}.$$

Put $\kappa+\beta = p\mu-\nu$ with $0\leqslant\nu\leqslant p-1$ and put $a = pc+d$ as above to get

$$\Sigma_5 \equiv \sum_{\beta=0}^{p-1}\sum_{\mu\in\mathbb{Z}}\sum_{\nu=0}^{p-1}\binom{\beta+q}{q}(-1)^{\mu+\nu+\beta}\binom{pr+p+q-pc-d}{p\mu+p-\nu-\beta}\binom{pc-pj-pl+d}{pk-p\mu+\nu}\binom{pc-pi-pk+d-q}{pj-p\mu+\nu};$$

we consider separately the cases $d=q$, $d>q$, $d<q$.

- $d=q$

  The second binomial coefficient is congruent to zero unless $\nu+\beta\equiv 0$, and the last binomial coefficient is zero unless $\nu=0$. Thus in this case we have

$$\begin{aligned}\Sigma_5 &\equiv \sum_{\mu\in\mathbb{Z}}(-1)\mu\binom{pr+p-pc}{p+p\mu}\binom{pc-pj-pl+d}{pk-p\mu}\binom{pc-pi-pk}{pj-p\mu}\\ &\equiv \sum_{\mu\in\mathbb{Z}}(-1)\mu\binom{r+1-c}{1+\mu}\binom{c-j-l}{k-\mu}\binom{c-i-k}{j-\mu}.\end{aligned}$$

- $d>q$

  The term with $\beta=\nu=0$ gives

$$(-1)^\mu\binom{r-c}{\mu+1}\binom{c-j-l}{k-\mu}\binom{c-i-k}{j-\mu} = X,$$

say. If $\beta+\nu>p$ then the reduction modulo p of the summand in Σ_5 has a factor $\binom{p+q-d}{2p-\beta-\nu}\binom{d-q}{\nu}$, which equals zero. So the sum of those terms with $\beta+\nu>0$ is

$$\sum_{\beta=0}^{p-1}\sum_{\mu\in\mathbb{Z}}\sum_{\nu=\max(1-\beta,0)}^{\min(p-\beta,p-1)}\binom{\beta+q}{q}(-1)^{\mu+\nu+\beta}\binom{r-c}{\mu}\binom{p+q-d}{p-\beta-\nu}\binom{c-j-l}{k-\mu}\binom{d}{\nu}\binom{c-i-k}{j-\mu}\binom{d-q}{\nu};$$

we may replace the range of summation for ν with $\sum_{\nu\in\mathbb{Z}}$, since $\binom{p+q-d}{p-\beta-\nu}\binom{d}{\nu}=0$ if $\beta=\nu=0$ or $\beta+\nu>p$ or $\nu<0$ or $\nu\geqslant p$. If we replace $(-1)^\beta$ with $\binom{p-1}{\beta}$, we may also replace the range of summation for β with $\mathbb{Z}$. Terms involving β then give

$$\begin{aligned}&\sum_{\beta\in\mathbb{Z}}\binom{\beta}{\zeta}\binom{q}{q-\zeta}\binom{p-1}{\beta}\binom{p+q-d}{p-\beta-\nu}\\ \equiv&\binom{p-1}{\zeta}\binom{q}{\zeta}\binom{2p-1+q-d-\zeta}{p-\nu-\zeta}\end{aligned}$$

by Lemma 2.3, whence

$$\Sigma_5\equiv X+\sum_{\mu,\nu,\zeta\in\mathbb{Z}}(-1)^{\mu+\nu}\binom{r-c}{\mu}\binom{c-j-l}{k-\mu}\binom{d}{\nu}\binom{c-i-k}{j-\mu}\binom{d-q}{\nu}\binom{p-1}{\zeta}\binom{q}{\zeta}\binom{2p-1+q-d-\zeta}{p-\nu-\zeta}.$$

Consider the possible values of ζ: if $\zeta<0$ then $\binom{q}{\zeta}=0$; if $\zeta>p+q-d-1$, then $\zeta>q$, so $\binom{q}{\zeta}=0$. If $0\leqslant\zeta\leqslant p+q-d-1$ and $\nu+\zeta>0$, then the reduction modulo p of $\binom{2p-1+q-d-\zeta}{p-\nu-\zeta}$ has a factor $\binom{p-1+q-d-\zeta}{p-\nu-\zeta}$, but $\binom{p-1+q-d-\zeta}{p-\nu-\zeta}\binom{d-q}{\nu}=0$. So the only non-zero contribution comes from $\zeta=\nu=0$, which gives

$$\Sigma_5\equiv X+\sum_{\mu\in\mathbb{Z}}(-1)^\mu\binom{r-c}{\mu}\binom{c-j-l}{k-\mu}\binom{c-i-k}{j-\mu}.$$

- $d<q$

  Here the term with $\mu=\beta=0$ gives

$$\sum_{\mu\in\mathbb{Z}}\binom{r+1-c}{\mu+1}\binom{c-j-l}{k-\mu}\binom{c-i-k}{j-\mu}=Y,$$

  say. If $\nu+\beta>p$, then again the summand in Σ_5 is zero. If $0<\nu+\beta\leqslant p$, then the reduction of modulo p of the summand in Σ_5 has factors $\binom{\beta+q}{q}\binom{q-d}{p-\beta-\nu}\binom{d}{\nu}$. If this is non-zero, then we must have

$$\begin{aligned}\beta+q&<p,\\ p-\beta-\nu&\leqslant q-d,\\ \nu&\leqslant d,\end{aligned}$$

  which gives a contradiction. Hence $\Sigma_5\equiv Y$.

Now we evaluate Σ_6; again we put $\kappa+\beta = p\mu-\nu$, and replace $\left(\sum_{\beta=0}^{p-1}(-1)^\beta\dots\right)$ with $\left(\sum_{\beta\in\mathbb{Z}}\binom{p-1}{\beta}\dots\right)$. We have

$$\begin{aligned}\Sigma_6 &\equiv \sum_{\beta\in\mathbb{Z}}\sum_{\mu\in\mathbb{Z}}\sum_{\nu=0}^{p-1}(-1)^\mu(-1)^\nu\binom{p-1}{\beta}\binom{pr+q-a}{p\mu-\nu-\beta}\binom{a-pj-pl}{pk+p-p\mu+\nu}\binom{a-pi-pk-q}{pj+p-p\mu+\nu}\\ &= \sum_{\mu\in\mathbb{Z}}\sum_{\nu=0}^{p-1}(-1)^\mu(-1)^\nu\binom{pr+p+q-a-1}{p\mu-\nu}\binom{a-pj-pl}{pk+p-p\mu+\nu}\binom{a-pi-pk-q}{pj+p-p\mu+\nu};\end{aligned}$$

for the summand to be non-zero modulo p, the mod p residue of $q-a-1$ must be at least that of $-\nu$, and the mod p residue of $a-q$ must be at least that of ν. This is only possible if $\nu=0$, so we get

$$\Sigma_6 \equiv \sum_{\mu\in\mathbb{Z}}\binom{pr+p+q-a-1}{p\mu}\binom{a-pj-pl}{pk+p-p\mu}\binom{a-pi-pk-q}{pj+p-p\mu}.$$

Replacing a with $pc+d$ and μ with $\mu+1$ then gives

$$\Sigma_6 \equiv \begin{cases}-X & (d\geqslant q)\\ -Y & (d<q),\end{cases}$$

where X and Y are as above. Adding Σ_5 to Σ_6 gives

$$C_{(a,b)}(\Psi(A)) \equiv \begin{cases}0 & (d<q)\\ \sum_{\mu\in\mathbb{Z}}(-1)^\mu\binom{r-c}{\mu}\binom{c-j-l}{k-\mu}\binom{c-i-k}{j-\mu} & (d\geqslant q)\end{cases}$$

as required. □

Proof of Theorem 3.4. Take $A\in M(r)$ as above. Each of the matrices in $\Psi(A)$ has first column sum equal to $p(i+k)+q$ and first row sum equal to $p(i+j)+q$, and so we have

$$\Psi(A)\circ x_s = \begin{cases}0 & (s\neq p(i+k)+q)\\ C_{(a,b)}(\Psi(A)).x_{p(i+j)+q} & (s=p(i+k)+q).\end{cases}$$

If $d<q$, then from Lemma 3.5 we have $\Psi(A)\circ x_s = 0$ for all A and s, so $F_{r,q}(\nabla(a,b))=0$ as required. Now suppose $d\geqslant q$. From above we have

$$F_{r,q}(\nabla(a,b)) \subset \langle x_s \mid s\equiv q \pmod p\rangle;$$

in fact equality holds: the identity element of $S(2,r)$ is $\sum_\epsilon\begin{pmatrix}\epsilon & 0\\ 0 & r-\epsilon\end{pmatrix}$; by Lemma 3.5, if $c\geqslant\epsilon\geqslant r-c$, we have $C_{(a,b)}(\Psi(\begin{pmatrix}\epsilon & 0\\ 0 & r-\epsilon\end{pmatrix}))\equiv C_{(c,r-c)}(\begin{pmatrix}\epsilon & 0\\ 0 & r-\epsilon\end{pmatrix})\equiv 1$, and so $e\circ x_s = x_s$ if $s\equiv q \pmod p$.

If $c<\frac{r}{2}$, then there is no x_s with $s\equiv q \pmod p$, and so $F_{r,j}(\nabla(a,b))=0$, as required. Otherwise, we define a bijection α from $\{s\mid a\geqslant s\geqslant b, s\equiv q \pmod p\}$ to $\{r-c, r-c+1,\dots,c\}$ by sending s to $\frac{s-q}{p}$, and then a linear isomorphism

$$f : F_{r,q}(\nabla(a,b)) \longrightarrow \nabla(c,r-c)$$

by

$$x_s \longmapsto x_{\alpha(s)}.$$

We need to show that this is a module isomorphism. But each matrix B involved in $\Psi(A)$ has $c_1(B) = s$ if and only if $c_1(A) = \alpha(s)$, and $r_1(B) = s$ if and only if $r_1(A) = \alpha(s)$, so by Lemma 3.5 we have $A \circ (f(x_s)) = \Psi(A) \circ x_s$ and the theorem is proved. □

4 Decomposition numbers for $S(2, r)$

In this section we show that we can use the results of the previous sections to recover the decomposition numbers $[\nabla(\lambda) : L(\mu)]$ for the Schur algebras $S(2, r)$. These were first found by Carter and Cline [1] in the context of the special linear group $\mathrm{SL}_n(k)$. First we need to identify the images of the simple modules $L(\lambda)$ under the functors $G_{d,r}$ and $F_{r,q}$. Then we can use the fact that both of these functors are exact to find the decomposition matrices of $S(2, r)$ by induction, given only a few of the decomposition numbers.

Lemma 4.1. *The image of a simple module under $G_{d,r}$ or $F_{r,q}$ is either simple or zero.*

Proof. We use contravariant duality. Given a module M for $S(2, r)$, define the *contravariant dual M°* of M to be the dual vector space M^* with transpose action

$$(A \circ \phi)(m) = \phi(A^{\mathrm{T}} \circ m)$$

for $A \in M(r), m \in M, \phi \in M^*$. The embedding $\Phi : S(2, d) \hookrightarrow S(2, r)$ commutes with transposition of matrices, i.e.

$$\Phi(A^{\mathrm{T}}) = (\Phi(A))^{\mathrm{T}}$$

for $A \in M(d)$, where transposition $^{\mathrm{T}}$ is extended linearly. Hence $G_{d,r}$ respects contravariant duality, i.e.

$$G_{d,r}(M^\circ) \cong G_{d,r}(M)^\circ$$

for $M \in \mathrm{mod}(S(n, r))$.

It is known ([6], Theorem 3.4.9) that $L(\lambda)$ is isomorphic to its contravariant dual; hence $G_{d,r}(L(a, b))$ is a contravariant self-dual submodule of $G_{d,r}(\nabla(a, b))$. If $a - b < m$ this gives $G_{d,r}(L(a, b)) = 0$, while if $a-b \geqslant m$, then $G_{d,r}(L(a, b))$ is a contravariant self-dual submodule of $\nabla(a-m, b)$. The latter has a unique simple submodule $L(a - m, b)$, which must constitute the socle of $G_{d,r}(L(a, b))$ if this is non-zero. But $L(a-m, b)$ is contravariant self-dual, and so the cosocle of $G_{d,r}(L(a, b))$ is also isomorphic to $L(a-m, b)$. $L(a - m, b)$ occurs only once as a composition factor of $\nabla(a - m, b)$, so if $G_{d,r}(L(a, b)) \neq 0$, then

$$\mathrm{soc}(G_{d,r}(L(a, b))) = \mathrm{cosoc}(G_{d,r}(L(a, b))) = G_{d,r}(L(a, b)) \cong L(a - m, b).$$

Similarly, for (a, b) a partition of $pr + q$ with $a = pc + d$, we have

$$F_{r,q}(L(a, b)) \cong L(c, r - c) \text{ or } 0.$$

□

Remark. In fact, for an algebra A with idempotent e, the functor $M \mapsto eM$ always sends simple modules to simple modules or to zero; moreover $\{eL \mid L \text{ a simple } A\text{-module}\}$ is a complete set of irreducibles for eAe ([6], Proposition 4.1.3). So we know that $G_{d,r}(L(a,b)) \cong L(a-m,b)$ if $a-b \geqslant m$. The situation for $F_{r,q}$ is more complicated.

Since we now know $G_{d,r}(L)$ for simple modules L, we have an immediate consequence for decomposition numbers.

Proposition 4.2. *Let d, r, m be as in Theorem 3.2, and let (a,b),(f,g) be partitions of r with $a-b \geqslant m, f-g \geqslant m$. Then*

$$[\nabla(a,b) : L(f,g)] = [\nabla(a-m,b) : L(f-m,g)].$$

We now recall the principle of column removal from [5].

Theorem 4.3 (James). *If λ, μ are partitions of r both with exactly n non-zero parts, then define the n-part partitions $\check{\lambda}$, $\check{\mu}$ by $\check{\lambda}_i = \lambda_i - 1$, $\check{\mu}_i = \mu_i - 1$. Then*

$$[\nabla(\lambda) : L(\mu)] = [\nabla(\check{\lambda}) : L(\check{\mu})].$$

In the case $n = 2$, suppose (a,b) and (f,g) are partitions of r, and put $\alpha = a-b, \beta = f-g$. The principle of column removal says that the number

$$d_{\alpha,\beta} = [\nabla(a,b) : L(f,g)]$$

is independent of r. So finding the decomposition numbers for all Schur algebras $S(2,r)$ is equivalent to finding all the numbers $d_{\alpha,\beta}$ for all pairs (α,β) of non-negative integers of the same parity. Since $[\nabla(\lambda) : L(\mu)] = 0$ unless λ dominates μ, we have $d_{\alpha,\beta} = 0$ for $\alpha < \beta$. We re-state Proposition 4.2 as follows.

Proposition 4.4. *Suppose $\alpha \geqslant \beta$ are non-negative integers of the same parity with $\alpha - \beta < 2p^s$ and $\beta \geqslant m \equiv 0 \pmod{p^s}$. Then*

$$d_{\alpha,\beta} = d_{\alpha-m,\beta-m}.$$

Now we show that we can find all the decomposition numbers for $S(2,r)$ provided we have the decomposition numbers for $S(2,r)$ with $r < 2p$. First we need to find the images of the simple modules under the functor $F_{r,q}$.

Proposition 4.5. *Given the decomposition numbers $d_{\alpha,\beta}$ for $\alpha,\beta < 2p$, we can find $F_{r,q}(L(a,b))$ for every partition (a,b) of $pr+q$.*

Proof. By Proposition 4.4 we can find all the decomposition numbers $d_{\alpha,\beta}$ with $\alpha - \beta < 2p$, that is, we can find the last p entries of each row of the decomposition matrix for $S(2, pr+q)$. We now find $F_{r,q}(L(a,b))$ by induction on a. For the smallest value of a, that is $a = \lceil \frac{pr+q}{2} \rceil$, we have $L(a,b) = \nabla(a,b)$, and we know $F_{r,q}(\nabla(a,b))$. Now suppose we know $F_{r,q}(L(a',b'))$ for $a' < a$. If $F_{r,q}(\nabla(a,b)) = 0$, then $F_{r,q}(L(a,b)) = 0$. Otherwise $F_{r,q}(\nabla(a,b)) = \nabla(c, r-c)$, so some composition factor of $\nabla(a,b)$ maps to $L(c, r-c)$ under $F_{r,q}$. But all composition factors of $\nabla(a,b)$ other than $L(a,b)$ have the form $L(a',b')$ for $a' < a$; we know the images under $F_{r,q}$ of these factors; in particular, we know (from the proof of Lemma 4.1) that $F_{r,q}(L(a',b')) \cong L(c,r-c)$ only if $a - a' < p$. And so $F_{r,q}(L(a,b)) \cong L(c,r-c)$ if and only if for every a' with $F_{r,q}(L(a',b')) \cong L(c,r-c)$ we have $[\nabla(a,b) : L(a',b')] = 0$. □

From the above proof, we immediately see the following.

Corollary 4.6. *If a is minimal such that $F_{r,q}(\nabla(a,b)) \cong \nabla(c, r-c)$, i.e. if $a \equiv q \pmod p$, then $F_{r,q}(L(a,b)) \cong L(c, r-c)$.*

Proposition 4.7. *Given the decomposition numbers $d_{\alpha,\beta}$ for $\alpha, \beta < 2p$, we can find all the decomposition numbers $d_{\alpha,\beta}$.*

Proof. We proceed by induction on $\alpha - \beta$, with the decomposition numbers for $\alpha - \beta < p$ following from Proposition 4.4. Given α, β of the same parity with $\alpha - \beta = k$, take R of the same parity as α and β with $R \equiv \beta \pmod p$ and sufficiently large that there exist partitions (a, b) and (f, g) of R with $a - b = \alpha$, $f - g = \beta$. Putting $R = rp + q$ with $0 \leqslant q < p$, we have $f = \frac{R+\beta}{2} \equiv q \pmod p$, so by Corollary 4.6 we have $F_{r,q}(L(f, g)) \cong L(c, r-c)$ (where now $c = \lfloor \frac{f}{p} \rfloor$). Any other simple modules $L(f', g')$ of $S(2, R)$ with $F_{r,q}(L(f', g')) \cong L(c, r-c)$ satisfy $f' > f$, so by induction we know the composition multiplicities of these simples in $\nabla(a, b)$. Since $F_{r,q}$ is exact, we have

$$[F_{r,q}(\nabla(a,b)) : L(c, r-c)] = \sum_{f'} [\nabla(a,b) : L(f', g')],$$

the sum being over all f' (including f) with $F_{r,q}(L(f', g')) \cong L(c, r-c)$. Hence we can find the decomposition number

$$[\nabla(a,b) : L(f,g)] = d_{\alpha,\beta}.$$ □

Example. Let k be a field of characteristic 2; we show that the above results give a very simple recursive formula for the decomposition numbers $d_{\alpha,\beta}$, given only the information

$$d_{2,0} = 1.$$

By Proposition 4.4, we have $d_{2\alpha,2\alpha-2} = 1$ for all α. Now consider the embedding $S(2, r) \hookrightarrow S(2, 2r)$. We have

$$F_{r,0}(\nabla(2a, 2b)) \cong \nabla(a, b),$$
$$F_{r,0}(\nabla(2a+1, 2b-1)) \cong \nabla(a, b);$$

since $d_{2\alpha,2\alpha-2} = 1$, we deduce

$$F_{r,0}(L(2a, 2b)) \cong L(a, b),$$
$$F_{r,0}(L(2a+1, 2b-1)) = 0.$$

Thus

$$[\nabla(2a, 2b) : L(2c, 2d)] = [\nabla(a, b) : L(c, d)] \quad (3)$$
$$[\nabla(2a+1, 2b-1) : L(2c, 2d)] = [\nabla(a, b) : L(c, d)]. \quad (4)$$

Next we consider the embedding $S(2, r) \hookrightarrow S(2, 2r+1)$. This gives

$$F_{r,1}(\nabla(2a+1, 2b)) \cong \nabla(a, b),$$
$$F_{r,1}(\nabla(2a, 2b+1)) = 0;$$

correspondingly

$$F_{r,1}(L(2a+1,2b)) \cong L(a,b),$$
$$F_{r,1}(L(2a,2b+1)) = 0.$$

Hence

$$[\nabla(2a+1,2b) : L(2c+1,2d)] = [\nabla(a,b) : L(c,d)], \quad (5)$$
$$[\nabla(2a,2b+1) : L(2c+1,2d)] = 0. \quad (6)$$

Applying (3–6) for all r, we find the recursive formula

$$d_{\alpha,\beta} = \begin{cases} d_{\frac{\alpha}{2},\frac{\beta}{2}} & \text{(if } \alpha \text{ and } \beta \text{ are even and congruent mod 4)} \\ d_{\frac{\alpha-2}{2},\frac{\beta}{2}} & \text{(if } \alpha \text{ and } \beta \text{ are even but not congruent mod 4)} \\ d_{\frac{\alpha-1}{2},\frac{\beta-1}{2}} & \text{(if } \alpha \text{ and } \beta \text{ are odd and congruent mod 4)} \\ 0 & \text{(if } \alpha \text{ and } \beta \text{ are odd but not congruent mod 4).} \end{cases}$$

5 Generalisations

Of course, we hope to be able to extend these methods to find Schur algebra embeddings for the Schur algebras $S(n,r)$ with n greater than two. $S(n,r)$ has a basis indexed by the set of $n \times n$ matrices with non-negative integer entries summing to r, and there is a multiplication rule which generalises that for $S(2,r)$. We have a conjectured embedding of $S(n,r)$ in $S(n,rp)$ for all n and r, and hope that this together with other results could elucidate the symmetries of the decomposition matrices for these Schur algebras.

The results ought also to extend to the quantum Schur algebra $S_q(n,r)$. In the case $n = 2$, the decomposition matrices are known to have the same structure as for the classical case, but depending on $e = \min\{r \mid p|(1+\ldots+q^{r-1})\}$ rather than on p. But we cannot find the subalgebra embeddings suggested by the decomposition matrices; $S_q(n,r)$ does not have the same natural basis as in the classical case.

References

[1] R. Carter & E. Cline, *The submodule structure of Weyl modules for groups of type* A_1, Proceedings of the Conference on Finite Groups, Univ. Utah, Park City, Academic Press, New York, 1976, pp. 301–11.

[2] M. Fayers & S. Martin, 'Schur subalgebras', *J. Algebra* **240** (2001), 859–73.

[3] J. Green, *Polynomial representations of* GL_n, Lecture Notes in Mathematics **830**, Springer-Verlag, New York/Berlin, 1980.

[4] A. Henke, 'Schur subalgebras and an application to the symmetric group', *J. Algebra* **233** (2000), 342–62.

[5] G. James, 'On the decomposition matrices of the symmetric groups III', *J. Algebra* **71** (1981), 115–22.

[6] S. Martin, *Schur algebras and representation theory*, Cambridge Tracts in Mathematics **112**, Cambridge Univ. Press, 1993.