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Schur subalgebras

M. Fayers*
Trinity College, Cambridge, CB2 1TQ, U.K.

S. Martin
Magdalene College, Cambridge, CB3 0AG, U.K.

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Abstract

Let k be a field of prime characteristic and let r be a positive integer. In this paper, we study the Schur algebra $S(2, r)$ over k and consider certain natural subalgebras.

1 Introduction

Let k be an infinite field of characteristic p, and let n be a positive integer and r a non-negative integer. The Schur algebra $S(n, r)$ over k is a finite-dimensional associative algebra whose module category is equivalent to the category of r-homogeneous polynomial representations of the general linear group $\mathrm{GL}_n(k)$. There are several equivalent definitions of the Schur algebra; we shall use the definition in terms of a basis and a multiplication rule given by Green.

For the essential results concerning the representation theory of the Schur algebra, the reader is urged to consult the books of Green [2] and Martin [8]; we recall some important points.

For each partition λ of r with at most n parts, one defines a module $\nabla(\lambda)$ for $S(n, r)$, the *dual Weyl module*. In the case $p = 0$, $S(n, r)$ is semi-simple, and the $\nabla(\lambda)$ are precisely the simple modules of $S(n, r)$. In positive characteristic, $\nabla(\lambda)$ has a simple socle $L(\lambda)$, and the $L(\lambda)$ are precisely the simple modules of $S(n, r)$. The *decomposition matrix* of $S(n, r)$ records the composition multiplicities $[\nabla(\lambda) : L(\mu)]$.

The decomposition numbers for $S(n, r)$ are known to be closely related to those for the symmetric groups; for the theory of the latter, see the book by James [7]. James determined the decomposition numbers for the symmetric groups corresponding to partitions with at most two parts ([5, 6]), and following this Carter and Cline [1] explicitly determined the decomposition matrix of $S(n, r)$ in the case $n = 2$. Define the function $\hat{}\colon \{0, \dots, 2(p-1)\} \to \{0, \dots, p-1\}$ by

$$\hat{m} = \begin{cases} m & (m \leqslant p-1) \\ 2(p-1) - m & (m \geqslant p-1); \end{cases}$$

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now say that a natural number t has an *admissible decomposition with respect to s* if there exists an expression

$$t = \sum_{i \geqslant 0} m_i p^i$$

where $0 \leqslant m_i \leqslant 2(p-1)$ for each i, and such that

$$s = \sum_{i \geqslant 0} \hat{m}_i p^i.$$

We then have the following.

Theorem 1.1 (Carter/Cline). $[\nabla(r-a,a) : L(r-b,b)] = 1$ *if there exists an admissible decomposition of $r-2a$ with respect to $r-2b$. Otherwise* $[\nabla(r-a,a) : L(r-b,b)] = 0$.

Henke [3] observed that in certain cases the decomposition matrix for $S(2,d)$ occurs as a submatrix in the bottom right-hand corner of the decomposition matrix for $S(2,r)$; she then proved by module-theoretic means in [4] that there are corresponding algebra embeddings $S(2,d) \hookrightarrow S(2,r)$. Her results are as follows.

Theorem 1.2 (Henke). *Let $d < r$ be positive integers of the same parity such that for some s we have $d < p^s$ and $d \equiv r \pmod{p^s}$. Then for $0 \leqslant a, b \leqslant d/2$,*

$$[\nabla(r-a,a) : L(r-b,b)] = [\nabla(d-a,a) : L(d-b,b)].$$

Furthermore, there exists an idempotent $e \in S(2,r)$ such that

$$eS(2,r)e \cong S(2,d)$$

as k-algebras.

In fact a slightly stronger version of this result is true; we prove this stronger version by elementary means.

The self-similar nature of the decomposition matrices for $S(2,r)$ also suggests the existence of embeddings of Schur algebras $S(2,r)$ which correspond to 'dilations' of the decomposition matrices. These are described in Section 4; an interpretation in terms of modules is reserved for a later paper.

2 The Schur algebra $S(2,r)$

2.1 Green's notation

We use the definition of the Schur algebra in terms of a basis given by Green in [2]. Let $I(n,r)$ denote the set of functions from $\{1,\dots,r\}$ to $\{1,\dots,n\}$, which we normally write as *multi-indices* $i_1 \dots i_r$. Let $\mathfrak{S}_r$ act in the natural way on $I(n,r)$ and on $I(n,r) \times I(n,r)$, and use the symbol $\sim$ to indicate $\mathfrak{S}_r$-conjugacy in both sets. Take a basis $\{\xi_{i,j}\}$ indexed by ordered pairs (i,j) with $\xi_{i,j}$ regarded as the same as $\xi_{k,l}$ iff $(j,k) \sim (k,l)$. Given $i,j,k,l,p,q \in I(n,r)$, define $Z(i,j,k,l,p,q)$ to be the number of $s \in I(n,r)$ such that

$$(i,j) \sim (p,s)$$

and

$$(s,q) \sim (k,l).$$

Now define a multiplication rule for basis elements by

$$\xi_{i,j}\xi_{k,l} = \sum_{(p,q)} Z(i,j,k,l,p,q)1_k\xi_{p,q}$$

where the sum is over a set of representatives (p,q) of $\mathfrak{S}_r$-orbits on $I(n,r) \times I(n,r)$. Taking a k-vector space with basis $\{\xi_{i,j}\}$ and extending this multiplication rule linearly gives the Schur algebra $S_k(n,r)$.

2.2 New notation

From now on, we restrict to the case $n = 2$. Let $M(r)$ denote the set of 2×2 matrices with non-negative integer entries summing to r. Given $i, j \in I(2,r)$, we define m_{uv} to be the number of $x \in \{1,\dots,r\}$ such that $i(x) = u, j(x) = v$ for $u, v = 1, 2$. We then define a function $f : I(n,r) \to M(r)$ by sending (i, j) to the matrix with entries m_{uv}. Now $f((i, j)) = f((k,l))$ iff $(i, j) \sim (k,l)$, and so we may index our basis of $S(2,r)$ by $M(r)$. In fact we let $M(r)$ be a basis for $S(2,r)$ by identifying $\xi_{i,j}$ with $f((i, j))$. We now hope to write the multiplication rule for $S(2,r)$ in terms of the matrices in $M(r)$; we shall write this as $A \circ B$ to avoid any confusion with ordinary matrix multiplication.

For $A \in M(r)$, denote by $r_1(A)$, $r_2(A)$ the first and second row sums of A, and by $c_1(A)$, $c_2(A)$ the first and second column sums of A. Now for $A, B \in M(r)$, define $N(A, B)$ to be the set of matrices $C \in M(r)$ with $r_1(C) = r_1(A)$ and $c_1(C) = c_1(B)$. In addition, if $c_1(A) = r_1(B)$, define $R(A, B)$ to be the set of 2×2 matrices D with (possibly negative) integer entries such that $r_u(D) = a_{u1}$, $c_v(D) = b_{1v}$ for $u, v = 1, 2$.

For any 2×2 matrices C, D with integer coefficients (non-negative in C), we now define

$$\binom{C}{D} = \prod_{u,v=1,2} \binom{c_{uv}}{d_{uv}}.$$

Proposition 2.1. *The multiplication rule for the Schur algebra $S(2,r)$ is given in terms of the basis elements $A \in M(r)$ by*

$$A \circ B = \begin{cases} 0 & (c_1(A) \neq r_1(B)) \\ \sum_{C \in N(A,B)} (\sum_{D \in R(A,B)} \binom{C}{D}).1_k.C & (c_1(A) = r_1(B)). \end{cases}$$

Proof. Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Since $Z(i, j, k,l, p, q) = 0$ unless $j \sim k$, we have $A \circ B = 0$ unless $a + c = e + f$ (and hence $b + d = g + h$). So suppose this holds, and consider the coefficient c_C of $C = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ in $A \circ B$. Now $Z(i, j, k,l, p, q) = 0$ unless $i \sim p$ and $l \sim q$, so if $c_C \neq 0$ we must have $a + b = w + x$ and $e + g = w + y$, i.e. $C \in N(A, B)$. If all these equalities hold, then we write (without loss)

$$A = \xi_{1^a1^b2^c2^d,1^a2^b1^c2^d},$$

$$B = \xi_{1^e1^f2^g2^h,1^e2^f1^g2^h},$$

$$C = \xi_{1^w1^x2^y2^z,1^w2^x1^y2^z}.$$

Hence c_C is the number (modulo char k) of $s \in I(n,r)$ such that

1. $(1^a1^b2^c2^d, 1^a2^b1^c2^d) \sim (1^w1^x2^y2^z, s)$, and
2. $(1^e1^f2^g2^h, 1^e2^f1^g2^h) \sim (s, 1^w2^x1^y2^z)$.

Condition (1) holds iff there are exactly a 1s among $s_1, \dots, s_{w+x}$ and exactly c 1s among $s_{w+x+1}, \dots, s_r$. Condition (2) holds iff there are exactly e 1s among $s_1, \dots, s_w, s_{w+x+1}, \dots, s_{w+x+y}$ and exactly f 1s among $s_{w+1}, \dots, s_{w+x}, s_{w+x+y+1}, \dots, s_r$. If there are exactly i 1s among $s_1, \dots, s_w$, then there are exactly $a-i$ 1s among $s_{w+1}, \dots, s_{w+x}$, exactly $e-i$ 1s among $s_{w+x+1}, \dots, s_{w+x+y}$ and exactly $i+c-e$ 1s among $s_{w+x+y}, \dots, s_r$. Hence there are

$$\binom{w}{i}\binom{x}{a-i}\binom{y}{e-i}\binom{z}{i+c-e} = \binom{C}{D_i}$$

possibilities for s, where $D_i = \begin{pmatrix} i & a-i \\ e-i & i+c-e \end{pmatrix} \in R(A,B)$. It is easily seen that the set of $D \in R(A,B)$ with non-negative entries is precisely the set of such D_i, and so summing over i we obtain

$$c_C = \sum_i \binom{C}{D_i} = \sum_{D \in R(A,B)} \binom{C}{D},$$

since those matrices in $R(A,B)$ with some entries negative do not affect the above sum. The result follows. □

Example. Suppose $r = 5$, and take $A = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$. Then we have

$$N(A,B) = \{\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}\},$$

$$R(A,B) = \{\begin{pmatrix} 1+\beta & 1-\beta \\ 0-\beta & 2+\beta \end{pmatrix} | \, \beta \in \mathbb{Z}\},$$

and

$$\begin{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix} = 2, \begin{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \\ \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \end{pmatrix} = 0, \begin{pmatrix} \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix} = 0, \begin{pmatrix} \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \end{pmatrix} = 3.$$

Hence

$$\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = 2\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} + 3\begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}.$$

We now make an observation to be used later on. Given $A \in M(r)$, let A^c be a with its columns interchanged, and let A^r be A with its rows interchanged. If we extend the functions ^{c} and ^{r} linearly, we have the following.

Lemma 2.2.

1. $A \circ B^c = (A \circ B)^c$*;*

2. $A^r \circ B = (A \circ B)^r$*;*

3. $A^c \circ B^r = A \circ B$*.*

Proof. We have $N(A, B^c) = \{C^c | \, C \in N(A, B)\}$, and $R(A, B^c) = \{D^c | \, D \in R(A, B)\}$; since $\binom{C^c}{D^c} = \binom{C}{D}$, (1) follows. (2) is similar.

For (3) we note first that $N(A^c, B^r) = N(A, B)$; given $C \in N(A, B)$, recall that we have $r_i(C) = r_i(A)$ and $c_i(C) = c_i(B)$, and so defining

$$d_{ij} \longmapsto c_{ij} - d_{ij}$$

gives a map $R(A, B) \to R(A^c, B^r)$, which is bijective. Since $\binom{c_{ij}}{c_{ij}-d_{ij}} = \binom{c_{ij}}{d_{ij}}$, we find that the coefficient of C in $A^c \circ B^r$ is the same as in $A \circ B$. □

3 Subalgebra embeddings

We begin with a lemma concerning binomial coefficients.

Lemma 3.1. *Let* X, Y, s, m *be integers with* $X, s, m \geqslant 0$*, and let* p *be a prime.*

1. If $Y < p^s$*, then*

$$\binom{X+mp^s}{Y} \equiv \binom{X}{Y} \pmod{p}.$$

2. If $Y > X - p^s$*, then*

$$\binom{X+mp^s}{Y+mp^s} \equiv \binom{X}{Y} \pmod{p}.$$

Proof. We have $\binom{X+mp^s}{Y} = \sum_{i=0}^{Y} \binom{X}{Y-i}\binom{mp^s}{i}$. $\binom{mp^s}{i}$ is divisible by p except when $i = 0$, and this gives (1). (2) follows immediately. □

We now use our new basis of the Schur algebra $S(2, r)$ to re-prove Henke's result and embed $S(2, d)$ in $S(2, r)$ for certain $d < r$. Strictly speaking we do not embed $S(2, d)$ as a subalgebra, since $S(2, r)$ and $S(2, d) \subset S(2, r)$ will not have the same identity element; in fact we embed it as a subset of the form $eS(2, r)e$ for e an idempotent.

Suppose $d < r$ with $r - d = m$. Define $\phi : M(d) \to M(r)$ by mapping

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix} \longmapsto \begin{cases} \begin{pmatrix} e+m & f \\ g & h \end{pmatrix} & \text{(if } e+f \geqslant g+h \text{ and } e+g \geqslant f+h\text{)} \\ \begin{pmatrix} e & f+m \\ g & h \end{pmatrix} & \text{(if } e+f \geqslant g+h \text{ and } e+g < f+h\text{)} \\ \begin{pmatrix} e & f \\ g+m & h \end{pmatrix} & \text{(if } e+f < g+h \text{ and } e+g \geqslant f+h\text{)} \\ \begin{pmatrix} e & f \\ g & h+m \end{pmatrix} & \text{(if } e+f < g+h \text{ and } e+g < f+h\text{).} \end{cases}$$

(ϕ could be described informally as 'adding m to the heaviest corner of each matrix'.) By linear extension of ϕ, we obtain a map $\Phi : S(2,d) \to S(2,r)$. We claim that under certain circumstances this is an embedding.

Theorem 3.2. *Let $p = \operatorname{char} k$, and let s be any non-negative integer. If $d < r$, $d < 2.p^s$ and $d \equiv r \pmod{p^s}$, then the map Φ defined above is an embedding of $S(2,d)$ in $S(2,r)$.*

Proof. We need to show that for $A, B \in M(d)$, $\Phi(A) \circ \Phi(B) = \Phi(A \circ B)$. Suppose that

$$A = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, B = \begin{pmatrix} i & j \\ k & l \end{pmatrix}.$$

If $e + g \neq i + j$ the result is obvious, so assume $e + g = i + j$. By interchanging rows and columns and using Lemma 2.2, we may assume wlog that $e + f \geqslant g + h, e + g \geqslant f + h$ (and hence $i + j \geqslant k + l$) and $i + k \geqslant j + l$, so that

$$\Phi(A) = \begin{pmatrix} e+m & f \\ g & h \end{pmatrix}, \Phi(B) = \begin{pmatrix} i+m & j \\ k & l \end{pmatrix}$$

where $m = r - d$ is a multiple of p^s. Let $\psi : M(d) \to M(r)$ be the function which adds m to the top left entry of each matrix.

Next we explicitly calculate $N(A, B)$, $N(\Phi(A), \Phi(B))$, $R(A, B)$ and $R(\Phi(A), \Phi(B))$. Since $e + f \geqslant r/2 \geqslant j + l$ we have

$$N(A, B) = \{\begin{pmatrix} e+f-j-l+\alpha & j+l-\alpha \\ g+h-\alpha & \alpha \end{pmatrix} \mid 0 \leqslant \alpha \leqslant \min(j+l, g+h)\}$$

and

$$N(\Phi(A), \Phi(B)) = \{\begin{pmatrix} m+e+f-j-l+\alpha & j+l-\alpha \\ g+h-\alpha & \alpha \end{pmatrix} \mid 0 \leqslant \alpha \leqslant \min(j+l, g+h)\};$$

we also have

$$R(A, B) = \{\begin{pmatrix} e-j+\beta & j-\beta \\ g-\beta & \beta \end{pmatrix} \mid \beta \in \mathbb{Z}\}$$

and

$$R(\Phi(A), \Phi(B)) = \{\begin{pmatrix} m+e-j+\beta & j-\beta \\ g-\beta & \beta \end{pmatrix} \mid \beta \in \mathbb{Z}\}.$$

So we have $N(\Phi(A), \Phi(B)) = \{\psi(C) \mid C \in N(A, B)\}$ and $R(\Phi(A), \Phi(B)) = \{\psi(D) \mid D \in R(A, B)\}$. Since $\Phi(C) = \psi(C)$ for $C \in N(A, B)$, it remains to show that for $C \in N(A, B), D \in R(A, B)$ we have

$$\binom{\psi(C)}{\psi(D)} \equiv \binom{C}{D} \pmod{p}.$$

Let $C \in N(A, B)$ and $D \in R(A, B)$ be defined in terms of α, β as above. There are two cases to consider.

1. If $\alpha > \beta + l$, then $\binom{C}{D} = \binom{\psi(C)}{\psi(D)} = 0$, since they both have a factor $\binom{j+l-\alpha}{j-\beta}$.

2. If $\alpha \leqslant \beta + l$, then we consider the upper left entries

$$X = e + f - j - l + \alpha, Y = e - j + \beta$$

of C, D respectively. We must show that

$$\binom{X+m}{Y+m} \equiv \binom{X}{Y} \pmod{p};$$

by Lemma 3.1, this follows provided $X - Y < p^s$. But

$$\begin{aligned} X - Y &= f - l + \alpha - \beta \\ &\leqslant f \\ &< p^s \end{aligned}$$

since $d < 2p^s$.

The result follows. □

Remark. It is easily seen that if we put

$$e = \sum_{i=0}^{d} \Phi(\begin{pmatrix} i & 0 \\ 0 & d-i \end{pmatrix})$$

then e is an idempotent in $S(2, r)$ and $\Phi(S(2, d)) = eS(2, r)e$. Hence the map Φ is as promised.

4 More subalgebra embeddings

We now construct embeddings of Schur algebras $S(2, r)$ which reflect more of the self-similarity of the decomposition matrices. We need another lemma concerning binomial coefficients.

Lemma 4.1. *Let p be a prime, and let $i, j, k, l, w, x, y, z, \epsilon$ be non-negative integers with $\epsilon < p$. Then*

$$\sum_{\alpha,\beta,\gamma=0}^{p-1} \begin{pmatrix} \begin{pmatrix} pi+\epsilon & pj-\epsilon \\ pk-\epsilon & pl+\epsilon \end{pmatrix} \\ \begin{pmatrix} pw+\alpha & px-\alpha+\beta \\ py-\alpha+\gamma & pz+\alpha-\beta-\gamma \end{pmatrix} \end{pmatrix} \equiv \begin{pmatrix} \begin{pmatrix} i & j \\ k & l \end{pmatrix} \\ \begin{pmatrix} w & x \\ y & z \end{pmatrix} \end{pmatrix} \pmod{p}.$$

Proof. We use Lucas's Lemma that

$$\binom{pa+b}{pc+d} \equiv \binom{a}{c}\binom{b}{d} \pmod{p}$$

for $b, d < p$. This has corollaries

$$\binom{pa}{pc} \equiv \binom{a}{c} \pmod{p}$$

and

$$\binom{pa}{e} \equiv 0 \pmod{p}$$

for e not divisible by p. This enables us to dismiss immediately the case $\epsilon = 0$, so we assume that $\epsilon > 0$. In order to use Lucas's Lemma we need to know the greatest multiple of p less than each of the entries of $\begin{pmatrix} pw+\alpha & px-\alpha+\beta \\ py-\alpha+\gamma & pz+\alpha-\beta-\gamma \end{pmatrix}$, and so we split into cases. Note first that if $\beta+\gamma-\alpha > p$, then $\beta > \alpha$ and $\gamma > \alpha$, so we must have $p-\epsilon \geqslant \beta-\alpha$ and $p-\epsilon \geqslant \gamma-\alpha$ to get a non-zero residue. We must also have $\epsilon \geqslant \alpha$ and $\epsilon \geqslant 2p+\alpha-\beta-\gamma$, and these inequalities taken together give a contradiction. So we may assume that the lower right entry $pz+\alpha-\beta-\gamma$ is always at least $p(z-1)$, and we split into cases; put

$$B_{\alpha,\beta,\gamma} = \begin{pmatrix} \begin{pmatrix} pi+\epsilon & pj-\epsilon \\ pk-\epsilon & pl+\epsilon \end{pmatrix} \\ \begin{pmatrix} pw+\alpha & px-\alpha+\beta \\ py-\alpha+\gamma & pz+\alpha-\beta-\gamma \end{pmatrix} \end{pmatrix}$$

and then write the sum as

$$\sum_{\alpha,\beta,\gamma=0}^{p-1} B_{\alpha,\beta,\gamma} = S_1+S_2+S_3+S_4+S_5+S_6+S_7+S_8$$

with the sums S_i as follows.

$$\begin{aligned}
S_1 &= \sum_{\alpha=0}^{p-1}\sum_{\beta=0}^{\alpha-1}\sum_{\gamma=0}^{\min(\alpha-1,\alpha-\beta)} B_{\alpha,\beta,\gamma}\\
S_2 &= \sum_{\alpha=0}^{p-1}\sum_{\beta=0}^{\alpha-1}\sum_{\gamma=\alpha}^{\alpha-\beta} B_{\alpha,\beta,\gamma}\\
S_3 &= \sum_{\alpha=0}^{p-1}\sum_{\beta=0}^{\alpha-1}\sum_{\gamma=\alpha-\beta+1}^{\alpha-1} B_{\alpha,\beta,\gamma}\\
S_4 &= \sum_{\alpha=0}^{p-1}\sum_{\beta=0}^{\alpha-1}\sum_{\gamma=\max(\alpha,\alpha-\beta+1)}^{p-1} B_{\alpha,\beta,\gamma}\\
S_5 &= \sum_{\alpha=0}^{p-1}\sum_{\beta=\alpha}^{p-1}\sum_{\gamma=0}^{\min(\alpha-1,\alpha-\beta)} B_{\alpha,\beta,\gamma}\\
S_6 &= \sum_{\alpha=0}^{p-1}\sum_{\beta=\alpha}^{p-1}\sum_{\gamma=\alpha}^{\alpha-\beta} B_{\alpha,\beta,\gamma}\\
S_7 &= \sum_{\alpha=0}^{p-1}\sum_{\beta=\alpha}^{p-1}\sum_{\gamma=\alpha-\beta+1}^{\alpha-1} B_{\alpha,\beta,\gamma}\\
S_8 &= \sum_{\alpha=0}^{p-1}\sum_{\beta=\alpha}^{p-1}\sum_{\gamma=\max(\alpha,\alpha-\beta+1)}^{p-1} B_{\alpha,\beta,\gamma}.
\end{aligned}$$

Now

$$S_6 = B_{0,0,0}$$

$$\equiv \begin{pmatrix} \begin{pmatrix} i & j-1 \\ k-1 & l \end{pmatrix} \\ \begin{pmatrix} w & x \\ y & z \end{pmatrix} \end{pmatrix}.$$

We can write S_1 as

$$S_1 = \sum_{\alpha=0}^{p-1} \sum_{\beta=0}^{\alpha-1} \sum_{\gamma=0}^{\min(\alpha-1,\alpha-\beta)} \begin{pmatrix} \begin{pmatrix} i & j-1 \\ k-1 & l \end{pmatrix} \\ \begin{pmatrix} w & x-1 \\ y-1 & z \end{pmatrix} \end{pmatrix} \binom{\epsilon}{\alpha} \binom{p-\epsilon}{p-\alpha+\beta} \binom{p-\epsilon}{p-\alpha+\gamma} \binom{\epsilon}{\alpha-\beta-\gamma};$$

note that if $\gamma < 0$ or $\gamma > \alpha - \beta$ or $\gamma \geqslant \alpha$ then $\binom{\epsilon}{\alpha}\binom{p-\epsilon}{p-\alpha+\gamma}\binom{\epsilon}{\alpha-\beta-\gamma} = 0$, so we have

$$\begin{aligned} \binom{\epsilon}{\alpha} \sum_{\gamma=0}^{\min(\alpha-1,\alpha-\beta)} \binom{p-\epsilon}{p-\alpha+\gamma} \binom{\epsilon}{\alpha-\beta-\gamma} &= \binom{\epsilon}{\alpha} \sum_{\gamma\in\mathbb{Z}} \binom{p-\epsilon}{p-\alpha+\gamma} \binom{\epsilon}{\alpha-\beta-\gamma} \\ &= \binom{\epsilon}{\alpha} \binom{p}{p-\beta} \end{aligned}$$

and so

$$S_1 \equiv \sum_{\alpha=0}^{p-1} \sum_{\beta=0}^{\alpha-1} \begin{pmatrix} \begin{pmatrix} i & j-1 \\ k-1 & l \end{pmatrix} \\ \begin{pmatrix} w & x-1 \\ y-1 & z \end{pmatrix} \end{pmatrix} \binom{\epsilon}{\alpha} \binom{p-\epsilon}{p-\alpha+\beta} \binom{p}{p-\beta};$$

now if $\beta < 0$ or $\beta \geqslant \alpha$ then $\binom{p-\epsilon}{p-\alpha+\beta}\binom{p}{p-\beta} = 0$, so we have

$$\begin{aligned} \sum_{\beta=0}^{\alpha-1} \binom{p-\epsilon}{p-\alpha+\beta} \binom{p}{p-\beta} &= \sum_{\beta\in\mathbb{Z}} \binom{p-\epsilon}{p-\alpha+\beta} \binom{p}{p-\beta} \\ &= \binom{2p-\epsilon}{2p-\alpha}; \end{aligned}$$

thus

$$S_1 \equiv \sum_{\alpha=0}^{p-1} \begin{pmatrix} \begin{pmatrix} i & j-1 \\ k-1 & l \end{pmatrix} \\ \begin{pmatrix} w & x-1 \\ y-1 & z \end{pmatrix} \end{pmatrix} \binom{\epsilon}{\alpha} \binom{2p-\epsilon}{2p-\alpha} \equiv \begin{pmatrix} \begin{pmatrix} i & j-1 \\ k-1 & l \end{pmatrix} \\ \begin{pmatrix} w & x-1 \\ y-1 & z \end{pmatrix} \end{pmatrix}.$$

Similarly we show that

$$\begin{aligned} S_2 &\equiv \sum_{\alpha=0}^{p-1} \begin{pmatrix} \begin{pmatrix} i & j-1 \\ k-1 & l \end{pmatrix} \\ \begin{pmatrix} w & x-1 \\ y & z \end{pmatrix} \end{pmatrix} \binom{\epsilon}{\alpha} \binom{p-\epsilon}{p-\alpha} \\ &\equiv \begin{pmatrix} \begin{pmatrix} i & j-1 \\ k-1 & l \end{pmatrix} \\ \begin{pmatrix} w & x-1 \\ y & z \end{pmatrix} \end{pmatrix}. \end{aligned}$$

and

$$S_5 \equiv \sum_{\alpha=0}^{p-1} \begin{pmatrix} \begin{pmatrix} i & j-1 \\ k-1 & l \end{pmatrix} \\ \begin{pmatrix} w & x \\ y-1 & z \end{pmatrix} \end{pmatrix} \binom{\epsilon}{\alpha}\binom{p-\epsilon}{p-\alpha}$$

$$\equiv \begin{pmatrix} \begin{pmatrix} i & j-1 \\ k-1 & l \end{pmatrix} \\ \begin{pmatrix} w & x \\ y-1 & z \end{pmatrix} \end{pmatrix}.$$

By further similar arguments, we can show that $S_3 \equiv S_4 \equiv S_7 \equiv S_8 \equiv 0 \pmod p$.

Thus we have

$$\begin{aligned} \sum_{\alpha,\beta,\gamma=0}^{p-1} B_{\alpha,\beta,\gamma} &\equiv \binom{i}{w}\binom{j-1}{x}\binom{k-1}{y}\binom{l}{z} \\ &+ \binom{i}{w}\binom{j-1}{x-1}\binom{k-1}{y-1}\binom{l}{z} \\ &+ \binom{i}{w}\binom{j-1}{x-1}\binom{k-1}{y}\binom{l}{z} \\ &+ \binom{i}{w}\binom{j-1}{x}\binom{k-1}{y-1}\binom{l}{z} \\ &\equiv \binom{i}{w}\binom{j}{x}\binom{k}{y}\binom{l}{z} \end{aligned}$$

as required. □

4.1 The embedding

We embed $S(2,r)$ in $S(2,rp)$ where p is the characteristic of k. Sadly we cannot express $S(2,r)$ as an algebra of the form $eS(2,rp)e$ for an idempotent e; in fact it is a subalgebra (with 1) of such an algebra.

Theorem 4.2. *Let $p = \text{char } k$. There exists an embedding of $S(2,r)$ in $S(2,rp)$ defined on basis elements by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{cases} \begin{pmatrix} pa & pb \\ pc & pd \end{pmatrix} & \text{(if } b \text{ or } c \text{ equals } 0\text{)} \\ \sum_{\epsilon=0}^{p-1} \begin{pmatrix} pa+\epsilon & pb-\epsilon \\ pc-\epsilon & pd+\epsilon \end{pmatrix} & \text{(otherwise).} \end{cases}$$

Proof. We need to check that the multiplication rule is preserved, i.e. that the coefficient of $\begin{pmatrix} pi & pj \\ pk & pl \end{pmatrix}$ in

$$\left(\sum_{\zeta=0}^{p-1} \begin{pmatrix} pa+\zeta & pb-\zeta \\ pc-\zeta & pd+\zeta \end{pmatrix}\right) \circ \left(\sum_{\eta=0}^{p-1} \begin{pmatrix} pe+\eta & pf-\eta \\ pg-\eta & ph+\eta \end{pmatrix}\right)$$

is the same as that of $\begin{pmatrix} i & j \\ k & l \end{pmatrix}$ in $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, and that if j and k are positive then the coefficient of $\begin{pmatrix} pi+\epsilon & pj-\epsilon \\ pk-\epsilon & pl+\epsilon \end{pmatrix}$ in the above product is the same as well, for $\epsilon < p$. Note that using the above product remains valid even if one of b, c, f, g is zero, for we may extend our definition of the product $\circ$ to matrices with negative entries, and it will always give zero.

Set $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, $A_\zeta = \begin{pmatrix} pa+\zeta & pb-\zeta \\ pc-\zeta & pd+\zeta \end{pmatrix}$, $B_\eta = \begin{pmatrix} pe+\eta & pf-\eta \\ pg-\eta & ph+\eta \end{pmatrix}$. This gives

$$R(A_\zeta, B_\eta) = \{\begin{pmatrix} pw+\alpha & px-\alpha+\zeta \\ py-\alpha+\eta & pz+\alpha-\zeta-\eta \end{pmatrix} | \begin{pmatrix} w & x \\ y & z \end{pmatrix} \in R(A,B), 0 \leqslant \alpha \leqslant p-1\}.$$

The result now follows by taking Lemma 4.1 and summing over $\begin{pmatrix} w & x \\ y & z \end{pmatrix} \in R(A,B)$. □

Remark. By putting

$$e = \sum_{i-0}^{r} \begin{pmatrix} pi & 0 \\ 0 & p(r-i) \end{pmatrix}$$

we obtain an idempotent in $S(2, rp)$ and hence an algebra $eS(2, rp)e$; $S(2, r)$ is a subalgebra of this, with identity e.

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