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Spin characters of the symmetric group which are proportional to linear characters in characteristic 2

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Abstract

For a finite group, it is interesting to determine when two ordinary irreducible representations have the same p -modular reduction; that is, when two rows of the decomposition matrix in characteristic p are equal, or equivalently when the corresponding p -modular Brauer characters are the same. We complete this task for the double covers of the symmetric group when $p = 2$, by determining when the 2-modular reduction of an irreducible spin representation coincides with a 2-modular Specht module. In fact, we obtain a more general result: we determine when an irreducible spin representation has 2-modular Brauer character proportional to that of a Specht module. In the course of the proof, we use induction and restriction functors to construct a function on generalised characters which has the effect of swapping runners in abacus displays for the labelling partitions.

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1 Introduction

When an irreducible spin representation of the symmetric group is reduced modulo 2, it becomes a representation of the symmetric group in the usual sense, and it is an interesting problem to say exactly which representation is obtained. In this paper, we determine when the result is “a multiple of a Specht module”, in the sense of their images in the Grothendieck group or equivalently of their Brauer characters. That is, we determine when the 2-modular reduction of an irreducible spin representation has Brauer character which is a multiple of that of a Specht module.

We show that the following pair of conditions on the labelling partition are necessary and sufficient for a spin representation to have a 2-modular Brauer character proportional to that of a Specht module. For a strict partition α , we say that:

- ◇ α is 4-stepped if for every part $\alpha_r > 4$ the integer $\alpha_r - 4$ is also a part of α ; and
- ◇ α is 4-semicongruent if the odd parts of α are congruent modulo 4.

If α satisfies both conditions, we abbreviate this by saying that α is 4-stepped-and-semicongruent.

To obtain the partition labelling the corresponding Specht module, observe that if α is 4-stepped-and-semicongruent then there are unique (up to reordering) 2-core partitions σ and τ such that the even parts of α comprise the partition $2(\sigma + \tau)$ (see §1.2 below for details). We define α_\circ to be the partition of $|\alpha|$ (unique up to conjugation) whose 2-quotient has the form (σ, τ) .

We denote by $\chi(\lambda)$ the character of the Specht module labelled by a partition λ , and denote by $\langle \alpha \rangle$ the character of an irreducible spin representation labelled by a strict partition α (when α has an odd number of even parts, there is actually an associate pair of characters $\langle \alpha \rangle_+$ and $\langle \alpha \rangle_-$ labelled by α , but they have the same 2-modular reduction, so we write $\langle \alpha \rangle$ to denote either of them). For any ordinary character φ , we write $\bar{\varphi}$ for the Brauer character of the 2-modular reduction of the corresponding representation.

Our main result is the following.

Theorem 1.1. *Let λ be a partition of n and let α be a strict partition of n . Then $\overline{\langle \alpha \rangle}$ is proportional to $\overline{\chi(\lambda)}$ if and only if α is 4-stepped-and-semicongruent and $\lambda \in \{\alpha_\circ, \alpha'_\circ\}$. In this case, $\overline{\langle \alpha \rangle} = 2^{\lfloor e/2 \rfloor} \overline{\chi(\lambda)}$, where e is the number of even parts of α .*

Equality of Brauer characters occurs when $e \in \{0, 1\}$, giving the following corollary. The *double* of a strict partition α , denoted $\text{dbl}(\alpha)$, is the partition obtained by replacing every odd part $2k + 1$ with parts $k + 1$ and k , and every even part $2k$ with two parts equal to k . Cores and quotients are defined in §2.5 and §2.6. Given partitions μ and ν , we write $\mu \sqcup \nu$ for the partition obtained by combining all parts of μ and ν and reordering as necessary.

Corollary 1.2. *Let λ be a partition of n and let α be a strict partition of n . Then $\overline{\langle \alpha \rangle} = \overline{\chi(\lambda)}$ if and only if:*

- ◇ $\alpha = \kappa \sqcup \eta$, where κ is of the form $(\dots, 9, 5, 1)$ or $(\dots, 11, 7, 3)$ and $\eta \in \{\emptyset, (2), (4)\}$, and
- ◇ λ has 2-core $\text{dbl}(\kappa)$ and 2-quotient (σ, τ) , where $\sigma, \tau \in \{\emptyset, (1)\}$ and $2(\sigma_1 + \tau_1) = \eta_1$.

Corollary 1.2 implies in particular that equality of Brauer characters can occur only when the characters lie in blocks of weight 0, 1 or 2. In the cases of weights 0 and 1, the equality of Brauer characters in fact implies isomorphism of representations, since in this case the modular reductions remain simple.

We wish to highlight the “runner-swapping function” (Definition 5.4) we introduce in the course of the proof as being of independent interest. This function acts (up to a sign) on Specht modules by swapping a pair of runners in an abacus display for the labelling partition, or equivalently by adding all addable nodes and removing all removable nodes of a particular residue in the Young diagram (see Example 5.7, Theorem 6.1 and Remark 6.7). This action can be understood as the affine Weyl group acting on the set of partitions (see [Fay14]). Our approach exhibits this action as a combination of induction and restriction functors. Our expression is new; although expressing the action via induction and restriction functors has been achieved before by [CR08], we believe our expression is simpler, and allows us to also compute its action also on spin representations.

1.1 Discussion on main problem

Fix a prime p . For an ordinary character χ of a double cover $\hat{\mathfrak{S}}_n$ of the symmetric group, the Brauer character $\bar{\chi}$ of the p -modular reduction of the representation affording χ can be found by restricting χ to the p -regular conjugacy classes of $\hat{\mathfrak{S}}_n$. We are interested in the question of when two ordinary irreducible characters $\chi \neq \psi$ of $\hat{\mathfrak{S}}_n$ can give $\bar{\chi} = \bar{\psi}$, or equivalently whether an ordinary irreducible character is determined by its values on p -regular conjugacy classes.

Our question naturally splits into three cases, two of which are (almost) already answered. We call a representation or character of $\hat{\mathfrak{S}}_n$ *spin* if the central element of order 2 acts non-trivially, and call it *linear* otherwise (see §2.3) (unfortunately the term *linear character* can also be used to mean a character of degree 1; we will not use it in that sense in this paper).

- ◇ The case where χ and ψ are both linear characters reduces to the corresponding question for the symmetric group, which was answered by Wildon [Wil08, Theorem 1.1.1]: if $\lambda, \mu \in \mathcal{P}(n)$ are distinct, then $\overline{\chi(\lambda)} = \overline{\chi(\mu)}$ if and only if $p = 2$ and $\mu = \lambda'$ is the conjugate of λ (see §2.1.2).
- ◇ In the case where χ and ψ are both spin characters and $p \neq 3$, this question has been answered by the second author [McD24, Theorem 1.2]: $\bar{\chi}$ and $\bar{\psi}$ are equal if and only if χ and ψ are an associate pair $\langle \alpha \rangle_+$ and $\langle \alpha \rangle_-$, where α is a strict partition with an odd number of even parts and a part divisible by p .

It remains to consider the case where exactly one of χ and ψ is a spin character. In this case, if p is odd, it is never the case that $\bar{\chi} = \bar{\psi}$: the composition factors of a p -modular reduction of an ordinary spin representation are always spin representations, so (the modules affording) $\bar{\chi}$ and $\bar{\psi}$ will have no composition factors in common. Thus we restrict attention to the case $p = 2$ from now on.

It will be helpful to interpret our main theorem also in terms of decomposition numbers (see §2.3 for details). It is a standard fact from modular representation theory that if χ and ψ are ordinary characters, then $\bar{\chi} = \bar{\psi}$ if and only if $[\chi : \varphi] = [\psi : \varphi]$ for all modular irreducible characters φ . Thus our question can be phrased as asking when two rows of the decomposition matrix are equal.

Example 1.3. The decomposition matrix of $\hat{\mathfrak{S}}_4$ in characteristic 2 is given as follows, with rows corresponding to ordinary characters and columns to 2-modular irreducibles.

	(4)	$(3,1)$
$\chi(4)$	1	·
$\chi(3,1)$	1	1
$\chi(2^2)$	·	1
$\chi(2,1^2)$	1	1
$\chi(1^4)$	1	·
$\langle 4 \rangle_+$	·	1
$\langle 4 \rangle_-$	·	1
$\langle 3,1 \rangle$	2	1

We can see that if χ and ψ are distinct ordinary irreducible characters, then $\bar{\chi} = \bar{\psi}$ if and only if $\{\chi, \psi\} = \{\chi(4), \chi(1^4)\}$, $\{\chi, \psi\} = \{\chi(3,1), \chi(2,1^2)\}$, or $\{\chi, \psi\} \subset \{\chi(2^2), \langle 4 \rangle_+, \langle 4 \rangle_-\}$. Of these:

- (i) the pairs of linear characters $\{\chi(4), \chi(1^4)\}$ and $\{\chi(3,1), \chi(2,1^2)\}$ are conjugate pairs, as predicted by [Wil08];
- (ii) the pair of spin characters $\{\langle 4 \rangle_+, \langle 4 \rangle_-\}$ is an associate pair labelled by a partition with a part divisible by 2, as predicted by [McD24];
- (iii) the equality of $\overline{\chi(2^2)}$ with $\overline{\langle 4 \rangle_+}$ and $\overline{\langle 4 \rangle_-}$ is predicted by our main theorem.

In fact, it turns out to be rare to have $\bar{\chi} = \bar{\psi}$ when only one of χ and ψ is a spin character, because the spin characters typically have much larger degrees than the characters of the Specht modules. So we address a more general question in this paper: when are $\bar{\chi}$ and $\bar{\psi}$ proportional to each other? In other words, when is one row of the decomposition matrix proportional to another? Given two characters χ and ψ whose 2-modular reductions are proportional, we can easily find the constant of proportionality using the regularisation theorems (see §2.5.4 and Lemma 3.1), and therefore extract the answer to our original question. Working with proportionality will actually make many of our calculations easier, because we will be able to neglect constants occurring.

Having broadened our scope to proportionality, one may ask whether p -modular reductions of pairs of linear characters, or of pairs of spin characters, can be proportional. The answer, except for pairs of spin characters when $p = 3$, is “no”: by a straightforward extension of their arguments, the results cited above that classify equality between such pairs also rule out non-trivial proportionality. Thus our result is in fact a characterisation of all proportional pairs of p -modular characters of $\hat{\mathfrak{S}}_n$ for $p \neq 3$. For $p = 3$, it is possible for the p -modular reductions of irreducible spin characters to be proportional but unequal: for example, $\overline{\langle 6,3,2 \rangle} = 2\overline{\langle 8,2,1 \rangle}$.

Our main theorem claims that $\overline{\langle 12, 8, 7, 4, 3, 2 \rangle} = 4\chi(\overline{12, 9, 6, 3^2, 1^3}) = 4\chi(\overline{8, 5^2, 3^3, 2^3, 1^3})$.

1.3 Structure of paper

Relevant background is collected in §2. The proofs of the “if” and “only if” directions of our main theorem are logically independent, and can be read in either order.

The proof of the “only if” direction (that is, that a spin character proportional to a linear one modulo 2 must be labelled by a 4-stepped-and-semicongruent strict partition) is contained in §3 and §4. The proof is by induction, using Robinson’s induction and restriction functors and the Murnaghan–Nakayama–Morris rules to constrain the partition labelling a proportional character.

The proof of the “if” direction (that is, that 4-stepped-and-semicongruent strict partitions do label spin characters proportional to linear ones modulo 2) is contained in §5–7. We first show the claim for *homogeneous* spin characters, then use certain combinations of Robinson’s induction and restriction functors to propagate the property of being proportional.

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2 Background

In this paper, \mathbb{N} denotes the set of positive integers. If $\epsilon \in \{0, 1\}$, we write $\bar{\epsilon} = 1 - \epsilon$.

2.1 Partitions and Young diagrams

We recall some simple combinatorial concepts relating to representations of symmetric groups.

2.1.1 Partitions

A *partition* is an infinite weakly-decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of integers which are eventually zero. The integers $\lambda_1, \lambda_2, \dots$ are the *parts* of λ . Given a partition λ and a natural number a , we write $a \in \lambda$ to mean that a is a part of λ .

We write $|\lambda|$ for the sum $\lambda_1 + \lambda_2 + \dots$, and say that λ is a partition of $|\lambda|$. We write \mathcal{P} for the set of all partitions, and $\mathcal{P}(n)$ for the set of partitions of n . When writing partitions, we omit trailing zeroes and group together equal parts with a superscript. We write \emptyset for the unique partition of 0. For any partition λ , we write $l(\lambda)$ for the number of non-zero parts of λ , which we call the *length* of λ .

We use some natural notation for combining partitions. Given partitions λ and μ , we write:

- ◊ $a\lambda = (a\lambda_1, a\lambda_2, \dots)$ for $a \in \mathbb{N}$;
- ◊ $\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$;
- ◊ $\lambda \sqcup \mu$ for the partition obtained by combining all the parts of λ and μ and writing them in decreasing order;
- ◊ $\lambda \setminus \mu$ for the partition obtained by retaining only those parts of λ which are not parts of μ .

A *bipartition* is an ordered pair $\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)})$ of partitions, which we call the *components* of $\underline{\lambda}$. We write $|\underline{\lambda}| = |\lambda^{(0)}| + |\lambda^{(1)}|$, and say that $\underline{\lambda}$ is a bipartition of $|\underline{\lambda}|$.

2.1.2 Young diagrams

The *Young diagram* of a partition λ is the set

$$[\lambda] = \{ (r, c) \in \mathbb{N}^2 \mid c \leq \lambda_r \},$$

drawn in the plane using the English convention. We call the elements of the Young diagram the *nodes* of λ . In general, a node means an element of \mathbb{N}^2 . We may not distinguish between a partition and its Young diagram; for example, given partitions λ and μ , we write:

- ◇ $\lambda \cap \mu$ for the partition with Young diagram the intersection of the Young diagrams of λ and μ ;
- ◇ $\mu \subseteq \lambda$ to mean that every node of μ is also a node of λ ;
- ◇ λ / μ for the set of nodes of λ which are not nodes of μ , provided that $\mu \subseteq \lambda$.

The *conjugate* of a partition λ , denoted λ' , is the partition whose Young diagram is obtained by reflecting that of λ in the main diagonal. Thus the parts of λ' are the lengths of the columns of λ .

A node (r, c) of λ is *removable* if it can be removed from $[\lambda]$ to leave the Young diagram of a smaller partition (that is, if $c = \lambda_r > \lambda_{r+1}$), while a node not in λ is an *addable* node of λ if it can be added to $[\lambda]$ to produce a Young diagram of a larger partition. The *rim* of a partition λ is the set of nodes $(r, c) \in [\lambda]$ such that $(r+1, c+1) \notin [\lambda]$. All removable nodes lie in the rim, but not all nodes in the rim are removable. We denote by $\lambda^{-\epsilon}$ the partition obtained by removing all removable ϵ -nodes.

If $\lambda, \mu \in \mathcal{P}(n)$, then we say that λ *dominates* μ (and write $\lambda \triangleright \mu$) if

$$\lambda_1 + \cdots + \lambda_r \geq \mu_1 + \cdots + \mu_r$$

for every $r \geq 0$. Another way of saying this is that $[\mu]$ can be obtained from $[\lambda]$ by moving some nodes to lower positions.

2.1.3 Residues, diagonals and ladders

The *residue* of a node (r, c) is the residue of $c - r$ modulo 2. For $\epsilon \in \{0, 1\}$, an ϵ -*node* means a node of residue ϵ .

For d in \mathbb{Z} , the d th *diagonal* in \mathbb{N}^2 is the set of all nodes (r, c) for which $c - r = d$ (that is, a top-left to bottom-right line of nodes of constant residue). We write \mathcal{D}_d for the d th diagonal.

For $l \in \mathbb{N}$, the l th *ladder* in \mathbb{N}^2 is the set of all nodes (r, c) for which $r + c = l + 1$ (that is, a bottom-left to top-right line of nodes of constant residue). We write \mathcal{L}_l for the l th ladder, and say that \mathcal{L}_m is *larger* than \mathcal{L}_l when $m > l$.

If $\lambda \in \mathcal{P}$, the *2-regularisation* of λ is the partition obtained by sliding the nodes of λ as far up their ladders as they will go. We write this partition as λ^{reg} . It is easy to see that if $\lambda, \mu \in \mathcal{P}$ then $\lambda^{\text{reg}} = \mu^{\text{reg}}$ if and only if $|\lambda \cap \mathcal{L}_l| = |\mu \cap \mathcal{L}_l|$ for every l .

2.2 Strict partitions

Now we outline the combinatorial notions relating to spin representations of the double covers of the symmetric group.

A *strict partition* is a partition α such that $\alpha_i > \alpha_{i+1}$ for all $i \leq l(\alpha)$. We write \mathcal{D} for the set of all strict partitions, and $\mathcal{D}(n)$ for the set of strict partitions of n .

As in the introduction, the *double* of a strict partition α is the partition

$$\text{dbl}(\alpha) = \left(\left\lceil \frac{\alpha_1}{2} \right\rceil, \left\lfloor \frac{\alpha_1}{2} \right\rfloor, \left\lceil \frac{\alpha_2}{2} \right\rceil, \left\lfloor \frac{\alpha_2}{2} \right\rfloor, \dots \right).$$

The *spin-residue* of a node (r, c) is the residue of $\lfloor c/2 \rfloor$ modulo 2. For $\epsilon \in \{0, 1\}$, an ϵ -*spin-node* means a node of spin-residue ϵ .

A node (r, c) of α is *spin-removable* if it can be removed, possibly together with other nodes of the same spin-residue, to leave a strict partition. A node (r, c) which is not a node of α is *spin-addable* if it can be added, possibly together with other nodes of the same spin-residue, to obtain a strict partition. An ϵ -*spin-removable node* means a spin-removable ϵ -spin-node, and likewise with “addable” in place of “removable”. We denote by $\alpha^{-\epsilon}$ the strict partition obtained by removing from α all ϵ -spin-removable nodes.

For example, let $\alpha = (7, 5, 4, 1)$. The spin-residues of the nodes of α are illustrated in the following diagram.

0	1	1	0	0	1	1
0	1	1	0	0		
0	1	1	0			
0						

The 0-spin-removable nodes are $(2, 5)$, $(3, 4)$ and $(4, 1)$, and $\alpha^{-0} = (7, 4, 3)$; the 0-spin-addable nodes are $(1, 8)$ and $(1, 9)$.

2.3 Irreducible characters of the symmetric group and its double cover

For $n \geq 1$, let $\hat{\mathfrak{S}}_n$ denote the group with generators t_1, \dots, t_{n-1}, z and defining relations

$$\begin{aligned}
 t_1^2 &= \dots = t_{n-1}^2 = z^2 = 1, \\
 t_i z &= z t_i && \text{for } 1 \leq i \leq n-1, \\
 t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1} && \text{for } 1 \leq i \leq n-2, \\
 t_i t_j &= z t_j t_i && \text{for } 1 \leq i \leq j-1 \leq n-2.
 \end{aligned}$$

Then $\hat{\mathfrak{S}}_n$ is a double cover of the symmetric group \mathfrak{S}_n , and is a Schur cover of \mathfrak{S}_n provided $n \geq 4$. The ordinary representation theory of $\hat{\mathfrak{S}}_n$ is described in detail in the book by Hoffman and Humphreys [HH92], and the modular theory is addressed in Kleshchev's book [Kle05]. We will be concerned with representations in characteristic 0 and characteristic 2. (We remark that there is another double cover of \mathfrak{S}_n , obtained by replacing the relations $t_i^2 = 1$ with $t_i^2 = z$; our results apply equally to either double cover.)

The element z is a central involution, so acts as a scalar ± 1 on any irreducible representation (over any field). The irreducible representations on which z acts as 1 are precisely the lifts of the irreducible representations of \mathfrak{S}_n . Irreducible representations on which z does not act as the identity are called *spin representations*, and (in characteristic 0) their characters are called *spin characters*. The irreducible characters which are not spin characters are called *linear characters*. We write $(:)$ for the usual inner product on ordinary characters, extended bilinearly to generalised characters.

The classification of ordinary irreducible representations of \mathfrak{S}_n is very well known. For $\lambda \in \mathcal{P}(n)$, let S^λ denote the *Specht module* defined over a field of characteristic 0 (see [Jam78a, Section 4], for example), and let $\chi(\lambda)$ denote its character. Then the set $\{\chi(\lambda) \mid \lambda \in \mathcal{P}(n)\}$ is a complete irredundant list of ordinary irreducible characters of \mathfrak{S}_n . We will treat S^λ as a module for $\hat{\mathfrak{S}}_n$ by letting z act as the identity, and correspondingly treat $\chi(\lambda)$ as a character for $\hat{\mathfrak{S}}_n$.

The classification of ordinary irreducible spin characters of $\hat{\mathfrak{S}}_n$ goes back to Schur [Sch11] (although, remarkably, a construction for the corresponding representations was not found until 1990, by Nazarov [Naz90]). Let $\alpha \in \mathcal{D}(n)$, and say that α is *even* or *odd* as the number of positive even parts of α is even or odd. If α is even, then Schur constructs an irreducible character $\langle \alpha \rangle$ corresponding to α . If α is odd, then Schur gives two characters $\langle \alpha \rangle_+$ and $\langle \alpha \rangle_-$ corresponding to α . These characters together give a complete irredundant list of ordinary irreducible spin characters of $\hat{\mathfrak{S}}_n$.

For studying 2-modular reductions, it will be helpful to adopt some notation introduced in [Fay20] for spin characters. For any $\alpha \in \mathcal{D}$, we define the generalised character

$$\langle\langle \alpha \rangle\rangle = \begin{cases} \langle \alpha \rangle & \text{if } \alpha \text{ is even,} \\ \frac{1}{\sqrt{2}}(\langle \alpha \rangle_+ + \langle \alpha \rangle_-) & \text{if } \alpha \text{ is odd.} \end{cases}$$

Using the generalised characters $\langle\langle \alpha \rangle\rangle$ simplifies several results for spin representations, such as Schur's degree formula [Sch11], the Bessenrodt–Olsson regularisation theorem (Theorem 2.4 below), and the branching rule (Theorem 2.6 below). Since the 2-modular reductions of $\langle \alpha \rangle_+$ and $\langle \alpha \rangle_-$ are equal (as they differ only on elements of even order), and since we are concerned only with proportionality, there is no harm in working with these generalised characters $\langle\langle \alpha \rangle\rangle$.

We also need to recall the classification of irreducible $\hat{\mathfrak{S}}_n$ -modules in characteristic 2. Obviously in characteristic 2 there are no irreducible spin representations, so the irreducible representations of $\hat{\mathfrak{S}}_n$ are precisely the lifts of the irreducible representations of \mathfrak{S}_n . These are afforded by the *James modules* D^μ for $\mu \in \mathcal{D}$: the James module D^μ can be constructed as the unique simple quotient of the Specht module S^μ . We write $\varphi(\mu)$ for the Brauer character of D^μ , and for an ordinary character χ , we define $[\chi : \varphi(\mu)]$ to be the multiplicity of $\varphi(\mu)$ as a composition factor of a 2-modular reduction of $\bar{\chi}$. These multiplicities, as χ ranges over the ordinary irreducible characters, are the *decomposition numbers* for $\hat{\mathfrak{S}}_n$.

2.4 Hooks, bars and character values

In this section we recall the Murnaghan–Nakayama rule for computing character values for the symmetric group, and its spin version, proved by Morris. In fact for our purposes we can make do with very simplified versions of these rules.

2.4.1 Hooks and the Murnaghan–Nakayama rule

We begin by recalling the well-known concept of rim-hooks of partitions. Take $k \in \mathbb{N}$ and λ a partition. A *k-rim-hook*, or rim-hook of length k , is a connected subset of the rim of λ comprising k nodes which can be removed to leave the Young diagram of a smaller partition. For a rim-hook δ in λ , the smaller partition obtained by removing δ is denoted $\lambda \setminus \delta$.

Let $R(\lambda, k)$ be the set of partitions that can be obtained by removing a k -rim-hook from λ . Given $\nu \in \mathcal{P}(n)$, let $\chi(\lambda)(\nu)$ be the value of the character $\chi(\lambda)$ on elements of \mathfrak{S}_n of cycle type ν . Now a simplified version of the Murnaghan–Nakayama rule (see for example [Jam78a, 21.1]) can be stated as follows.

Theorem 2.1. *Suppose $\lambda \in \mathcal{P}(n)$ and $k \in \mathbb{N}$. For each $\mu \in R(\lambda, k)$ there is a non-zero constant c_μ such that for any $\nu \in \mathcal{P}(n - k)$,*

$$\chi(\lambda)(\nu \sqcup (k)) = \sum_{\mu \in R(\lambda, k)} c_\mu \chi(\mu)(\nu).$$

2.4.2 Odd bars and Morris's rule

Now we consider spin characters, and explain Morris's rule (as given by Hoffman and Humphreys [HH92, Theorem 10.1]). Because we are concerned with 2-modular Brauer characters in this paper, we need only consider 2-regular conjugacy classes in $\hat{\mathfrak{S}}_n$. In fact these are in one-to-one correspondence with 2-regular conjugacy classes in \mathfrak{S}_n : for each 2-regular conjugacy class C in \mathfrak{S}_n , there are two conjugacy classes in $\hat{\mathfrak{S}}_n$ of elements that map to C under the quotient map $\hat{\mathfrak{S}}_n \rightarrow \mathfrak{S}_n$, and only one of these consists of elements of odd order. So 2-regular conjugacy classes in $\hat{\mathfrak{S}}_n$ can be labelled

by partitions of n into odd parts. Given $\alpha \in \mathcal{D}$ and a partition ν into odd parts, we write $\langle\!\langle \alpha \rangle\!\rangle(\nu)$ for the value of $\langle\!\langle \alpha \rangle\!\rangle$ on the 2-regular conjugacy class corresponding to ν .

The appropriate analogue of rim-hooks for strict partitions is *bars*. Take $k \in \mathbb{N}$ odd and $\alpha \in \mathcal{D}$. A k -bar in α is one of the following:

- ◊ a part of size equal to k ;
- ◊ a pair of parts whose sizes sum to k ;
- ◊ the last k nodes in a part of size $\alpha_r > k$ such that $\alpha_r - k$ is not already a part of α .

Removing a k -bar from α means removing the bar and, if necessary, reordering the remaining parts to form a strict partition.

Let $S(\alpha, k)$ denote the set of strict partitions that can be obtained from α by removing a k -bar. Now a simplified version of Morris's rule can be stated as follows.

Theorem 2.2. *Suppose $\alpha \in \mathcal{D}(n)$ and $k \in \mathbb{N}$ is odd. For each $\beta \in S(\alpha, k)$ there is a non-zero constant d_β such that for any partition ν of $n - k$ into odd parts,*

$$\langle\!\langle \alpha \rangle\!\rangle(\nu \sqcup (k)) = \sum_{\beta \in S(\alpha, k)} d_\beta \langle\!\langle \beta \rangle\!\rangle(\nu).$$

2.5 Blocks

The 2-block structure of $\hat{\mathfrak{S}}_n$ was determined by Bessenrodt and Olsson [BO97], extending the classical result of Brauer and Robinson (the “Nakayama Conjecture”) for \mathfrak{S}_n . We summarise the essential points here.

2.5.1 Cores and weights

Suppose $\lambda \in \mathcal{P}$ and $k \in \mathbb{N}$. The k -core of λ is the partition obtained by successively removing k -rim-hooks until no more remain. It is well-known that this partition is independent of the choice of k -rim-hook removed at each stage. The k -weight of λ is the number of k -rim-hooks removed to reach the k -core, or equivalently the number of rim-hooks of λ of size divisible by k . By the Murnaghan–Nakayama rule (Theorem 2.1), the k -weight of λ is the maximum w such that the character value $\chi(\lambda)(k^w, 1^{n-kw})$ is non-zero.

When $k = 2$, it is easy to see that the 2-cores are precisely the partitions of the form $(r, r - 1, \dots, 1)$ for $r \geq 0$. In this paper we will write $\kappa_r = (r, r - 1, \dots, 1)$.

The Brauer–Robinson Theorem (which applies equally well for $\hat{\mathfrak{S}}_n$) says that $\chi(\lambda)$ and $\chi(\mu)$ lie in the same k -block if and only if λ and μ have the same k -core. This result also gives the distribution of the irreducible Brauer characters $\varphi(\mu)$ into blocks, since $\varphi(\mu)$ is a composition factor of $\overline{\chi(\mu)}$.

This means that we can define the k -core and k -weight of a block of $\hat{\mathfrak{S}}_n$ to be the common k -core and k -weight of the partitions λ labelling characters $\chi(\lambda)$ in that block.

2.5.2 Bar-cores and bar-weights

Now we come to spin characters. For $\alpha \in \mathcal{D}$ and k odd, the k -bar-core of α is the strict partition obtained by successively removing k -bars until none remain, and the k -bar-weight of α is the number of k -bars removed to reach the k -bar-core. By Morris's rule (Theorem 2.2), the k -bar-weight of α is the maximum w such that the character value $\langle\!\langle \alpha \rangle\!\rangle(k^w, 1^{n-kw})$ is non-zero.

To work in characteristic 2, we require a slightly different definition: for $\alpha \in \mathcal{D}$, the 4-bar-core of α is obtained by repeatedly applying moves of the following types:

- ◇ deleting an even part;
- ◇ deleting two parts whose sum is a multiple of 4;
- ◇ replacing an odd part $\alpha_r > 4$ with $\alpha_r - 4$, if $\alpha_r - 4$ is not already a part of α .

(This permits removing parts of size equal to 2 in addition to “4-bars” defined analogously to the odd case.) The 4-bar-weight of α is the total number of nodes removed in this way divided by 2.

The possible 4-bar-cores are the partitions $(4t - 1, 4t - 5, \dots, 3)$ and $(4t + 1, 4t - 3, \dots, 1)$ for $t \geq 0$. For $r \geq 1$, we write $\bar{\kappa}_r$ for the 4-bar-core with largest part $2r - 1$, and we set $\bar{\kappa}_0 = \emptyset$. Note that the function $\tau \mapsto \text{dbl}(\tau)$ gives a bijection from the set of 4-bar-cores to the set of 2-cores, sending $\bar{\kappa}_r$ to κ_r for every r . (The inverse of this bijection is given by $\nu \mapsto \text{dbl}(\nu)'$.)

Now we can describe the distribution of spin characters into blocks [BO97, Theorem 4.1]: suppose $\alpha \in \mathcal{D}$, and let γ be the 4-bar-core of α ; then the spin character $\langle\!\langle \alpha \rangle\!\rangle$ lies in the 2-block of $\hat{\mathfrak{S}}_n$ with 2-core $\text{dbl}(\gamma)$.

2.5.3 RoCK blocks

Say that the block with 2-core $\kappa_a = (a, a - 1, \dots, 1)$ and 2-weight w is *RoCK* if $a \geq w - 1$. RoCK blocks, which are often also called *Rouquier blocks*, are particularly well understood, and part of the proof of our main theorem will involve reducing to the case of RoCK blocks.

2.5.4 Regularisation theorems

A very useful tool in the study of decomposition numbers for the symmetric groups is James’s regularisation theorem [Jam76, Theorem A]. For the double cover $\hat{\mathfrak{S}}_n$ in characteristic 2, a useful analogue was proved by Bessenrodt and Olsson [BO97, Theorem 5.2]. These results are as follows, where the double of a strict partition is as defined in §2.2 and the regularisation of a partition is as defined in §2.1.3.

Theorem 2.3. *Suppose $\lambda \in \mathcal{P}$. Then $[\chi(\lambda) : \varphi(\lambda^{\text{reg}})] = 1$, while $[\chi(\lambda) : \varphi(\mu)] = 0$ unless $\mu \triangleright \lambda^{\text{reg}}$.*

Theorem 2.4. *Suppose $\alpha \in \mathcal{D}$, and let e be the number of even parts of α . Then $[\langle\!\langle \alpha \rangle\!\rangle : \varphi(\text{dbl}(\alpha)^{\text{reg}})] = 2^{e/2}$, while $[\langle\!\langle \alpha \rangle\!\rangle : \varphi(\mu)] = 0$ unless $\mu \triangleright \text{dbl}(\alpha)^{\text{reg}}$.*

2.5.5 Content and spin-content

The block classification can alternatively be expressed in terms of residues of nodes. Define the *content* of a partition λ to be the multiset of residues of its nodes. It was proved by Littlewood that two partitions of n have the same 2-core if and only if they have the same content [Lit51, p. 347]. So given a block of $\hat{\mathfrak{S}}_n$, we may define its content to be the common contents of the partitions labelling characters $\chi(\lambda)$ in the block.

To extend this idea to spin characters, define the *spin-content* of a strict partition α to be the multiset of spin-residues of its nodes. Then $\langle\!\langle \alpha \rangle\!\rangle$ lies in the block whose content coincides with the spin-content of α (as can be shown, for example, by observing that the spin-content of a partition equals the content of its double [BO97, Lemma 3.9] and using Theorem 2.4 above).

2.6 Abacus displays and 2-quotients

Given $\lambda \in \mathcal{P}$, take an integer $r \geq l(\lambda)$, and define the *beta-numbers* $\beta_i = \lambda_i + r - i$ for $i = 1, \dots, r$. Take an abacus with two vertical runners; label these runners 0 and 1 from left to right. Mark positions $0, 2, 4, \dots$ from the top down on runner 0, and $1, 3, 5, \dots$ from the top down on runner 1. Now

place r beads on the abacus, at positions β_1, \dots, β_r . The resulting configuration is the r -bead abacus display for λ .

The 2-quotient of a partition λ is a bipartition constructed from an abacus display for λ by viewing each runner in isolation as a 1-runner abacus and reading off the corresponding partitions. This requires a choice of convention with regard to the number of beads used in the abacus display and which runner corresponds to which component of the bipartition. In this paper we follow a slightly unusual convention: we choose the number of beads r so that runner 1 has at least as many beads as runner 0 (or equivalently so that the number of beads is congruent modulo 2 to the length of the 2-core of λ), and then define the ϵ th component of the 2-quotient from the ϵ th runner (explicitly, the 2-quotient is $(\lambda^{(0)}, \lambda^{(1)})$ where $\lambda_i^{(\epsilon)}$ is the number of empty spaces above the i th lowest bead on runner ϵ). This can alternatively be described as follows: given any r -bead abacus display, let the 1st component $\lambda^{(1)}$ correspond to the runner with strictly more beads if such a runner exists, or to runner 1 if the numbers of beads on the runners are equal.

(The more usual convention for the 2-quotient (see for example [JK81, Section 2.7]) is to take r always to be even and let the ϵ th component $\lambda^{(\epsilon)}$ correspond to runner ϵ ; our choice will be more convenient when we consider RoCK partitions. The analogue for odd primes of our convention is used in [Fay16], where it is called the *ordered p -quotient*.)

It is well-known (and easy to see on the abacus) that a partition is defined by its 2-core and 2-quotient. Conversely, given a 2-core κ and a bipartition $\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)})$, there is a partition with 2-core κ and 2-quotient $\underline{\lambda}$. We write this as $[\kappa; (\lambda^{(0)}, \lambda^{(1)})]$.

2.7 Induction and restriction

Induction and restriction between symmetric groups of different sizes is one of the most important tools in the representation theory of these groups, and the same applies to the double covers. Robinson's i -induction and $-$ restriction functors, which give more powerful results in positive characteristic, likewise extend to $\tilde{\mathfrak{S}}_n$. We summarise the results we will need here, following the account in [Fay18], where references and further details can be found.

Given a character χ for $\tilde{\mathfrak{S}}_n$, we write $\chi \downarrow_{\tilde{\mathfrak{S}}_{n-1}}$ for its restriction to $\tilde{\mathfrak{S}}_{n-1}$, and $\chi \uparrow^{\tilde{\mathfrak{S}}_{n+1}}$ for the induced character for $\tilde{\mathfrak{S}}_{n+1}$. Now suppose χ lies in a single block, and write the content of this block as $\{0^a, 1^b\}$. Define $e_0\chi$ to be the component of $\chi \downarrow_{\tilde{\mathfrak{S}}_{n-1}}$ lying in the block with content $\{0^{a-1}, 1^b\}$ if there is such a block, and $e_0\chi = 0$ otherwise. Define $f_0\chi$ to be the component of $\chi \uparrow^{\tilde{\mathfrak{S}}_{n+1}}$ lying in the block with content $\{0^{a+1}, 1^b\}$ if there is such a block, or 0 otherwise. Define the characters $e_1\chi$ and $f_1\chi$ similarly. It follows from the branching rules for $\tilde{\mathfrak{S}}_n$ and the block classification that $\chi \downarrow_{\tilde{\mathfrak{S}}_{n-1}} = e_0\chi + e_1\chi$ and $\chi \uparrow^{\tilde{\mathfrak{S}}_{n+1}} = f_0\chi + f_1\chi$ for any χ .

Fix $\epsilon \in \{0, 1\}$ for the rest of this subsection. The functor e_ϵ is defined for any $n > 0$, so we can define the power e_ϵ^r and the divided power $e_\epsilon^{(r)} = e_\epsilon^r / r!$, and similarly for f_ϵ . These functors can be applied either to ordinary characters or 2-modular Brauer characters, and commute with reduction modulo 2.

The classical branching rule for induction and restriction of ordinary irreducible characters of symmetric groups extends to the double covers in characteristic 2, and can be phrased in terms of the functors e_i as follows.

Theorem 2.5. *Suppose $\lambda \in \mathcal{P}(n)$ and $\mu \in \mathcal{P}(n - r)$, and $\epsilon \in \{0, 1\}$. If μ is obtained from λ by removing r ϵ -nodes, then $(e_\epsilon^{(r)}\chi(\lambda) : \chi(\mu)) = (f_\epsilon^{(r)}\chi(\mu) : \chi(\lambda)) = 1$; otherwise $(e_\epsilon^{(r)}\chi(\lambda) : \chi(\mu)) = (f_\epsilon^{(r)}\chi(\mu) : \chi(\lambda)) = 0$.*

The branching rule for spin characters is given as follows (slightly rephrased from [Fay20]).

Theorem 2.6 [Fay20, Proposition 4.11]. Suppose $\alpha \in \mathcal{D}(n)$ and $\beta \in \mathcal{D}(n-r)$, and $\epsilon \in \{0,1\}$. Then $(e_\epsilon^{(r)} \langle \alpha \rangle : \langle \beta \rangle)$ and $(f_\epsilon^{(r)} \langle \beta \rangle : \langle \alpha \rangle)$ are non-zero if and only if β is obtained by removing r ϵ -spin-nodes from α . If this is the case, let b be the number of values $c \geq 2$ such that α/β contains a node in column c but does not contain any nodes in columns $c-1$ and $c+1$. Then

$$(e_\epsilon^{(r)} \langle \alpha \rangle : \langle \beta \rangle) = (f_\epsilon^{(r)} \langle \beta \rangle : \langle \alpha \rangle) = 2^{b/2}.$$

We remark that the integer b in Theorem 2.6 can also be characterised as the number of even integers which are parts of α or β but not both.

3 First results on proportionality

Our first step towards proving the “only if” direction of the main theorem, as well as establishing the claimed constant of proportionality, is to examine the consequences of the regularisation theorems Theorems 2.3 and 2.4. To say that $\overline{\chi(\lambda)}$ is proportional to $\overline{\langle \alpha \rangle}$ is to say that there is a scalar a such that $[\langle \alpha \rangle : \varphi(\mu)] = a[\chi(\lambda) : \varphi(\mu)]$ for every $\mu \in \mathcal{D}$. The regularisation theorems substantially restrict the possible pairs (λ, α) for which this can happen, as well as determining the constant a . The following result is immediate from Theorems 2.3 and 2.4.

Lemma 3.1. Suppose $\lambda \in \mathcal{P}$ and $\alpha \in \mathcal{D}$, with $\overline{\langle \alpha \rangle} \propto \overline{\chi(\lambda)}$. Then:

- (i) $\lambda^{\text{reg}} = \text{dbl}(\alpha)^{\text{reg}}$ (that is, λ can be obtained from $\text{dbl}(\alpha)$ by sliding nodes along the 2-ladders);
- (ii) $\overline{\langle \alpha \rangle} = 2^{e/2} \overline{\chi(\lambda)}$, where e is the number of even parts of α .

Part (ii) of this lemma establishes the constant of proportionality claimed in our main theorem Theorem 1.1.

Next we examine the consequences of the Murnaghan–Nakayama–Morris rules for character values given in §2.4. First we compare k -weight and k -bar-weight.

Lemma 3.2. Let $\lambda \in \mathcal{P}$ and $\alpha \in \mathcal{D}$, with $\overline{\langle \alpha \rangle} \propto \overline{\chi(\lambda)}$. Then for any odd integer k , the k -weight of λ equals the k -bar-weight of α .

Proof. The characters $\langle \alpha \rangle$ and $\chi(\lambda)$ vanish on precisely the same 2-regular classes. In particular, the k -weight of λ and the k -bar-weight of α are both equal to the largest w such that $\langle \alpha \rangle$ and $\chi(\lambda)$ are non-zero on elements of cycle type $(k^w, 1^{n-kw})$. \square

Observe that any strict partition has a unique largest bar of odd length (if all parts are odd, it is the largest part; if all parts are even, it is the largest part minus one node; otherwise, it is the largest odd part together with the largest even part).

Proposition 3.3. Suppose $\alpha \in \mathcal{D}$ has at least one odd part. Let π be the unique largest bar of odd length in α , and let k be its length. If $\lambda \in \mathcal{P}$ with $\overline{\langle \alpha \rangle} \propto \overline{\chi(\lambda)}$, then λ has a unique k -rim-hook δ , and $\overline{\langle \alpha \setminus \pi \rangle} \propto \overline{\chi(\lambda \setminus \delta)}$.

Proof. It is easy to see that $\alpha \setminus \pi$ has no k -bars, since the largest bar of odd length in $\alpha \setminus \pi$ is at most the sum of the largest odd part and largest even part of $\alpha \setminus \pi$ (taking these to be zero if no such parts exist), which is strictly less than the sum of those for α . So α has k -bar-weight 1, and therefore Lemma 3.2 gives that λ has k -weight 1. This implies in particular that λ has a unique rim-hook of length k , which we denote δ . Then $R(\lambda, k) = \{\lambda \setminus \delta\}$, while $S(\alpha, k) = \{\alpha \setminus \pi\}$.

Let a be such that $\overline{\langle\langle\alpha\rangle\rangle} = a\overline{\chi(\lambda)}$. Then Theorems 2.1 and 2.2 give

$$\overline{\langle\langle\alpha \setminus \pi\rangle\rangle}(\nu) = \frac{1}{d_{\alpha \setminus \pi}} \overline{\langle\langle\alpha\rangle\rangle}(\nu \sqcup (k)) = \frac{a}{d_{\alpha \setminus \pi}} \overline{\chi(\lambda)}(\nu \sqcup (k)) = \frac{ac_{\lambda \setminus \delta}}{d_{\alpha \setminus \pi}} \overline{\chi(\lambda \setminus \delta)}(\nu)$$

for any partition ν of $n - k$ into odd parts, and therefore

$$\overline{\langle\langle\alpha \setminus \pi\rangle\rangle} = \frac{ac_{\lambda \setminus \delta}}{d_{\alpha \setminus \pi}} \overline{\chi(\lambda \setminus \delta)}.$$

□

Finally we derive some consequences of the modular branching rules. Recall, given $\lambda \in \mathcal{P}$, $\alpha \in \mathcal{D}$ and $\epsilon \in \{0, 1\}$, that $\lambda^{-\epsilon}$ denotes the partition obtained by removing all the removable ϵ -nodes from λ , and $\alpha^{-\epsilon}$ denotes the strict partition obtained by removing all the ϵ -spin-removable nodes from α .

Proposition 3.4. *Let $\lambda \in \mathcal{P}$ and $\alpha \in \mathcal{D}$ with $\overline{\langle\langle\alpha\rangle\rangle} \propto \overline{\chi(\lambda)}$. Let $\epsilon \in \{0, 1\}$. Then $\overline{\langle\langle\alpha^{-\epsilon}\rangle\rangle} \propto \overline{\chi(\lambda^{-\epsilon})}$.*

Proof. Let r be the number of ϵ -spin-removable nodes of α . Theorem 2.6 shows that r is maximal such that $e_{\epsilon}^{(r)} \overline{\langle\langle\alpha\rangle\rangle} \neq 0$, and so r is also maximal such that $e_{\epsilon}^{(r)} \overline{\chi(\lambda)} \neq 0$. Theorem 2.5 then shows that r is the number of removable ϵ -nodes of λ , and that $e_{\epsilon}^{(r)} \overline{\chi(\lambda)} = \overline{\chi(\lambda^{-\epsilon})}$. At the same time, Theorem 2.6 shows that $e_{\epsilon}^{(r)} \overline{\langle\langle\alpha\rangle\rangle}$ is a non-zero multiple of $\overline{\langle\langle\alpha^{-\epsilon}\rangle\rangle}$. So

$$\overline{\langle\langle\alpha^{-\epsilon}\rangle\rangle} \propto e_{\epsilon}^{(r)} \overline{\langle\langle\alpha\rangle\rangle} \propto e_{\epsilon}^{(r)} \overline{\chi(\lambda)} = \overline{\chi(\lambda^{-\epsilon})}.$$

□

4 Proportional implies 4-stepped-and-semicongruent

Our aim in this section is to prove the following statement. We say that a strict partition α is *proportional* if there exists a partition λ such that $\overline{\langle\langle\alpha\rangle\rangle} \propto \overline{\chi(\lambda)}$; the definition of 4-stepped-and-semicongruent was given at the start of the Introduction.

Theorem 4.1. *Suppose $\alpha \in \mathcal{D}$. If α is proportional, then α is 4-stepped-and-semicongruent.*

This is enough to prove the “only if” claim of Theorem 1.1 when given also the “if” direction (Theorem 5.1). Indeed, if $\overline{\langle\langle\alpha\rangle\rangle}$ is proportional to $\overline{\chi(\lambda)}$, then Theorem 4.1 says that α is 4-stepped-and-semicongruent, and we must furthermore deduce that $\lambda \in \{\alpha_{\circ}, \alpha'_{\circ}\}$; having that α is 4-stepped-and-semicongruent, Theorem 5.1 will tell us that $\overline{\langle\langle\alpha\rangle\rangle}$ is proportional to $\overline{\chi(\alpha_{\circ})}$; since the constants of proportionality are the same (by Lemma 3.1(ii)), we find that $\overline{\chi(\alpha_{\circ})} = \overline{\chi(\lambda)}$. Then a theorem of Wildon [Wil08, Theorem 1.1.1(ii)] gives $\lambda \in \{\alpha_{\circ}, \alpha'_{\circ}\}$.

The remainder of this section comprises the proof of Theorem 4.1. The proof is by induction on the size of α : let α be a proportional strict partition; suppose that all smaller proportional strict partitions are 4-stepped-and-semicongruent; and our goal is to show that α is 4-stepped-and-semicongruent.

We achieve our goal principally by applying the inductive hypothesis to three strict partitions: α^{-0} , α^{-1} and $\alpha \setminus \pi$, where π denotes the unique largest bar of odd length in α . We also make use of requirements on $\text{dbl}(\alpha)^{\text{reg}}$ (Lemma 3.1(i)) and the bar-weights of α (Lemma 3.2).

The strict partitions α^{-0} and α^{-1} are proportional by Proposition 3.4, and so whenever they are smaller than α the inductive hypothesis applies. Meanwhile $\alpha \setminus \pi$ is proportional whenever α contains an odd part by Proposition 3.3 (whilst we quickly deal with the case of α having all parts even in the first subsection below).

Throughout, let λ be a partition such that $\overline{\langle\langle\alpha\rangle\rangle}$ is proportional to $\overline{\chi(\lambda)}$. We summarise below the assumptions just made.

Assumptions in force for the rest of §4:

- ◇ $\alpha \in \mathcal{D}$ is proportional;
- ◇ π is the unique largest odd bar in α ;
- ◇ $\lambda \in \mathcal{P}$ is such that $\overline{\langle\langle\alpha\rangle\rangle}$ is proportional to $\overline{\chi(\lambda)}$;
- ◇ any proportional strict partition β with $|\beta| < |\alpha|$ is 4-stepped-and-semicongruent.

4.1 Case of all parts having same parity

First we consider the case where the parts of α all have the same parity.

Lemma 4.2. *Suppose α consists of a single odd part. Then α is 4-stepped-and-semicongruent (that is, $\alpha \in \{(1), (3)\}$).*

Proof. Write $\alpha_1 = 2k + 1$. Then α has a $(2k + 1)$ -bar, so by Lemma 3.2 λ has a $(2k + 1)$ -rim-hook. But λ is a partition of $2k + 1$, so λ must itself be a rim-hook. Meanwhile $\lambda^{\text{reg}} = \text{dbl}(\alpha)^{\text{reg}} = (k + 1, k)$ by Lemma 3.1(i), so the only possibility is $\lambda = (k + 1, 1^k)$.

If $k \geq 1$ then α has also a $(2k - 1)$ -bar, so λ has a $(2k - 1)$ -rim-hook (by Lemma 3.2). But the second-largest rim-hook of $\lambda = (k + 1, k)$ is of length k , so we must have $k \geq 2k - 1$. This gives $k \leq 1$, and so $\alpha_1 = 2k + 1 \leq 3$ as required. \square

Proposition 4.3. *Suppose α has all parts odd. Then α is 4-stepped-and-semicongruent (that is, it is a 4-bar-core).*

Proof. The case $l(\alpha) = 1$ is Lemma 4.2, so suppose $l(\alpha) > 1$. The unique largest odd bar π is the largest part of α . By Proposition 3.3 and the inductive hypothesis, $\alpha \setminus \pi$ is 4-stepped-and-semicongruent, and since it has all parts odd $\alpha \setminus \pi$ is 4-bar-core. We can write $\alpha_1 = \alpha_2 + 2k$ for some positive integer k ; we are required to show $k = 2$. Let ϵ be the spin-residue of the node at the end of row 2 of α (and hence the spin-residue of the nodes at the end of every row except possibly row 1).

Suppose towards a contradiction $k \geq 3$. Then $\alpha^{-\epsilon} = (\alpha_2 + r, \alpha_2 - 2, \alpha_3 - 2, \dots)$ where $r \in \{2k, 2k - 2\}$. Then $\alpha_2 + r > 4$ but $\alpha_2 + r - 4 \notin \alpha^{-\epsilon}$, so $\alpha^{-\epsilon}$ is not 4-stepped, a contradiction.

Now suppose towards a contradiction $k = 1$. Then the node at the end of row 1 of α is removable and has spin-residue $\bar{\epsilon}$, giving $\alpha^{-\bar{\epsilon}} = (\alpha_2 + 1, \alpha_2, \alpha_3, \dots)$. If $\alpha_2 > 3$, then $\alpha_1^{-\bar{\epsilon}} > 4$ but $\alpha^{-\bar{\epsilon}} - 4 \notin \alpha^{-\bar{\epsilon}}$, contradicting the assumption that $\alpha^{-\bar{\epsilon}}$ is 4-stepped. If $\alpha_2 \in \{1, 3\}$, then $\alpha \in \{(3, 1), (5, 3)\}$ and it is easily checked that these strict partitions are not proportional (for example by inspecting the character tables in [JK81] and [HH92], or by identifying differences in odd weights and removable nodes between α and candidate partitions). \square

Proposition 4.4. *Suppose α has all parts even. Then α is 4-stepped-and-semicongruent.*

Proof. Trivially any strict partition with all parts even is 4-semicongruent; it remains to verify the 4-stepped property.

Let i be such that $\alpha_i > 4$ (hence $\alpha_i \geq 6$). Let ϵ be the spin-residue of the node at the end of row i of α . Then $\alpha_i^{-\epsilon} = \alpha_i - 1 > 4$ and $\alpha^{-\epsilon}$ is 4-stepped-and-semicongruent by the inductive hypothesis. Thus $\alpha_i - 5 \in \alpha^{-\epsilon}$. But since $\alpha_i - 5$ is odd, it cannot be a part of α , and likewise nor can $\alpha_i - 3$; thus a part of size $\alpha_i - 5$ must have been obtained by removing a single ϵ -spin-node from a part of size $\alpha_i - 4 \in \alpha$. \square

Given the above two propositions, we assume for the remainder of this section that α has both odd and even parts. In particular, the unique largest odd bar π comprises the largest even part and the largest odd part, and by Proposition 3.3 the partition λ has a unique rim-hook δ of length $|\pi|$, and $\overline{\langle\alpha \setminus \pi\rangle}$ is proportional to $\overline{\chi(\lambda \setminus \delta)}$. Then by the inductive hypothesis $\alpha \setminus \pi$ is 4-stepped-and-semi-congruent.

Additional assumption in force for the rest of §4:

- ◇ α has both odd and even parts;
- ◇ δ is the unique rim-hook of length $|\pi|$ in λ , satisfying $\overline{\langle\alpha \setminus \pi\rangle} \propto \overline{\chi(\lambda \setminus \delta)}$.

We deal with one additional configuration here.

Lemma 4.5. *Suppose 1 is the only odd part of α and that $2 \notin \alpha \setminus \pi$. Then α is 4-stepped-and-semi-congruent.*

Proof. The hypotheses imply that $\alpha \setminus \pi$ does not have 1, 2 or 3 as a part. Then since $\alpha \setminus \pi$ is 4-stepped-and-semi-congruent, it is of the form $(4k, 4k-4, \dots, 4)$ for some nonnegative integer k . So $\alpha = (\alpha_1, 4k, 4k-4, \dots, 4, 1)$ for some even integer $\alpha_1 > 4k$. If $\alpha_1 = 4k+4$, or if $k=0$ and $\alpha_1 = 4k+2$, then α is 4-stepped-and-semi-congruent, as required.

Suppose towards a contradiction either $\alpha_1 \geq 4k+6$ or $k > 0$ and $\alpha_1 = 4k+2$. In either case, α has at least one 0-spin-removable node, and $\alpha^{-0} = (\alpha_1^{-0}, 4k-1, 4k-5, \dots, 3)$, where α_1^{-0} is either equal to α_1 or to $\alpha_1 - 1$; in either case $\alpha_1^{-0} > 4$ but $\alpha_1^{-0} - 4$ is not a part of α^{-0} . Then α^{-0} is not 4-stepped, a contradiction. \square

4.2 Initial bounds on differences between parts of α

We begin the proof for general α by showing that consecutive parts of α cannot be too far apart. Clearly the parts except the largest odd and even parts cannot differ from their neighbouring parts by more than 4, since $\alpha \setminus \pi$ is 4-stepped. But furthermore the largest odd and even parts cannot be too much larger than the others, or else we could remove nodes from these parts and find α^{-0} or α^{-1} to have significantly differing parts, and hence not be 4-stepped. This is made precise by the following lemma.

Lemma 4.6. *Let $1 \leq i \leq l(\alpha)$.*

- (i) $\alpha_i - \alpha_{i+1} \leq 5$.
- (ii) $\alpha_i - \alpha_{i+1} \leq 4$ unless $\alpha_1 \not\equiv \alpha_2 \pmod{2}$ and either $i = 1$, or $i = 2$, α_2 is odd and $\alpha_1 - \alpha_2 = 3$.

Proof. Let ϵ be the spin-residue of the node (i, α_i) (the node at the end of row i).

- (i) Suppose towards a contradiction that $\alpha_i - \alpha_{i+1} > 5$.

First consider the case where α_i is even. Then the ϵ -spin-node at the end of row i of α is spin-removable, and thus $\alpha_i^{-\epsilon} = \alpha_i - 1$. But then $\alpha_i^{-\epsilon} - 4 = \alpha_i - 5 > \alpha_{i+1} \geq \alpha_{i+1}^{-\epsilon}$, contradicting the assumption that $\alpha^{-\epsilon}$ is 4-stepped.

Now consider the case where α_i is odd. If $\alpha_i - \alpha_{i+1} > 6$, then the two ϵ -spin-nodes at the end of row i of α are spin-removable, and thus $\alpha_i^{-\epsilon} = \alpha_i - 2$. But then $\alpha_i^{-\epsilon} - 4 = \alpha_i - 6 > \alpha_{i+1} \geq \alpha_{i+1}^{-\epsilon}$, contradicting the assumption that $\alpha^{-\epsilon}$ is 4-stepped.

If $\alpha_i - \alpha_{i+1} = 6$, then the node at the end of row $i+1$ has residue $\bar{\epsilon}$ distinct from that of row i , and $\alpha_i^{-\bar{\epsilon}} - \alpha_{i+1}^{-\bar{\epsilon}} \geq \alpha_i - \alpha_{i+1} > 4$. Thus $\alpha^{-\bar{\epsilon}}$ is not 4-stepped, so to avoid a contradiction with the

inductive hypothesis we have $\alpha^{-\bar{\epsilon}} = \alpha$ (that is, α has no $\bar{\epsilon}$ -spin-removable nodes). In order for row $i + 1$ to have no $\bar{\epsilon}$ -spin-removable nodes, we must have $\alpha_{i+1} \geq 3$ and $\alpha_{i+1} - 1, \alpha_{i+1} - 2 \in \alpha$. But then $\alpha_{i+1}, \alpha_{i+1} - 2 \in \alpha \setminus \pi$ (with α_{i+1} odd), contradicting the assumption that $\alpha \setminus \pi$ is 4-semicongruent.

- (ii) Since $\alpha \setminus \pi$ is 4-stepped, we have $\alpha_i - \alpha_{i+1} \leq 4$ for all i except possibly when α_i is the largest part of its parity. If α_i is largest of its parity but $i \geq 3$, then α_{i-1} is not largest of its parity, and so $\alpha_{i-1} \in \alpha \setminus \pi$ but $\alpha_i \notin \alpha \setminus \pi$. Then $\alpha_{i-1} - \alpha_{i+1} \leq 4$ and hence $\alpha_i - \alpha_{i+1} \leq 4$. So we may assume $i \in \{1, 2\}$.

If $\alpha_1 \equiv \alpha_2 \pmod{2}$, then by part (i) we have $\alpha_1 - \alpha_2 \leq 4$ (resolving the case $i = 1$), while α_2 is not the largest part of its parity (resolving the case $i = 2$). So we may assume $\alpha_1 \not\equiv \alpha_2 \pmod{2}$.

Consider the case $i = 2$, α_1 odd and α_2 even. We have $\alpha_2 - \alpha_3 \leq 5$ by part (i). Suppose towards a contradiction that $\alpha_3 = \alpha_2 - 5$. We argue as in the final paragraph of part (i): the node at the end of row 3 has residue $\bar{\epsilon}$, and thus $\alpha_2^{-\bar{\epsilon}} - \alpha_3^{-\bar{\epsilon}} \geq \alpha_2 - \alpha_3 > 4$; therefore $\alpha^{-\bar{\epsilon}}$ is not 4-stepped, and so $\alpha^{-\bar{\epsilon}} = \alpha$; in order for row 3 to have no $\bar{\epsilon}$ -spin-removable nodes, we have $\alpha_3 \geq 3$ and $\alpha_3 - 1, \alpha_3 - 2 \in \alpha$; but then $\alpha_3, \alpha_3 - 2 \in \alpha \setminus \pi$, contradicting the assumption that $\alpha \setminus \pi$ is 4-semicongruent. So $\alpha_2 - \alpha_3 \leq 4$ in this case.

The only remaining case to consider is when $i = 2$, α_1 even and α_2 odd, where we must show that if $\alpha_2 - \alpha_3 = 5$ then $\alpha_1 - \alpha_2 = 3$. Indeed, if $\alpha_2 - \alpha_3 = 5$ then the nodes at the ends of rows 2 and 3 both have spin-residue ϵ , and so $\alpha_2^{-\epsilon} - \alpha_3^{-\epsilon} = \alpha_2 - \alpha_3 > 4$. Thus $\alpha^{-\epsilon}$ is not 4-stepped, so to avoid a contradiction with the inductive hypothesis we have $\alpha^{-\epsilon} = \alpha$ (that is, α has no ϵ -spin-removable nodes). In particular, we cannot have $\alpha_1 = \alpha_2 + 5$ (for then row 1 would have a $\bar{\epsilon}$ -spin-removable node). So $\alpha_1 \in \{\alpha_2 + 1, \alpha_2 + 3\}$. But if $\alpha_1 = \alpha_2 + 1$, then $\alpha_1^{-\epsilon} = \alpha_1 = \alpha_2 + 1 > 4$, but $\alpha_2^{-\epsilon} = \alpha_2 - 2$ and $\alpha_3^{-\epsilon} \leq \alpha_3 = \alpha_2 - 5$, contradicting the assumption that $\alpha^{-\epsilon}$ is 4-stepped. So $\alpha_1 - \alpha_2 = 3$ as required. \square

4.3 Improving bounds by considering the last fully-occupied ladder

We now tighten the bounds from the previous section by considering restrictions on the *fully-occupied ladders* in $\text{dbl}(\alpha)$ imposed by the regularisation requirement (Lemma 3.1(i)).

Recall from §2.1.3 that \mathcal{L}_l denotes the l th ladder in \mathbb{N}^2 . For a partition μ , we say a ladder \mathcal{L}_l is *fully-occupied* in μ if $\mathcal{L}_l \subseteq \mu$. For a strict partition β , let $f(\beta)$ be maximal such that $\mathcal{L}_{f(\beta)}$ is fully-occupied in $\text{dbl}(\beta)$, and set $f(\emptyset) = 0$. Trivially we find $f(\beta) \leq \text{dbl}(\beta)_1 = \lceil \beta_1/2 \rceil$. We will show that for α and $\alpha \setminus \pi$, this bound is in fact attained.

Lemma 4.7. *Suppose $\beta \in \mathcal{D}$, and let $1 \leq i \leq l(\beta) - 1$.*

- (i) *Suppose either β_i is odd and $\beta_i - \beta_{i+1} \leq 4$, or β_i is even and $\beta_i - \beta_{i+1} \leq 5$. Then the ladder that contains the node $(2i - 1, \lceil \beta_i/2 \rceil)$ (that is, the final node in row $2i - 1$ of $\text{dbl}(\beta)$) intersects $\text{dbl}(\beta)$ also in rows $2i, 2i + 1$ and $2i + 2$ (ignoring the last if $\beta_{i+1} = 1$).*
- (ii) *Suppose β_i is odd and $\beta_i - \beta_{i+1} \leq 6$. Then the ladder that contains the node $(2i - 1, \lfloor \beta_i/2 \rfloor)$ (that is, the penultimate node in row $2i - 1$ of $\text{dbl}(\beta)$) intersects $\text{dbl}(\beta)$ also in rows $2i, 2i + 1$ and $2i + 2$ (ignoring the last if $\beta_{i+1} = 1$).*

Proof.

- (i) The ladder that contains $(2i - 1, \lceil \beta_i/2 \rceil)$ is $\mathcal{L}_{\lceil \beta_i/2 \rceil + 2i - 2}$, so $\text{dbl}(\beta)$ meets this ladder in row $j > 2i - 1$ if and only if $\text{dbl}(\beta)_j \geq \lceil \beta_i/2 \rceil + 2i - 1 - j$. Clearly

$$\text{dbl}(\beta)_{2i} = \lfloor \beta_i/2 \rfloor \geq \lceil \beta_i/2 \rceil - 1,$$

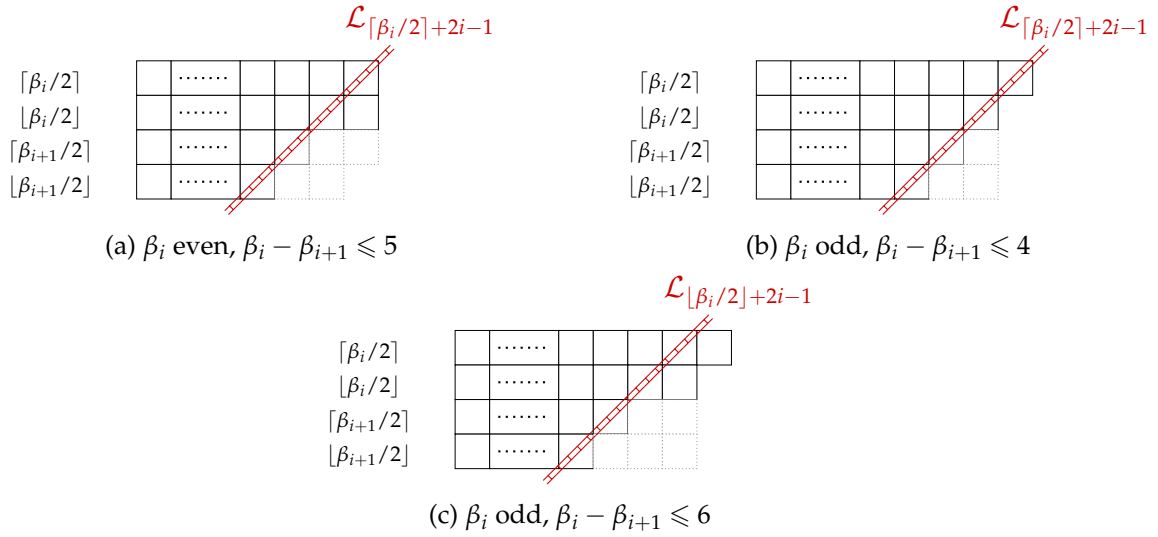


Figure 1: An illustration of the ladder considered in Lemma 4.7 hitting the following three rows. The dashed boxes indicate nodes that may or may not be present in the Young diagram.

while as illustrated in Figures 1a and 1b the assumptions imply that

$$\begin{aligned} \text{dbl}(\beta)_{2i+1} &= \lceil \beta_{i+1}/2 \rceil \geq \lceil \beta_i/2 \rceil - 2, \\ \text{dbl}(\beta)_{2i+2} &= \lfloor \beta_{i+1}/2 \rfloor \geq \lceil \beta_i/2 \rceil - 3. \end{aligned}$$

- (ii) The ladder that contains $(2i - 1, \lfloor \beta_i/2 \rfloor)$ is $\mathcal{L}_{\lfloor \beta_i/2 \rfloor + 2i - 2}$, so $\text{dbl}(\beta)$ meets this ladder in row $j > 2i - 1$ if and only if $\text{dbl}(\beta)_j \geq \lfloor \beta_i/2 \rfloor + 2i - 1 - j$. Clearly

$$\text{dbl}(\beta)_{2i} = \lfloor \beta_i/2 \rfloor \geq \lfloor \beta_i/2 \rfloor - 1,$$

while as illustrated in Figure 1c the assumptions imply that

$$\begin{aligned} \text{dbl}(\beta)_{2i+1} &= \lceil \beta_{i+1}/2 \rceil \geq \lfloor \beta_i/2 \rfloor - 2, \\ \text{dbl}(\beta)_{2i+2} &= \lfloor \beta_{i+1}/2 \rfloor \geq \lfloor \beta_i/2 \rfloor - 3. \end{aligned}$$

□

The next lemma gives some basic results about $f(\alpha)$ and $f(\alpha \setminus \pi)$.

Lemma 4.8.

- (i) $f(\alpha \setminus \pi) = \text{dbl}(\alpha \setminus \pi)_1 = \lceil \frac{1}{2}(\alpha \setminus \pi)_1 \rceil$.
- (ii) $f(\alpha) \in \{\text{dbl}(\alpha)_1, \text{dbl}(\alpha)_1 - 1\}$.
- (iii) If α_1 is even, then $f(\alpha) = \text{dbl}(\alpha)_1 = \frac{1}{2}\alpha_1$.

Proof. The strict partition $\alpha \setminus \pi$ is 4-stepped and hence satisfies the conditions of Lemma 4.7(i) for every i . Thus the ladder $\mathcal{L}_{\text{dbl}(\alpha \setminus \pi)_1}$ intersects $\text{dbl}(\alpha \setminus \pi)$ in every row, and so $f(\alpha \setminus \pi) = \text{dbl}(\alpha \setminus \pi)_1$ as required.

By Lemma 4.6, the strict partition α also satisfies the conditions of Lemma 4.7(i), except possibly for a single value of $i \in \{1, 2\}$ with α_i odd, $\alpha_1 \not\equiv \alpha_2 \pmod{2}$, and $\alpha_1 - \alpha_2 = 3$ if $i = 2$.

If the condition fails for $i = 1$, then α_1 is odd and $\alpha_1 - \alpha_2 = 5$, and so by Lemma 4.7(ii) the ladder $\mathcal{L}_{\text{dbl}(\alpha)_1 - 1}$ intersects rows 2, 3 and 4 of $\text{dbl}(\alpha)$, and hence all rows of $\text{dbl}(\alpha)$ by Lemma 4.7(i).

If the condition fails for $i = 2$, then α_1 is even and $\alpha_1 - \alpha_2 = 3$. So by Lemma 4.7(i), the ladder $\mathcal{L}_{\text{dbl}(\alpha)}$ intersects rows 2, 3 and 4 of $\text{dbl}(\alpha)$, and moreover since $\alpha_1 - \alpha_2 = 3$ it is easily seen that it intersects nodes in these rows strictly before the last. Then by Lemma 4.7(ii), this ladder intersects also rows 5 and 6 of $\text{dbl}(\alpha)$, and hence all rows of $\text{dbl}(\alpha)$ by Lemma 4.7(i).

Thus in either case $f(\alpha) \geq \text{dbl}(\alpha)_1 - 1$, and if α_1 is even then $f(\alpha) = \text{dbl}(\alpha)_1$. \square

The following is the key observation that, combined with the values of $f(\alpha)$ and $f(\alpha \setminus \pi)$ identified above, allows us to improve the bounds on differences between parts of α .

Lemma 4.9. $f(\alpha) \leq f(\alpha \setminus \pi) + 2$.

Proof. By Lemma 3.1(i), the partitions λ and $\text{dbl}(\alpha)$ have the same 2-regularisation, so $f(\alpha)$ is maximal such that $\mathcal{L}_{f(\alpha)} \subseteq \lambda$. Similarly $f(\alpha \setminus \pi)$ is maximal such that $\mathcal{L}_{f(\alpha \setminus \pi)} \subseteq \lambda \setminus \delta$. Thus we can find a node $(r, c) \in \mathcal{L}_{f(\alpha \setminus \pi)+1}$ which is not contained in $\lambda \setminus \delta$. Then the node $(r+1, c+1)$ cannot be contained in λ , because δ contains at most one node on any diagonal, and hence cannot contain both (r, c) and $(r+1, c+1)$. The node $(r+1, c+1)$ lies in $\mathcal{L}_{f(\alpha \setminus \pi)+3}$, and so we deduce $\mathcal{L}_{f(\alpha \setminus \pi)+3} \not\subseteq \lambda$, and therefore $f(\alpha) \leq f(\alpha \setminus \pi) + 2$. \square

We can now deduce tighter bounds on the differences between parts of α .

Lemma 4.10.

(i) If α_1 is even and α_2 is odd, then $\alpha_1 - \alpha_3 \leq 5$.

(ii) If α_1 is odd and α_2 is even, then $\alpha_1 - \alpha_3 \leq 6$.

Proof. In both cases, $\alpha_1 \not\equiv \alpha_2 \pmod{2}$ and so $(\alpha \setminus \pi)_1 = \alpha_3$. Then combining Lemma 4.8(i) and Lemma 4.9 gives $f(\alpha) \leq \frac{1}{2}(\alpha_3 + 1) + 2$.

(i) Suppose α_1 is even and α_2 is odd. Then Lemma 4.8(iii) gives $f(\alpha) = \frac{1}{2}\alpha_1$. Thus $\frac{1}{2}\alpha_1 \leq \frac{1}{2}(\alpha_3 + 1) + 2$, which gives the claim.

(ii) Suppose α_1 is odd and α_2 is even. Then Lemma 4.8(ii) gives $f(\alpha) \geq \text{dbl}(\alpha)_1 - 1 = \frac{1}{2}(\alpha_1 - 1)$. Thus $\frac{1}{2}(\alpha_1 - 1) \leq \frac{1}{2}(\alpha_3 + 1) + 2$, which gives the claim. \square

Proposition 4.11.

(i) Let $1 \leq i \leq l(\alpha)$. Then $\alpha_i - \alpha_{i+1} \leq 4$.

(ii) $f(\alpha) = \text{dbl}(\alpha)_1 = \lceil \frac{1}{2}\alpha_1 \rceil$.

(iii) If α_1 is odd and α_2 is even, then $\alpha_1 - \alpha_3 \leq 4$.

Proof.

(i) The only cases not covered by Lemma 4.6 are as follows.

α_1 is even, α_2 is odd, $i \in \{1, 2\}$: By Lemma 4.10(i) we have $\alpha_1 - \alpha_3 \leq 5$. Then $\alpha_1 - \alpha_2 \leq 4$ and $\alpha_2 - \alpha_3 \leq 4$ as required.

α_1 is odd, α_2 is even, $i = 1$: By Lemma 4.10(ii) we have $\alpha_1 - \alpha_3 \leq 6$. Suppose towards a contradiction $\alpha_1 - \alpha_2 = 5$; then $\alpha_3 = \alpha_1 - 6$. The nodes at the end of rows 1 and 2 have the same spin-residue, say ϵ , and the node at the end of row 3 has distinct spin-residue $\bar{\epsilon}$. If the node at the end of row 3 is spin-removable, then $\alpha^{-\bar{\epsilon}}$ is not 4-stepped, contradicting the inductive hypothesis. So we must have $\alpha_3 \geq 3$ and $\alpha_3 - 1, \alpha_3 - 2 \in \alpha$. But then $\alpha_3, \alpha_3 - 2 \in \alpha \setminus \pi$, contradicting the assumption that $\alpha \setminus \pi$ is 4-semicongruent.

- (ii) By part (i), we now have that the conditions of Lemma 4.7(i) are satisfied for α for every i , and so as in the proof of Lemma 4.8 the ladder $\mathcal{L}_{\text{dbl}(\alpha)_1}$ intersects every row of $\text{dbl}(\alpha)$.
- (iii) Suppose α_1 is odd and α_2 is even. As in the proof of Lemma 4.10, combining Lemma 4.8(i) and Lemma 4.9 gives $f(\alpha) \leq \frac{1}{2}(\alpha_3 + 1) + 2$. Part (ii) gives $f(\alpha) = \text{dbl}(\alpha)_1 = \frac{1}{2}(\alpha_1 + 1)$. Thus $\frac{1}{2}(\alpha_1 + 1) \leq \frac{1}{2}(\alpha_3 + 1) + 2$, which gives the claim. \square

4.4 Deducing 4-semicongruence by considering the last partially-occupied ladder

We also gain information on α by considering the ladders which non-trivially intersect $\text{dbl}(\alpha)$.

For a strict partition β , let $t(\beta)$ be maximal such that $\text{dbl}(\beta) \cap \mathcal{L}_{t(\beta)}$ is nonempty, and set $t(\emptyset) = 0$. Trivially

$$t(\beta) \geq l(\text{dbl}(\beta)) = \begin{cases} 2l(\beta) & \text{if } 1 \notin \beta, \\ 2l(\beta) - 1 & \text{if } 1 \in \beta. \end{cases}$$

We will show that for α and $\alpha \setminus \pi$ this bound is attained or nearly attained.

Lemma 4.12. *Let β be any strict partition, and let $1 \leq i \leq l(\beta) - 1$. Suppose either β_i is even and $\beta_i - \beta_{i+1} \leq 4$, or β_i is odd and $\beta_i - \beta_{i+1} \leq 5$. Then the largest ladder that meets row $2i - 1$ or $2i$ of $\text{dbl}(\beta)$ is smaller than or equal to the largest ladder that meets row $2i + 1$ or $2i + 2$ of $\text{dbl}(\beta)$.*

Proof. The largest ladder that meets row $2i - 1$ or $2i$ is ladder $\mathcal{L}_{\lfloor \beta_i/2 \rfloor + 2i - 1}$, while the largest ladder that meets rows $2i + 1$ or $2i + 2$ is ladder $\mathcal{L}_{\lfloor \beta_{i+1}/2 \rfloor + 2i + 1}$. The assumptions on β_i and β_{i+1} imply that $\lfloor \beta_i/2 \rfloor \leq \lfloor \beta_{i+1}/2 \rfloor + 2$, which is what we need, as illustrated in Figure 2. \square

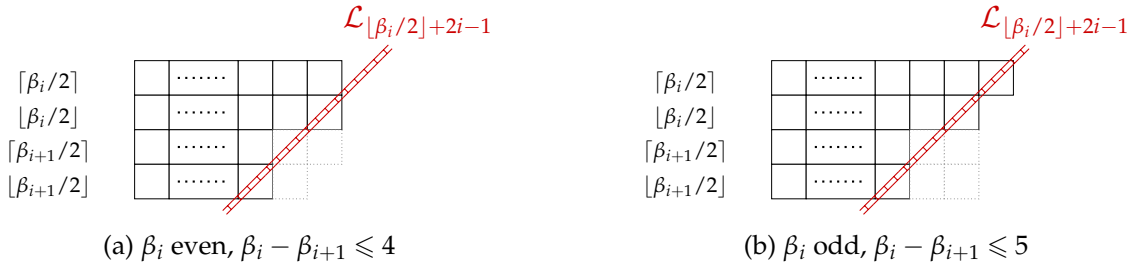


Figure 2: An illustration of the largest ladder meeting row $2i - 1$ or $2i$, considered in Lemma 4.12, being met and possibly exceeded by row $2i + 1$ or $2i + 2$. The dashed boxes indicate nodes that may or may not be present in the Young diagram.

Proposition 4.13. *Let $\beta \in \{\alpha, \alpha \setminus \pi\}$. Then*

$$t(\beta) = \begin{cases} l(\text{dbl}(\beta)) & \text{if } 1, 2 \text{ or } 3 \text{ is a part of } \beta; \\ l(\text{dbl}(\beta)) + 1 & \text{otherwise.} \end{cases}$$

In particular, if α has at least two odd parts, then $t(\alpha) = l(\text{dbl}(\alpha))$.

Proof. Both $\alpha \setminus \pi$ and α satisfy the conditions of Lemma 4.12 for every i : $\alpha \setminus \pi$ does so since it is 4-stepped, and α does so by Proposition 4.11(i). Thus the largest ladder that meets β meets it in its final row. If the final part of β is 1, 2 or 3 then this is the $l(\text{dbl}(\beta))$ ladder, while if the final part of β is 4 then this is the $l(\text{dbl}(\beta)) + 1$ ladder. The 4-stepped property of $\alpha \setminus \pi$ and the bounds on α imply that the final part of β is at most 4, so these are all possible cases.

The “in particular” follows since if α has at least two odd parts then $\alpha \setminus \pi$ has at least one odd part, and since it is 4-stepped therefore has 1 or 3 as a part, and hence so does α . \square

Lemma 4.14. $t(\alpha) \geq t(\alpha \setminus \pi) + 3$.

Proof. It suffices to show $l(\text{dbl}(\alpha)) \geq t(\alpha \setminus \pi) + 3$. If α has an odd part greater than 1, then $l(\text{dbl}(\alpha)) = l(\text{dbl}(\alpha \setminus \pi)) + 4$, and Proposition 4.13 gives $l(\text{dbl}(\alpha \setminus \pi)) + 1 \geq t(\alpha \setminus \pi)$, yielding the result. Otherwise, the only odd part of α is 1, and $l(\text{dbl}(\alpha)) = l(\text{dbl}(\alpha \setminus \pi)) + 3$. By Lemma 4.5, we may assume $2 \in \alpha \setminus \pi$ in this case. Then $l(\text{dbl}(\alpha \setminus \pi)) = t(\alpha \setminus \pi)$ by Proposition 4.13, and the result follows. \square

Now recall the rim-hook δ . Because $|\delta|$ is odd, we can write $|\delta| = 2a + 1$. The residues of the nodes in a rim-hook alternate between 0 and 1 from one end of the rim-hook to the other, so the multiset of residues of the nodes of δ is either $\{0^{a+1}, 1^a\}$ or $\{0^a, 1^{a+1}\}$.

Lemma 4.15. *The rim-hook δ meets $\mathcal{L}_{t(\alpha)}$ in the first row or column. In particular, the multiset of residues of δ is $\{0^{a+1}, 1^a\}$ if $t(\alpha)$ is odd, or $\{0^a, 1^{a+1}\}$ if $t(\alpha)$ is even.*

Proof. That δ meets the ladder $\mathcal{L}_{t(\alpha)}$ is clear from the fact that $t(\alpha) > t(\alpha \setminus \pi)$. Since furthermore $t(\alpha) \geq t(\alpha \setminus \pi) + 3$ by Lemma 4.14, we deduce that $\lambda \setminus \delta$ does not contain any nodes in the ladder $\mathcal{L}_{t(\alpha)-2}$, and hence the only possible nodes of $\mathcal{L}_{t(\alpha)}$ that can belong to δ are $(1, t(\alpha))$ and $(t(\alpha), 1)$. So δ meets $\mathcal{L}_{t(\alpha)}$ in the first row or column. \square

Proposition 4.16. *The odd parts of α are congruent modulo 4 (that is, α is 4-semicongruent).*

Proof. If α has only one odd part, this is trivially true. So suppose α has at least two odd parts. Since $\alpha \setminus \pi$ is 4-semicongruent, all the odd parts of α except possibly the largest are congruent modulo 4, so it suffices to show that the largest odd part of α is congruent to the smallest odd part of α . Write $2k + 1$ for the largest odd part of α . Note that the smallest odd part of α is the smallest odd part of $\alpha \setminus \pi$, and so (since $\alpha \setminus \pi$ is 4-stepped) is equal to 1 or 3.

Recall that $\langle \text{spin} \rangle$ -contents (i.e. multisets of $\langle \text{spin} \rangle$ -residues) of partitions characterise blocks (§2.5). Since $\langle \alpha \rangle$ and $\chi(\lambda)$ are proportional, they lie in the same block, so the spin-content of α is equal to the content of λ ; likewise the spin-content of $\alpha \setminus \pi$ is equal to the content of $\lambda \setminus \delta$. It follows that the multiset of spin-residues of π is equal to the multiset of residues of δ . Write $2a + 1$ for the size of δ .

Observe that an even part of α has equal numbers of nodes of spin-residue 0 and 1, a part congruent to 1 (mod 4) has one more node of spin-residue 0 than 1, and a part congruent to 3 (mod 4) has one more node of spin-residue 1 than 0. Thus we have

$$\begin{aligned}
 2k + 1 \equiv 1 \pmod{4} &\iff \pi \text{ has multiset of spin-residues } \{0^{a+1}, 1^a\} && \text{(by the above observation)} \\
 &\iff \delta \text{ has multiset of residues } \{0^{a+1}, 1^a\} && \text{(as contents characterise blocks)} \\
 &\iff t(\alpha) \text{ is odd} && \text{(by Lemma 4.15)} \\
 &\iff l(\text{dbl}(\alpha)) \text{ is odd} && \text{(by Proposition 4.13)} \\
 &\iff 1 \text{ is the smallest odd part of } \alpha.
 \end{aligned}$$

Therefore the largest odd part of α is congruent modulo 4 to the smallest, as required. \square

The congruence of the odd parts of α also allows us to make the following useful deductions.

Corollary 4.17.

- (i) Suppose $1 \leq i \leq l(\alpha)$. If α_i is odd, then the node (i, α_i) at the end of row i is spin-removable.
- (ii) Let $\epsilon \in \{0, 1\}$ and let $m \geq 2$ be even. If $m \in \alpha^{-\epsilon}$, then $m \in \alpha$.

Proof.

- (i) The only way the node (i, α_i) can fail to be spin-removable is if $\alpha_i \geq 3$ and both $\alpha_i - 1$ and $\alpha_i - 2$ are parts of α . But this contradicts Proposition 4.16.

- (ii) Let j be such that $\alpha_j^{-\epsilon} = m$. If the node (j, α_j) at the end of row j in α has spin-residue $\bar{\epsilon}$, then $\alpha_j = m$ also. Otherwise, (j, α_j) has spin-residue ϵ , and either $\alpha_j = m$, in which case the claim holds, or $\alpha_j = m + 1$. In this remaining case, the penultimate node in row j , the node (j, m) , is not spin-removable, and so either $\alpha_{j+1} = m$ or $\alpha_{j+1} = m - 1$. But if m is even then $m + 1$ and $m - 1$ cannot both be parts of α by Proposition 4.16, so $\alpha_{j+1} = m$. \square

4.5 Ruling out configurations using hook lengths

Before we can complete the proof of Theorem 4.1 by showing that α is 4-stepped, we first rule out several tricky configurations by considering restrictions on hook lengths in λ .

Recall that by assumption λ has a unique rim-hook δ of length $|\pi|$.

Lemma 4.18. *If $|\delta| \geq t(\alpha) + f(\alpha \setminus \pi) + 1$, then $f(\alpha) = f(\alpha \setminus \pi) + 2$.*

Proof. Observe that if δ meets all of the diagonals intersecting the ladders $\mathcal{L}_{f(\alpha \setminus \pi)+1}$ and $\mathcal{L}_{f(\alpha \setminus \pi)+2}$, then these ladders are both fully-occupied in λ (using the fact that $\mathcal{L}_{f(\alpha \setminus \pi)}$ is already fully-occupied in $\lambda \setminus \delta$). Thus if δ meets these diagonals, then $f(\alpha) \leq f(\alpha \setminus \pi) + 2$, which must then hold with equality by Lemma 4.9.

So suppose $|\delta| \geq t(\alpha) + f(\alpha \setminus \pi) + 1$, and it suffices to show that δ meets the diagonals intersecting the ladders $\mathcal{L}_{f(\alpha \setminus \pi)+1}$ and $\mathcal{L}_{f(\alpha \setminus \pi)+2}$, which are the diagonals

$$-(f(\alpha \setminus \pi) + 1), \dots, -1, 0, 1, \dots, f(\alpha \setminus \pi) + 1.$$

We know from Lemma 4.15 that δ meets the ladder $\mathcal{L}_{t(\alpha)}$ in either the first column or first row. Without loss of generality, suppose it is the first column; then the bottom-most node (the “foot node”) of δ is $(t(\alpha), 1)$, which lies on the $1 - t(\alpha)$ diagonal. Furthermore, since a rim-hook meets each diagonal at most once, δ meets all diagonals from $1 - t(\alpha)$ to $|\delta| - t(\alpha)$ inclusive. So provided $|\delta| \geq t(\alpha) + f(\alpha \setminus \pi) + 1$ as assumed (and also $t(\alpha) - 1 \geq f(\alpha \setminus \pi) + 1$, which holds since $t(\alpha) \geq t(\alpha \setminus \pi) + 3 \geq f(\alpha \setminus \pi) + 3$), the desired diagonals are met. \square

Proposition 4.19. *α is not of the form:*

- (i) $(4k + 2) \sqcup \gamma \sqcup (4k, 4k - 1, 4k - 4, 4k - 5, \dots, 4, 3)$ where $k \geq 1$ and γ is a strict partition with all parts congruent to 2 (mod 4) and strictly less than $4k - 2$; or
- (ii) $(4k + 4) \sqcup \gamma \sqcup (4k + 2, 4k + 1, 4k - 2, 4k - 3, \dots, 2, 1)$ where $k \geq 1$ and γ is a strict partition with all parts congruent to 0 (mod 4) and strictly less than $4k$.

Proof. In both cases we suppose that α is of the given form and find a contradiction with Lemma 4.18.

- (i) We compute (using Lemma 4.8 and Proposition 4.13 if desired)

$$\begin{aligned} t(\alpha) &= l(\text{dbl}(\alpha)) = 2(2k + 1 + l(\gamma)) \leq 6k, \\ f(\alpha) &= \text{dbl}(\alpha)_1 = 2k + 1, \\ f(\alpha \setminus \pi) &= \text{dbl}(\alpha \setminus \pi)_1 = 2k, \\ |\delta| &= |\pi| = 8k + 1. \end{aligned}$$

But then $t(\alpha) + f(\alpha \setminus \pi) + 1 \leq 8k + 1 = |\delta|$ but $f(\alpha) \neq f(\alpha \setminus \pi) + 2$, contradicting Lemma 4.18.

(ii) We compute (using Lemma 4.8 and Proposition 4.13 if desired)

$$\begin{aligned} t(\alpha) &= l(\text{dbl}(\alpha)) - 1 = 2(2k + 3 + l(\gamma)) - 1 \leq 6k + 3, \\ f(\alpha) &= \text{dbl}(\alpha)_1 = 2k + 2, \\ f(\alpha \setminus \pi) &= \text{dbl}(\alpha \setminus \pi)_1 = 2k + 1, \\ |\delta| &= |\pi| = 8k + 5. \end{aligned}$$

But then $t(\alpha) + f(\alpha \setminus \pi) + 1 \leq 8k + 5 = |\delta|$ but $f(\alpha) \neq f(\alpha \setminus \pi) + 2$, contradicting Lemma 4.18. \square

The following is in fact a statement about the longest odd hook length in any partition λ with given 2-regularisation.

Lemma 4.20. *In all cases, $|\delta| \leq 2t(\alpha) - 1$. Moreover:*

- (i) *if $\text{dbl}(\alpha)$ contains only a single node in $\mathcal{L}_{t(\alpha)}$, then $|\delta| < 2t(\alpha) - 1$;*
- (ii) *if $\text{dbl}(\alpha)$ contains only a single node in $\mathcal{L}_{t(\alpha)}$ and exactly two nodes in each of $\mathcal{L}_{t(\alpha)-1}$ and $\mathcal{L}_{t(\alpha)-2}$, then $|\delta| < 2t(\alpha) - 3$.*

Proof. The diagonals that intersect the first $t(\alpha)$ ladders are the diagonals

$$-(t(\alpha) - 1), \dots, -1, 0, 1, \dots, t(\alpha) - 1.$$

Thus λ meets at most $2t(\alpha) - 1$ diagonals. Since δ meets each diagonal at most once, we have $|\delta| \leq 2t(\alpha) - 1$. In order for equality to be achieved, δ must meet both the $-(t(\alpha) - 1)$ and $t(\alpha) - 1$ diagonals, but their intersection with the first $t(\alpha)$ ladders is with only $\mathcal{L}_{t(\alpha)}$; thus $\mathcal{L}_{t(\alpha)}$ must contain at least two nodes of λ , proving (i).

Now suppose $\text{dbl}(\alpha)$ is as described in (ii). By part (i), we have $|\delta| < 2t(\alpha) - 1$; suppose towards a contradiction λ contains a rim-hook ρ of length $2t(\alpha) - 3$ (because $|\delta|$ is necessarily odd, this will suffice). Then ρ must meet both the diagonals $t(\alpha) - 3$ and $-(t(\alpha) - 3)$, and hence λ must contain the nodes at start of each of these diagonals: $(1, t(\alpha) - 2)$ and $(t(\alpha) - 2, 1)$. These nodes lie in the ladder $\mathcal{L}_{t(\alpha)-2}$, which by assumption has exactly two nodes in $\text{dbl}(\alpha)$; thus the nodes $(1, t(\alpha) - 2)$ and $(t(\alpha) - 2, 1)$ are the only nodes in $\lambda \cap \mathcal{L}_{t(\alpha)-1}$ and furthermore are contained in ρ , and thus ρ is the largest rim-hook in λ . It follows that the nodes in λ in the ladders $\mathcal{L}_{t(\alpha)-1}$ and $\mathcal{L}_{t(\alpha)}$ must also be at the start of their diagonals (that is, in the first row or the first column), and ρ has length $2t(\alpha) - 2$, a contradiction. \square

Proposition 4.21. *α is not of the form:*

- (i) $(4k, 4k - 4, \dots, 4) \sqcup (5)$ where $k \geq 1$; or
- (ii) $(4k, 4k - 4, \dots, 4) \sqcup (5, 2)$ where $k \geq 1$.

Proof. In both cases we suppose that α is of the given form and find a contradiction with Lemma 4.20. In both cases $|\delta| = |\pi| = 4k + 5$.

- (i) Observe that $t(\alpha) = l(\text{dbl}(\alpha)) + 1 = 2k + 3$ and that $\text{dbl}(\alpha)$ contains only a single node of $\mathcal{L}_{t(\alpha)}$. But also $|\delta| = 2t(\alpha) - 1$, contradicting Lemma 4.20(i).
- (ii) Observe that $t(\alpha) = l(\text{dbl}(\alpha)) = 2k + 4$ and that $\text{dbl}(\alpha)$ contains only a single node of $\mathcal{L}_{t(\alpha)}$. Furthermore $\mathcal{L}_{t(\alpha)-1}$ and $\mathcal{L}_{t(\alpha)-2}$ contain exactly two nodes of $\text{dbl}(\alpha)$ each. But also $|\delta| = 2t(\alpha) - 3$, contradicting Lemma 4.20(ii). \square

4.6 4-stepped property

To complete our proof of Theorem 4.1, we show that $\alpha_i - 4 \in \alpha$ whenever $\alpha_i > 4$.

Lemma 4.22. *Let α_i be the largest odd part of α , and suppose $\alpha_i > 4$ and $\alpha_i - 4 \notin \alpha$. Then $\alpha_{i+1} = \alpha_i - 1$, and if $\alpha_i - 3 \in \alpha$ then $\alpha_i + 1 \notin \alpha$.*

Proof. Our bound on consecutive parts $\alpha_i - \alpha_{i+1} \leq 4$ from Proposition 4.11(i), together with the assumption $\alpha_i - 4 \notin \alpha$ and that $\alpha_i - 2 \notin \alpha$ by Proposition 4.16, gives $\alpha_{i+1} \in \{\alpha_i - 1, \alpha_i - 3\}$.

First suppose $i = 1$. Clearly $\alpha_1 + 1 \notin \alpha$. Meanwhile if $\alpha_1 - 1 \notin \alpha$ we have $\alpha_2 = \alpha_1 - 3$; then by Proposition 4.11(iii) we have $\alpha_3 = \alpha_1 - 4$, contradicting the assumption on α .

So suppose $i > 1$, and let ϵ be the spin-residue of the node at the end of row i in α . If $\alpha_i - 3 \notin \alpha$, then $\alpha_{i+1} = \alpha_i - 1$; it remains to suppose $\alpha_i - 3 \in \alpha$ and show that $\alpha_i - 1 \in \alpha$ and $\alpha_i + 1 \notin \alpha$. Observe that, since $\alpha_i - 4 \notin \alpha$, there is a $\bar{\epsilon}$ -spin-removable node in column $\alpha_i - 3$ (and so the inductive hypothesis applies to $\alpha^{-\bar{\epsilon}}$). Since also $\alpha_i - 2 \notin \alpha$, we deduce that $\alpha_i - 3 \notin \alpha^{-\bar{\epsilon}}$. Now, α_{i-1} is even and $\alpha_{i-1} - \alpha_i \leq 4$, so $\alpha_{i-1} \in \{\alpha_i + 1, \alpha_i + 3\}$. In either case, the node at the end of row $i - 1$ is not a $\bar{\epsilon}$ -spin-removable node, and thus $\alpha_{i-1}^{-\bar{\epsilon}} = \alpha_{i-1}$. So if $\alpha_{i-1} = \alpha_i + 1$, then $\alpha_{i-1}^{-\bar{\epsilon}} = \alpha_i + 1$ but $\alpha_i - 3 \notin \alpha^{-\bar{\epsilon}}$, contradicting the assumption that $\alpha^{-\bar{\epsilon}}$ is 4-stepped; thus $\alpha_i + 1 \notin \alpha$, and $\alpha_{i-1} = \alpha_i + 3$. Then $\alpha_{i-1}^{-\bar{\epsilon}} = \alpha_i + 3$ and hence $\alpha_i - 1 \in \alpha^{-\bar{\epsilon}}$, and so $\alpha_i - 1 \in \alpha$ by Corollary 4.17(ii). \square

Proposition 4.23. *If α_i is odd and $\alpha_i > 4$, then $\alpha_i - 4 \in \alpha$.*

Proof. Since $\alpha \setminus \pi$ is 4-stepped, the proposition holds for all odd parts except possibly the largest. So suppose α_i is the largest odd part of α . Suppose towards a contradiction $\alpha_i - 4 \notin \alpha$. By Lemma 4.22 we have $\alpha_{i+1} = \alpha_i - 1$. Let ϵ be the spin-residue of the nodes (i, α_i) and $(i + 1, \alpha_i - 1)$ at the ends of rows i and $i + 1$; note that these nodes are spin-removable, so $\alpha^{-\epsilon}$ is 4-stepped-and-semicongruent by the inductive hypothesis.

Consider first the case $\alpha_i \geq 7$. We have $\alpha_i^{-\epsilon} = \alpha_i - 1$ and $\alpha_{i+1}^{-\epsilon} = \alpha_i - 2$. Then since $\alpha^{-\epsilon}$ is 4-stepped and $\alpha_i \geq 7$, we have $\alpha_i - 5, \alpha_i - 6 \in \alpha^{-\epsilon}$. So α contains at least two of the integers $\alpha_i - 4, \alpha_i - 5$ and $\alpha_i - 6$. But $\alpha_i - 6 \notin \alpha$ by Proposition 4.16, so $\alpha_i - 4 \in \alpha$ as required.

Now consider the case $\alpha_i = 5$ (and hence $\epsilon = 0$). Since any part congruent to 2 or 3 modulo 4 has final node of spin-residue 1, the parts of α congruent to 2 modulo 4 are precisely the same as the parts of α^{-0} congruent to 2 modulo 4. Since α^{-0} is 4-stepped, these parts must therefore be $2, 6, \dots, 4r - 2$ for some $r \geq 0$. But by Lemma 4.22 we have $r \leq 1$. Thus α is of the form $\alpha = (4k, 4k - 4, \dots, 8, 5, 4)$ or $\alpha = (4k, 4k - 4, \dots, 8, 5, 4, 2)$ for some $k \geq 1$, contradicting Proposition 4.21. \square

Proposition 4.24. *If α_i is even and $\alpha_i > 4$, then $\alpha_i - 4 \in \alpha$.*

Proof. Since $\alpha \setminus \pi$ is 4-stepped, the claim holds for all even parts except possibly the largest. So assume α_i is the largest even part of α . Let ϵ be the spin-residue of the node (i, α_i) .

By Corollary 4.17(ii), it suffices to show that either $\alpha^{-\epsilon}$ or $\alpha^{-\bar{\epsilon}}$ has a part equal to $\alpha_i - 4$. Since $\alpha_i \in \alpha^{-\bar{\epsilon}}$, it also suffices to show α has a $\bar{\epsilon}$ -spin-removable node, for then the inductive hypothesis implies that $\alpha^{-\bar{\epsilon}}$ is 4-stepped and hence $\alpha_i - 4 \in \alpha^{-\bar{\epsilon}}$.

If α has any part congruent to $\alpha_i - 1 \pmod{4}$, then α has a $\bar{\epsilon}$ -spin-removable node by Corollary 4.17(i), and we are done. Meanwhile, if α has any part $\alpha_j \equiv \alpha_i - 2 \pmod{4}$ with $\alpha_j - 1 \notin \alpha$, then α has a $\bar{\epsilon}$ -spin-removable node, so again we are done.

Consider the case $i > 1$. Then α_{i-1} is odd and $\alpha_{i-1} - \alpha_i \leq 4$ (by Proposition 4.11(i)), so $\alpha_{i-1} \in \{\alpha_i + 1, \alpha_i + 3\}$. If $\alpha_{i-1} = \alpha_i + 3$ then we can use the argument from the last paragraph, so assume $\alpha_{i-1} = \alpha_i + 1$. Then by Corollary 4.17(i) α has a ϵ -spin-removable node in row $i - 1$, so the inductive hypothesis implies that $\alpha^{-\epsilon}$ is 4-stepped. Since both $\alpha_i, \alpha_i + 1 \in \alpha$ we have $\alpha_i \in \alpha^{-\epsilon}$, so $\alpha_i - 4 \in \alpha^{-\epsilon}$ and we are done.

So we may assume $i = 1$. Proposition 4.11(i) gives $\alpha_1 - \alpha_2 \leq 4$. Clearly if $\alpha_2 = \alpha_1 - 4$ the lemma holds, while if $\alpha_2 = \alpha_1 - 1$ we can use the argument from earlier in the proof. So assume $\alpha_2 \in \{\alpha_1 - 2, \alpha_1 - 3\}$.

Suppose $\alpha_2 = \alpha_1 - 3$. Then α_2 is odd and, using Lemma 4.10(i), we have $\alpha_3 \in \{\alpha_1 - 4, \alpha_1 - 5\}$. But we cannot have $\alpha_1 - 3, \alpha_1 - 5 \in \alpha$ by Proposition 4.16, so $\alpha_3 = \alpha_1 - 4$ and we are done.

We are left with the case $\alpha_2 = \alpha_1 - 2$. Then $\alpha_1 - 2 \in \alpha \setminus \pi$, and hence $\alpha_1 - 2, \alpha_1 - 6, \dots \in \alpha$. From the third paragraph of the proof, we can also assume $\alpha_1 - 3, \alpha_1 - 7, \dots \in \alpha$. Thus if the proposition is not true, then α is of the form

$$\alpha = (4k + 2, 4k, 4k - 1, 4k - 4, 4k - 5, \dots, 4, 3) \sqcup \gamma$$

where $k \geq 1$, and γ is a strict partition with all parts congruent to 2 (mod 4) and strictly less than $4k - 2$, or

$$\alpha = (4k + 4, 4k + 2, 4k + 1, 4k - 2, 4k - 3, \dots, 2, 1) \sqcup \gamma$$

where $k \geq 1$, and γ is a strict partition with all parts congruent to 0 (mod 4) and strictly less than $4k$. But these are the cases ruled out by Proposition 4.19. \square

This completes the proof of Theorem 4.1.

5 4-stepped-and-semicongruent implies proportional

We now turn towards proving the “if” direction of our main theorem, stated below. We remind the reader that the construction of the partition α_\circ was given in §1.2.

Theorem 5.1 (“If” direction of Theorem 1.1). *Let α be a 4-stepped-and-semicongruent strict partition, and let $\lambda \in \{\alpha_\circ, \alpha'_\circ\}$. Then $\langle\langle \alpha \rangle\rangle$ is proportional to $\overline{\chi(\lambda)}$.*

Our strategy is to first show the result for spin characters which are *homogeneous*, meaning that their composition factors in characteristic 2 are all isomorphic. We then use (linear combinations of) induction and restriction functors to propagate the property of being proportional. For this second step, in this section we only state the actions of our functions on relevant characters and show how they allow us to deduce Theorem 5.1, deferring the proofs of these actions to the following sections.

5.1 Homogeneous case

Our proof starts with homogeneous characters. We recall some notation from §2: κ_a is the 2-core partition $(a, a - 1, \dots, 1)$; $\bar{\kappa}_a$ is the 4-bar-core whose double is κ_a ; and given two partitions σ and τ , we write $[\kappa_a; (\sigma, \tau)]$ for the partition with 2-core κ_a and 2-quotient (σ, τ) .

Proposition 5.2. *Let $\alpha = \bar{\kappa}_a \sqcup 2\kappa_r$ for some integers $a, r \geq 0$ with $a \geq r - 1$. Then:*

- (i) $\langle\langle \alpha \rangle\rangle$ is homogeneous;
- (ii) $\overline{\chi(\alpha_\circ)}$ is irreducible;
- (iii) $\alpha_\circ = \text{dbl}(\alpha)^{\text{reg}}$;
- (iv) $\langle\langle \alpha \rangle\rangle \propto \overline{\chi(\alpha_\circ)}$.

Proof.

- (i) The homogeneous spin characters labelled by *separated* partitions were classified in [Fay20, Theorem 5.11]; in fact since the even parts of α are twice a 2-core, we need only use [Fay20, Proposition 5.8] to deduce that $\overline{\langle\langle\alpha\rangle\rangle}$ is homogeneous.
- (ii) The irreducible Specht modules were classified in [JM99]; in fact since $\alpha_\circ = [\kappa_a; (\emptyset, \kappa_r)] = \kappa_a + \kappa_r$ is 2-regular, we need only use [Jam78b, Theorem 2.13] to deduce that $\overline{\chi(\alpha_\circ)}$ is irreducible.
- (iii) It is easily seen that $\text{dbl}(\alpha) = \kappa_a \sqcup \kappa_r \sqcup \kappa_r$. Observe that this partition is 2-restricted (i.e. consecutive parts differ by at most 1), and hence its regularisation is equal to its conjugate. Note also for any partitions λ, μ we have $(\lambda \sqcup \mu)' = \lambda' + \mu'$. Thus

$$\text{dbl}(\alpha)^{\text{reg}} = (\kappa_a \sqcup \kappa_r \sqcup \kappa_r)' = \kappa'_a + \kappa'_r + \kappa'_r = \kappa_a + 2\kappa_r = \alpha_\circ.$$

- (iv) By Theorem 2.4 and part (iii) of the present proposition, $\overline{\langle\langle\alpha\rangle\rangle}$ has least dominant composition factor given by $\varphi(\text{dbl}(\alpha)^{\text{reg}}) = \varphi(\alpha_\circ)$. Then since $\overline{\langle\langle\alpha\rangle\rangle}$ is homogeneous by (i), all its composition factors are isomorphic to $\varphi(\alpha_\circ)$. Finally by (ii) we have $\varphi(\alpha_\circ) = \overline{\chi(\alpha_\circ)}$, giving the result. \square

Remark 5.3. In fact, the spin characters $\langle\langle\alpha\rangle\rangle$ appearing in the above proposition are the only homogeneous spin characters proportional to a linear character in characteristic 2, with the exception of $\alpha = (4)$ (which satisfies $\overline{\langle\langle(4)\rangle\rangle} \propto \overline{\chi(2,2)} = \varphi(3,1)$). Indeed, suppose $\langle\langle\alpha\rangle\rangle$ is homogeneous and proportional to $\overline{\chi(\lambda)}$. Then $\overline{\chi(\lambda)}$ is also homogeneous, and hence $\overline{\chi(\lambda)} = \varphi(\lambda^{\text{reg}})$ is irreducible (since we know $\varphi(\lambda^{\text{reg}})$ occurs with multiplicity 1). James and Mathas's classification of irreducible linear characters [JM99] says that $\overline{\chi(\lambda)}$ is irreducible if and only if λ or λ' is 2-Carter (a property which implies 2-regular) or $\lambda = (2,2)$.

On the other hand, our main theorem gives $\lambda \in \{\alpha_\circ, \alpha'_\circ\}$ and hence both components of the 2-quotient of λ are 2-cores. If both of these 2-cores are nonempty, then in an abacus display for λ or λ' there are beads in adjacent positions, and so λ and λ' are both 2-singular (that is, having repeated parts) and hence not 2-Carter. Furthermore if the nonempty component of the 2-quotient has length greater than the length of the 2-core by 2 or more, then again an abacus display for λ or λ' has beads in adjacent positions. Thus the only possibility (up to conjugacy) is $\lambda = [\kappa_a; (\emptyset, \kappa_r)]$ for some $a \geq r - 1$ (which is precisely the situation of the lemma) or $\lambda = (2,2)$ (in which case $\alpha = (4)$).

5.2 Application of runner-swapping and quotient-redistributing functions

In order to prove Theorem 5.1 from Proposition 5.2, we apply certain combinations of the induction and restriction functors e_i, f_i . They are both functions on the Grothendieck group, or equivalently on generalised characters.

Definition 5.4. For $c \in \mathbb{Z}$ and $\epsilon \in \{0,1\}$, define the *runner-swapping function* to be

$$S_\epsilon^{(c)} = \sum_{a \geq \max\{0, -c\}} (-1)^{a+c} f_\epsilon^{(a+c)} e_\epsilon^{(a)}.$$

Definition 5.5. For $d \in \mathbb{Z}$ and $\epsilon \in \{0,1\}$, define the *quotient-redistributing function* to be

$$R_\epsilon^{(d)} = \sum_{a \geq \max\{0, -d\}} (-1)^{a+d} f_\epsilon^{(a+d)} f_{\bar{\epsilon}}^{(a+d)} e_{\bar{\epsilon}}^{(a)} e_\epsilon^{(a)}.$$

The actions of these functions on certain partitions of interest are given by the next four propositions, whose proofs are deferred to subsequent sections.

We begin with the action of the runner-swapping function. The following proposition is proved in a much more general form in §6.1.

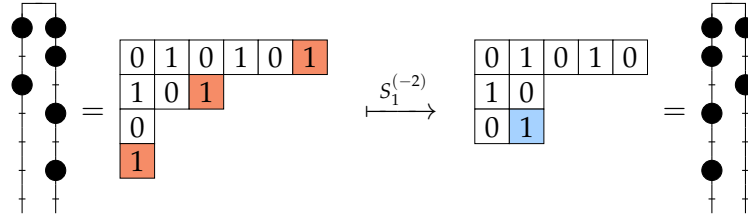
Proposition 5.6 (Action of runner-swapping function on certain characters). *Let $r, s \geq 0$ and let $a \geq 1$. Let ϵ be the residue of $a + 1$ modulo 2. Then*

$$S_\epsilon^{(-a)} \chi([\kappa_a; (\kappa_r, \kappa_s)]) = \pm \chi([\kappa_{a-1}; (\kappa_r, \kappa_s)]).$$

Example 5.7. Take $r = 1, s = 2, a = 2$ and hence $\epsilon = 1$. Then $[\kappa_2; (\kappa_1, \kappa_2)]$ is the partition $\lambda = (6, 3, 1^2)$. The maximum d for which $e_1^{(d)} \chi(\lambda) \neq 0$ is 3, so that

$$\begin{aligned} S_1^{(-2)} \chi(\lambda) &= e_1^{(2)} \chi(\lambda) - f_1 e_1^{(3)} \chi(\lambda) \\ &= \chi(6, 2, 1) + \chi(5, 3, 1) + \chi(5, 2, 1^2) - f_1 \chi(5, 2, 1) \\ &= -\chi(5, 2^2) \end{aligned}$$

and indeed $(5, 2^2) = [\kappa_1; (\kappa_1, \kappa_2)]$. This is illustrated with Young diagrams and abacus displays below; the removed and added nodes are coloured.



We have a corresponding statement for spin characters, proved in §6.2.

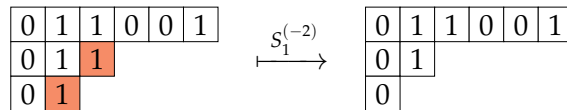
Proposition 5.8 (Action of runner-swapping function on certain spin characters). *Let $r, s \geq 0$ and let $a \geq 1$. Let ϵ be the residue of $a + 1$ modulo 2. Then*

$$S_\epsilon^{(-a)} \langle\langle \bar{\kappa}_a \sqcup 2(\kappa_r + \kappa_s) \rangle\rangle = \pm \langle\langle \bar{\kappa}_{a-1} \sqcup 2(\kappa_r + \kappa_s) \rangle\rangle.$$

Example 5.9. As in Example 5.7, we take $r = 1, s = 2, a = 2$ and hence $\epsilon = 1$. Now $\bar{\kappa}_2 = (3)$, so that $\bar{\kappa}_a \sqcup 2(\kappa_r + \kappa_s)$ is the partition $\alpha = (6, 3, 2)$. Now we can calculate

$$\begin{aligned} S_1^{(-2)} \langle\langle \alpha \rangle\rangle &= e_1^{(2)} \chi(\alpha) - f_1 e_1^{(3)} \chi(\alpha) \\ &= 2\langle\langle 5, 3, 1 \rangle\rangle + \langle\langle 6, 2, 1 \rangle\rangle - \sqrt{2} f_1 \langle\langle 5, 2, 1 \rangle\rangle \\ &= 2\langle\langle 5, 3, 1 \rangle\rangle + \langle\langle 6, 2, 1 \rangle\rangle - (2\langle\langle 6, 2, 1 \rangle\rangle + 2\langle\langle 5, 3, 1 \rangle\rangle) \\ &= -\langle\langle 6, 2, 1 \rangle\rangle \end{aligned}$$

and indeed $(6, 2, 1) = \bar{\kappa}_1 \sqcup 2(\kappa_1 + \kappa_2)$. This is illustrated with Young diagrams below; the removed nodes are coloured.



For the action of the quotient-redistributing function, we restrict attention to (spin) characters in RoCK blocks. Recall that the block with 2-core $\kappa_a = (a, a - 1, \dots, 1)$ and weight w is RoCK if $a \geq w - 1$. The following is proved in §7.1.

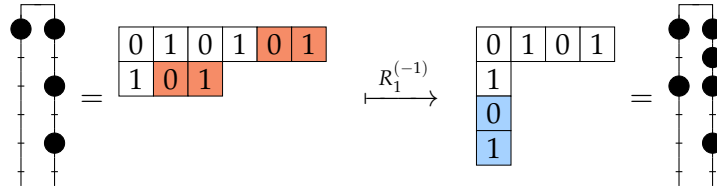
Proposition 5.10 (Action of quotient-redistributing function on certain RoCK characters). *Let $s > r \geq 0$ and let $a \geq \frac{1}{2}(r(r+1) + s(s+1)) - 1$. Let ϵ be the residue of $a+1$ modulo 2. Then*

$$R_\epsilon^{(-(s-r-1))} \chi([\kappa_a; (\kappa_r, \kappa_s)]) = \pm \chi([\kappa_a; (\kappa_{r+1}, \kappa_{s-1})]).$$

Example 5.11. Take $r = 0, s = 2$ and $a = 2$, and hence $\epsilon = 1$. Then $[\kappa_2; (\kappa_0, \kappa_2)]$ is the partition $(6, 3)$. We calculate

$$\begin{aligned} R_\epsilon^{(-1)} \chi(6, 3) &= e_0^{(1)} e_1^{(1)} \chi(6, 3) - f_1^{(1)} f_0^{(1)} e_0^{(2)} e_1^{(2)} \chi(6, 3) \\ &= \chi(6, 1) + \chi(4, 3) - f_1^{(1)} f_0^{(1)} \chi(4, 1) \\ &= -\chi(4, 1^3) \end{aligned}$$

and indeed $(4, 1^3) = [\kappa_1; (\kappa_1, \kappa_1)]$. This is illustrated with Young diagrams and abacus displays below; the removed and added nodes are coloured.



The corresponding statement for spin characters below is proved in §7.2.

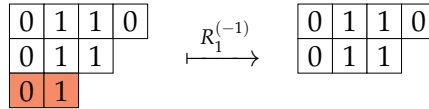
Proposition 5.12 (Action of quotient-redistributing function on certain RoCK spin characters). *Let $s > r \geq 0$ and let $a \geq \frac{1}{2}(r(r+1) + s(s+1)) - 1$. Let ϵ be the residue of $a+1$ modulo 2. Then*

$$R_\epsilon^{(-(s-r-1))} \langle \bar{\kappa}_a \sqcup 2(\kappa_r + \kappa_s) \rangle = \pm \sqrt{2} \langle \bar{\kappa}_a \sqcup 2(\kappa_{r+1} + \kappa_{s-1}) \rangle.$$

Example 5.13. As in Example 5.11, we take $r = 0, s = 2, a = 2$ and hence $\epsilon = 1$. Now $\bar{\kappa}_2 \sqcup 2(\kappa_0 + \kappa_2)$ is the strict partition $(4, 3, 2)$. We calculate

$$\begin{aligned} R_1^{(-1)} \langle 4, 3, 2 \rangle &= e_0^{(1)} e_1^{(1)} \langle 4, 3, 2 \rangle - f_1^{(1)} f_0^{(1)} e_0^{(2)} e_1^{(2)} \langle 4, 3, 2 \rangle \\ &= \sqrt{2} \langle 4, 3 \rangle - \sqrt{2} f_1^{(1)} f_0^{(1)} \langle 3, 2 \rangle \\ &= \sqrt{2} \langle 4, 3 \rangle - 2\sqrt{2} \langle 4, 3 \rangle \\ &= -\sqrt{2} \langle 4, 3 \rangle \end{aligned}$$

and indeed $(4, 3) = \bar{\kappa}_2 \sqcup 2(\kappa_1 + \kappa_1)$. This is illustrated with Young diagrams below; the removed nodes are coloured.



These allow us to complete the goal of this section, restated below.

Theorem 5.1 (“If” direction of Theorem 1.1). *Let α be a 4-stepped-and-semicongruent strict partition, and let $\lambda \in \{\alpha_\circ, \alpha'_\circ\}$. Then $\langle \alpha \rangle$ is proportional to $\chi(\lambda)$.*

Proof. By assumption there exist non-negative integers a, r, s with $s \geq r$ such that $\alpha = \bar{\kappa}_a \sqcup 2(\kappa_r + \kappa_s)$ and $\lambda = [\kappa_a; (\kappa_r, \kappa_s)]$. Choose $b \in \mathbb{N}$ such that $b \geq a$ and $b \geq |\kappa_{r+s}| - 1$. Applying Proposition 5.2 with $\alpha = \bar{\kappa}_b \sqcup 2\kappa_{r+s}$, we obtain

$$\langle \bar{\kappa}_b \sqcup 2\kappa_{r+s} \rangle \propto \chi([\kappa_b; (\emptyset, \kappa_{r+s})]).$$

Applying quotient-redistributing functions to each side r times and using Propositions 5.10 and 5.12 iteratively – the choice of b having been made to satisfy the hypotheses of these propositions – we obtain

$$\overline{\langle\langle \bar{\kappa}_b \sqcup 2(\kappa_r + \kappa_s) \rangle\rangle} \propto \overline{\chi([\kappa_b; (\kappa_r, \kappa_s)])}.$$

Then applying runner-swapping functions to each side $b - a$ times and using Propositions 5.6 and 5.8 iteratively yields

$$\overline{\langle\langle \bar{\kappa}_a \sqcup 2(\kappa_r + \kappa_s) \rangle\rangle} \propto \overline{\chi([\kappa_a; (\kappa_r, \kappa_s)])}$$

as required. \square

Remark 5.14. For characters in RoCK blocks, a possible alternative to our application of quotient-redistributing functions is to use the first author's explicit formulæ for the decomposition numbers of RoCK blocks in terms of symmetric functions [Fay20, Theorem 5.3]. Using these formulæ, it can be shown that Theorem 5.1 for RoCK blocks is equivalent to the following statement about symmetric functions: if σ and τ are 2-cores, then

$$P_{\sigma+\tau} = s_\sigma s_\tau,$$

where P_α denotes the Schur P-function labelled by a strict partition α , and s_λ is the Schur function corresponding to a partition λ (see Macdonald's book [Mac95] for basic definitions regarding symmetric functions). We have not been able to find this result in the literature; our main theorem yields a proof of this identity.

6 Runner-swapping functions

The purpose of this section is to describe the action of the runner-swapping functions $S_\epsilon^{(c)}$ on certain characters and spin characters. In fact we will establish the action in much more generality than that required in §5.2: we identify, for all characters and a large class of spin characters, a choice of parameters for which the image is (up to sign) a specified single (spin) character.

We recall the definition of the function from §5.

Definition 5.4. For $c \in \mathbb{Z}$ and $\epsilon \in \{0, 1\}$, define the *runner-swapping function* to be

$$S_\epsilon^{(c)} = \sum_{a \geq \max\{0, -c\}} (-1)^{a+c} f_\epsilon^{(a+c)} e_\epsilon^{(a)}.$$

6.1 Action on characters

First we consider the effect of $S_\epsilon^{(c)}$ on characters $\chi(\lambda)$. We prove the following result (recall for $r \geq 0$ we write κ_r for the 2-core partition $(r, r-1, \dots, 1)$, and given a bipartition $\underline{\lambda}$, we write $[\kappa_r; \underline{\lambda}]$ for the partition with 2-core κ_r and 2-quotient $\underline{\lambda}$).

Theorem 6.1. *Let $\lambda = [\kappa_a; \underline{\lambda}]$ be any partition.*

(i) *Let ϵ be the residue of a modulo 2. Then*

$$S_\epsilon^{(a+1)} \chi([\kappa_a; \underline{\lambda}]) = \pm \chi([\kappa_{a+1}; \underline{\lambda}]).$$

(ii) *Suppose $a \geq 1$, and let ϵ be the residue of $a+1$ modulo 2. Then*

$$S_\epsilon^{(-a)} \chi([\kappa_a; \underline{\lambda}]) = \pm \chi([\kappa_{a-1}; \underline{\lambda}]).$$

The special case of Theorem 6.1(ii) in which $\underline{\lambda} = (\kappa_r, \kappa_s)$ gives Proposition 5.6 required in the proof of our main theorem.

To prepare for the proof, we introduce some notation. Recall that \mathcal{D}_d is the d th diagonal in \mathbb{N}^2 . For a partition λ and $d \in \mathbb{Z}$, we write

$$\iota_d(\lambda) = \begin{cases} 1 & \text{if } \lambda \text{ has an addable node in } \mathcal{D}_d; \\ -1 & \text{if } \lambda \text{ has a removable node in } \mathcal{D}_d; \\ 0 & \text{otherwise.} \end{cases}$$

If μ is another partition, we write $\mu -_d \lambda$ for the number of nodes of μ in \mathcal{D}_d minus the number of nodes of λ in \mathcal{D}_d .

Lemma 6.2. *Let λ be a partition, $c \in \mathbb{Z}$, and $\epsilon \in \{0, 1\}$. If μ is a partition such that $\chi(\mu)$ appears with non-zero coefficient in $S_\epsilon^{(c)}\chi(\lambda)$, then $\mu -_d \lambda \leq \iota_d(\lambda)$ for all $d \in \mathbb{Z}$ with $d \equiv \epsilon \pmod{2}$.*

Proof. It is clear that if $\chi(\mu)$ appears with non-zero coefficient in $S_\epsilon^{(c)}\chi(\lambda)$ then μ is obtained from λ by removing and then adding ϵ -nodes. In particular, λ and μ agree on \mathcal{D}_d for every $d \not\equiv \epsilon \pmod{2}$. This implies that $-1 \leq \mu -_d \lambda \leq 1$, and $\mu -_d \lambda = 0$ if $\iota_d(\lambda) = 0$. This automatically gives the desired result for all d for which $\iota_d(\lambda) \geq 0$. So we take d with $\iota_d(\lambda) = -1$, and assume for a contradiction that $\mu -_d \lambda = 0$; that is, the removable node of λ in \mathcal{D}_d is also present in μ .

Let Λ be the set of partitions ν such that λ and μ can each be obtained from ν by adding ϵ -nodes. Then Λ is partitioned into pairs $\{\nu^0, \nu^1\}$, where the removable node of λ in \mathcal{D}_d is also a removable node of ν^0 , and ν^1 is obtained from ν^0 by removing this node. Each such pair satisfies $|\nu^1| = |\nu^0| - 1$, so the contributions to $(S_\epsilon^{(c)}\chi(\lambda) : \chi(\mu))$ from ν^0 and ν^1 cancel out. Summing over all such pairs, we find that $(S_\epsilon^{(c)}\chi(\lambda) : \chi(\mu)) = 0$, contrary to assumption. \square

Given a partition λ , let $\lambda^{*\epsilon}$ be the partition obtained from λ by simultaneously adding all the addable ϵ -nodes and removing all the removable ϵ -nodes. We can equivalently characterise $\lambda^{*\epsilon}$ as the unique partition such that:

- ◇ $\lambda^{*\epsilon}$ is obtained from λ by adding and/or removing ϵ -nodes, and
- ◇ $\lambda^{*\epsilon} -_d \lambda = \iota_d(\lambda)$ for every $d \equiv \epsilon \pmod{2}$.

Let $\Delta_\epsilon \lambda$ be the number of addable ϵ -nodes of λ minus the number of removable ϵ -nodes (that is, $\Delta_\epsilon \lambda = \sum_{d \equiv \epsilon \pmod{2}} \iota_d(\lambda) = |\lambda^{*\epsilon}| - |\lambda|$).

Proposition 6.3. *Let λ be a partition and $\epsilon \in \{0, 1\}$. Then $S_\epsilon^{(\Delta_\epsilon \lambda)}\chi(\lambda) = \pm \chi(\lambda^{*\epsilon})$.*

Proof. Suppose $\chi(\mu)$ appears with non-zero coefficient in $S_\epsilon^{(\Delta_\epsilon \lambda)}\chi(\lambda)$. In particular, μ can be obtained from λ by removing then adding ϵ -nodes, with a net gain of $\Delta_\epsilon \lambda$ -many nodes, and so $|\mu| - |\lambda| = \Delta_\epsilon \lambda$. By definition of $\Delta_\epsilon \lambda$, we therefore have

$$|\mu| - |\lambda| = \sum_{\substack{d \equiv \epsilon \\ \pmod{2}}} \iota_d(\lambda).$$

Meanwhile, by considering the net number of nodes added to each diagonal individually, we have

$$|\mu| - |\lambda| = \sum_{\substack{d \equiv \epsilon \\ \pmod{2}}} \mu -_d \lambda.$$

Equating these two expressions for $|\mu| - |\lambda|$ gives

$$\sum_{\substack{d \equiv \epsilon \\ (\text{mod } 2)}} \mu -_d \lambda = \sum_{\substack{d \equiv \epsilon \\ (\text{mod } 2)}} \iota_d(\lambda). \quad (\dagger)$$

But by Lemma 6.2, we have $\mu -_d \lambda \leq \iota_d(\lambda)$ for every $d \equiv \epsilon \pmod{2}$. Thus for (\dagger) to hold, the equality $\mu -_d \lambda = \iota_d(\lambda)$ must hold for every $d \equiv \epsilon \pmod{2}$. Then μ fulfils the characterisation of $\lambda^{*\epsilon}$ given directly above the statement of this proposition, so $\mu = \lambda^{*\epsilon}$.

If μ is any partition obtained from λ by removing then adding ϵ -nodes, then the net number of nodes added (i.e. the number of nodes added, minus the number of nodes removed) is

$$|\mu| - |\lambda| = \sum_{d \equiv \epsilon} \mu -_d \lambda.$$

Now suppose $\chi(\mu)$ appears with non-zero coefficient in $S_\epsilon^{(\Delta_\epsilon \lambda)} \chi(\lambda)$. Then $|\mu| - |\lambda| = \Delta_\epsilon \lambda = \sum_{d \equiv \epsilon} \iota_d(\lambda)$ (the latter equality being by definition). Also, Lemma 6.2 gives $\mu -_d \lambda \leq \iota_d(\lambda)$ for every $d \equiv \epsilon \pmod{2}$, and so we have

$$\sum_{d \equiv \epsilon} \mu -_d \lambda \leq \sum_{d \equiv \epsilon} \iota_d(\lambda).$$

But by definition $\Delta_\epsilon \lambda = \sum_{d \equiv \epsilon} \iota_d(\lambda)$, so the But the sum over all d of $\mu -_d \lambda$ is $|\mu| - |\lambda|$; this is the net number of nodes added by $S_\epsilon^{(\Delta_\epsilon \lambda)}$, which is $\Delta_\epsilon \lambda = \sum_{d \equiv \epsilon \pmod{2}} \iota_d(\lambda)$. Thus we have equality for every d , which forces $\mu = \lambda^{*\epsilon}$.

It remains to determine the coefficient of $\chi(\lambda^{*\epsilon})$ in $S_\epsilon^{(\Delta_\epsilon \lambda)} \chi(\lambda)$. It is easy to see that the only partition ν such that λ and $\lambda^{*\epsilon}$ are both obtained from ν by adding ϵ -nodes is $\lambda \cap \lambda^{*\epsilon}$, the partition obtained from λ by removing all removable ϵ -nodes. Thus

$$\left(S_\epsilon^{(\Delta_\epsilon \lambda)} \chi(\lambda) : \chi(\lambda^{*\epsilon}) \right) = (-1)^{|\lambda^{*\epsilon}| - |\lambda \cap \lambda^{*\epsilon}|}. \quad \square$$

In view of Proposition 6.3, we would like to understand the partition $\lambda^{*\epsilon}$. Another equivalent characterisation of $\lambda^{*\epsilon}$ is via the following standard abacus combinatorics, motivating the name “runner-swapping function” for $S_\epsilon^{(c)}$.

Proposition 6.4. *Let λ be a partition and let $\epsilon \in \{0, 1\}$. An abacus display for $\lambda^{*\epsilon}$ can be obtained by choosing an r -bead abacus display for λ where $r \equiv \bar{\epsilon} \pmod{2}$ and swapping the runners.*

Proof. It is well-known (and easy to see) how to identify addable and removable nodes in an abacus display: with the number of beads being congruent to $\bar{\epsilon}$ modulo 2, addable ϵ -nodes correspond to beads on runner 0 with no bead immediately to the right, and adding these nodes corresponds to moving these beads to the right; dually, removable ϵ -nodes correspond to beads on runner 1 with no bead immediately to the left, and removing these nodes corresponds to moving these beads to the left. So adding all the addable ϵ -nodes and removing all the removable ϵ -nodes can be accomplished on the abacus by moving every bead on runner 0 to the right and every bead on runner 1 to the left, which is the same as swapping the two runners. \square

The abacus interpretation in Proposition 6.4 enables us to describe $\lambda^{*\epsilon}$ in terms of 2-cores and 2-quotients as follows.

Lemma 6.5.

(i) *Let $a \geq 0$ and let $\epsilon \in \{0, 1\}$. Then*

$$\Delta_\epsilon \kappa_a = \begin{cases} a+1 & \text{if } \epsilon \equiv a \pmod{2}; \\ -a & \text{if } \epsilon \not\equiv a \pmod{2}, \end{cases} \quad \kappa_a^{*\epsilon} = \begin{cases} \kappa_{a+1} & \text{if } \epsilon \equiv a \pmod{2}; \\ \kappa_{a-1} & \text{if } \epsilon \not\equiv a \pmod{2}, \end{cases}$$

unless $a = 0$ and $\epsilon = 1$, in which case $\Delta_1 \kappa_0 = 0$ still holds but $(\kappa_0)^{*1} = \kappa_0$.

(ii) Let ν be a 2-core partition, let $\lambda^{(0)}, \lambda^{(1)}$ be partitions and let $\epsilon \in \{0, 1\}$. Then

$$[\nu; (\lambda^{(0)}, \lambda^{(1)})]^{*\epsilon} = [\nu^{*\epsilon}; (\lambda^{(0)}, \lambda^{(1)})]$$

unless $\nu = \emptyset$ and $\epsilon = 1$, in which case $[\emptyset; (\lambda^{(0)}, \lambda^{(1)})]^{*1} = [\emptyset; (\lambda^{(1)}, \lambda^{(0)})]$. In any case, we have $\Delta_\epsilon [\nu; (\lambda^{(0)}, \lambda^{(1)})] = \Delta_\epsilon \nu$.

Proof. Part (i) is routine (with or without Proposition 6.4). For part (ii) we apply Proposition 6.4: write $\lambda = [\nu; (\lambda^{(0)}, \lambda^{(1)})]$ and choose an r -bead abacus display for λ where $r \equiv \bar{\epsilon} \pmod{2}$, so that an r -bead abacus display for $\lambda^{*\epsilon}$ is obtained by swapping the runners. Then clearly the 2-core of $\lambda^{*\epsilon}$ has abacus display given by swapping the runners in an r -bead abacus display for ν , which by Proposition 6.4 again is precisely $\nu^{*\epsilon}$. Meanwhile by our unusual convention for the 2-quotient (§2.6), swapping the runners has no effect on the 2-quotient provided $\nu \neq \emptyset$ or $\epsilon = 0$, for then one runner has strictly more beads than the other and we take $\lambda^{(1)}$ to be given by the runner with more beads. In the case $\nu = \emptyset$ and $\epsilon = 1$, however, the runners have equal numbers of beads so $\lambda^{(1)}$ is given by runner 1 before and after swapping, so the components of the 2-quotient are swapped. \square

Theorem 6.1 now follows by combining Proposition 6.3 and Lemma 6.5.

Remark 6.6.

- (i) Our runner-swapping functions are not universal in the sense that for a given pair λ, ϵ , we need to choose a particular value c to obtain $S_\epsilon^{(c)} \chi(\lambda) = \pm \chi(\lambda^{*\epsilon})$. In fact, one can define a universal runner-swapping function S_ϵ such that $S_\epsilon \chi(\lambda) = \pm \chi(\lambda^{*\epsilon})$ for all λ , by

$$S_\epsilon = \exp(-f_\epsilon) \exp(e_\epsilon) \exp(-f_\epsilon) = \sum_{a,b,c \geq 0} (-1)^{a+c} f_\epsilon^{(a)} e_\epsilon^{(b)} f_\epsilon^{(c)}$$

(the expression $\exp(-f_\epsilon) \exp(e_\epsilon) \exp(-f_\epsilon)$ arises from the general theory of \mathfrak{sl}_2 -representations, and is taken from [CR08, §4.2]). But we prefer our family of runner-swapping functions $S_\epsilon^{(c)}$, which afford simpler calculations.

- (ii) In the case where λ is RoCK, so that the runner with more beads has a bead on every row that the other runner does, all but one term in $S_\epsilon^{(\Delta_\epsilon \lambda)} \chi(\lambda)$ vanishes and our function becomes a single (divided power of a) induction or restriction functor. In this case, Scopes showed that the functor in fact defines a Morita equivalence between the relevant blocks [Sco91]. Our application in §5.2, however, is to use runner-swapping functions to reach non-RoCK partitions.

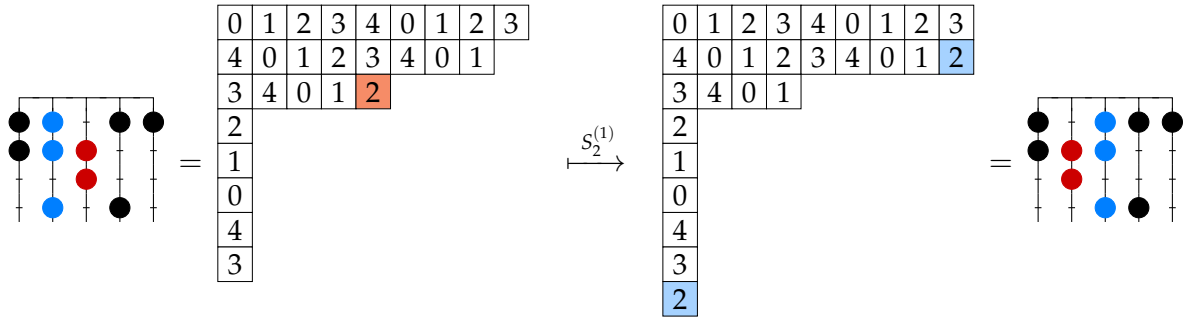
Remark 6.7. The proofs of Lemma 6.2 and Propositions 6.3 and 6.4 apply just as well with residues taken modulo p and abacus displays with p runners, for p an odd prime. That is, we have constructed a function that, with suitable choice of parameters, adds all addable nodes and removes all removable nodes of a given residue, or equivalently swaps a pair of adjacent runners in an abacus display. With the appropriate convention for the p -quotient (called the “ordered p -quotient” in [Fay16], where the reader can find more background on the appropriate combinatorics when p is odd), this function preserves the p -quotient – except (as in the exceptional case in Lemma 6.5(ii)) where the two runners in question have the same number of beads, in which case the corresponding components of the p -quotient are swapped.

For example, let $p = 5$ and consider the function $S_2^{(1)} = \sum_{a \geq 0} (-1)^{(a+1)} f_2^{(a+1)} e_2^{(a)}$ applied to the character labelled by the partition $(9, 8, 5^2, 2, 1^3)$. We have

$$\begin{aligned} S_2^{(1)} \chi(9, 8, 5, 1^5) &= f_2^{(1)} \chi(9, 8, 5, 1^5) - f_2^{(2)} e_2^{(1)} \chi(9, 8, 5, 1^5) \\ &= \chi(9, 8, 5, 1^6) + \chi(9^2, 5, 1^5) - f_2^{(2)} \chi(9, 8, 4, 1^5) \\ &= -\chi(9^2, 4, 1^6). \end{aligned}$$

The partitions $(9, 8, 5, 1^5)$ and $(9^2, 4, 1^6)$ both have 5-quotient $(\emptyset, \emptyset, (1^2), (2), (1))$ (with the appropriate convention), and their 5-cores, namely (2) and (3) , differ in the addition of a 2-node.

This is illustrated with Young diagrams and a choice of abacus display below. The removed and added nodes, and the swapped runners, are coloured. (Here we have chosen a number of beads congruent to 0 modulo 5, so that the parts with final nodes of residue 2 are given on runner 2, and it is runners 1 and 2 that are swapped.)



6.2 Action on spin characters

We next prove a corresponding result for spin characters. This is not quite as general as Theorem 6.1 (it doesn't apply to all strict partitions), though still more general than is needed in §5.2.

Theorem 6.8. *Let $\alpha = \bar{\kappa}_a \sqcup 2\eta$ be a strict partition whose odd parts form a 4-bar-core.*

(i) *Let ϵ be the residue of a modulo 2. Then*

$$S_\epsilon^{(a+1)} \langle\langle \bar{\kappa}_a \sqcup 2\eta \rangle\rangle = \pm \langle\langle \bar{\kappa}_{a+1} \sqcup 2\eta \rangle\rangle.$$

(ii) *Suppose $a \geq 1$, and let ϵ be the residue of $a + 1$ modulo 2. Then*

$$S_\epsilon^{(-a)} \langle\langle \bar{\kappa}_a \sqcup 2\eta \rangle\rangle = \pm \langle\langle \bar{\kappa}_{a-1} \sqcup 2\eta \rangle\rangle.$$

The special case of Theorem 6.8 in which $\eta = \kappa_r + \kappa_s$ gives Proposition 5.8 (required in the proof of our main theorem).

Our proof is similar to that of Theorem 6.1, but (because we need to consider spin-residues rather than residues) we consider the effect of the functor on pairs of adjacent columns rather than individual diagonals.

For $\alpha \in \mathcal{D}$ and an even non-negative integer d , let $i_d(\alpha)$ denote the number of spin-addable nodes in columns d and $d + 1$ minus the number of spin-removable nodes in columns d and $d + 1$. (When $d = 0$, we ignore any reference to column d .) Note that any particular column can contain at most one spin-addable node and at most one spin-removable node, and cannot contain both. Column 1 must contain either a spin-addable or a spin-removable node; meanwhile, for any other column of

residue ϵ , if there is a spin-addable or spin-removable node in that column, then the adjacent column of residue ϵ contains either a spin-addable or spin-removable node (not necessarily of the same type). Thus $i_d(\alpha) \in \{-2, 0, 2\}$ if $d > 0$, and $i_d(\alpha) \in \{-1, 1\}$ if $d = 0$.

If β is another strict partition, let $\beta \ominus_d \alpha$ be the number of nodes of β in columns d and $d+1$ minus the number of nodes of α in these columns, i.e. $\beta \ominus_d \alpha = \beta'_d + \beta'_{d+1} - \alpha'_d - \alpha'_{d+1}$ (ignoring α'_d and β'_d if $d = 0$).

Lemma 6.9. *Suppose $\alpha \in \mathcal{D}$, $c \in \mathbb{Z}$, and $\epsilon \in \{0, 1\}$. If $\beta \in \mathcal{D}$ and $\langle\!\langle\beta\rangle\!\rangle$ appears with non-zero coefficient in $S_\epsilon^{(c)}\langle\!\langle\alpha\rangle\!\rangle$, then $\beta \ominus_d \alpha \leq i_d(\alpha)$ for all $d \geq 0$ with $d \equiv 2\epsilon \pmod{4}$.*

Proof. The fact that $\langle\!\langle\beta\rangle\!\rangle$ appears in $S_\epsilon^{(c)}\langle\!\langle\alpha\rangle\!\rangle$ means that β is obtained from α by removing and then adding ϵ -spin-nodes. This implies in particular that $-2 \leq \beta \ominus_d \alpha \leq 2$ for every d , and $-1 \leq \beta \ominus_0 \alpha \leq 1$. Now it is immediate that $\beta \ominus_d \alpha \leq i_d(\alpha)$ if $i_d(\alpha) > 0$. So we consider d such that $i_d(\alpha) \leq 0$. We assume for a contradiction that $\beta \ominus_d \alpha > i_d(\alpha)$.

Let Γ be the set of strict partitions γ such that α and β can both be obtained from γ by adding ϵ -spin-nodes. Say that two partitions $\gamma^0, \gamma^1 \in \Gamma$ are *equivalent* if they agree outside columns $d, d+1$, i.e. $(\gamma^0)'_k = (\gamma^1)'_k$ for all $k \neq d, d+1$.

Now we consider the possible values of $i_d(\alpha)$ and $\beta \ominus_d \alpha$ satisfying our assumptions. There are four cases.

$i_d(\alpha) = -2$ and $\beta \ominus_d \alpha = 0$: In this case α and β agree in columns $d, d+1$, both having spin-removable nodes in both columns. Then each equivalence class in Γ has the form $\{\gamma^0, \gamma^1, \gamma^2\}$, where $\gamma^i \ominus_d \beta = -i$ for each i . Take such a class, and let $a = |\alpha| - |\gamma^0|$. We claim that

$$\sum_{i=0}^2 (-1)^{a+i+c} \left(e_\epsilon^{(a+i)} \langle\!\langle\alpha\rangle\!\rangle : \langle\!\langle\gamma^i\rangle\!\rangle \right) \left(f_\epsilon^{(a+i+c)} \langle\!\langle\gamma^i\rangle\!\rangle : \langle\!\langle\beta\rangle\!\rangle \right) = 0.$$

Using Theorem 2.6, we find that there are positive real numbers s, t such that

$$\begin{aligned} \left(e_\epsilon^{(a)} \langle\!\langle\alpha\rangle\!\rangle : \langle\!\langle\gamma^0\rangle\!\rangle \right) &= s, & \left(f_\epsilon^{(a+c)} \langle\!\langle\gamma^0\rangle\!\rangle : \langle\!\langle\beta\rangle\!\rangle \right) &= t, \\ \left(e_\epsilon^{(a+1)} \langle\!\langle\alpha\rangle\!\rangle : \langle\!\langle\gamma^1\rangle\!\rangle \right) &= \sqrt{2}s, & \left(f_\epsilon^{(a+1+c)} \langle\!\langle\gamma^1\rangle\!\rangle : \langle\!\langle\beta\rangle\!\rangle \right) &= \sqrt{2}t, \\ \left(e_\epsilon^{(a+2)} \langle\!\langle\alpha\rangle\!\rangle : \langle\!\langle\gamma^2\rangle\!\rangle \right) &= s, & \left(f_\epsilon^{(a+2+c)} \langle\!\langle\gamma^2\rangle\!\rangle : \langle\!\langle\beta\rangle\!\rangle \right) &= t, \end{aligned}$$

which yields the claim. Now summing over all equivalence classes gives $(S_\epsilon^{(c)}\langle\!\langle\alpha\rangle\!\rangle : \langle\!\langle\beta\rangle\!\rangle) = 0$, contrary to hypothesis.

$i_d(\alpha) = -2$ and $\beta \ominus_d \alpha = -1$: In this case α has spin-removable nodes in both columns d and $d+1$, one of which has been removed in β . Then each equivalence class in Γ has the form $\{\gamma^0, \gamma^1\}$, where again $\gamma^i \ominus_d \beta = -i$ for each i , and γ^1 is obtained from γ^0 by removing a node in column d or $d+1$. Now, letting $a = |\alpha| - |\gamma^0|$, there are s, t such that

$$\begin{aligned} \left(e_\epsilon^{(a)} \langle\!\langle\alpha\rangle\!\rangle : \langle\!\langle\gamma^0\rangle\!\rangle \right) &= \sqrt{2}s, & \left(f_\epsilon^{(a+c)} \langle\!\langle\gamma^0\rangle\!\rangle : \langle\!\langle\beta\rangle\!\rangle \right) &= t, \\ \left(e_\epsilon^{(a+1)} \langle\!\langle\alpha\rangle\!\rangle : \langle\!\langle\gamma^1\rangle\!\rangle \right) &= s, & \left(f_\epsilon^{(a+1+c)} \langle\!\langle\gamma^1\rangle\!\rangle : \langle\!\langle\beta\rangle\!\rangle \right) &= \sqrt{2}t, \end{aligned}$$

and again we get $(S_\epsilon^{(c)}\langle\!\langle\alpha\rangle\!\rangle : \langle\!\langle\beta\rangle\!\rangle) = 0$ and a contradiction.

$i_d(\alpha) = -1$ **and** $\beta \ominus_d \alpha > -1$: In this case $d = \epsilon = 0$ and α has a spin-removable node in column 1. This node cannot have been removed in β , or else then $\beta \ominus_d \alpha = -1$; so β also has a spin-removable node in column 1, and $\beta \ominus_d \alpha = 0$. Then each equivalence classes in Γ has the form $\{\gamma^0, \gamma^1\}$, where γ^1 is obtained from γ^0 by removing a node in column 1. Now there are s, t such that

$$\begin{aligned} (e_\epsilon^{(a)} \langle \alpha \rangle : \langle \gamma^0 \rangle) &= s, & (f_\epsilon^{(a+c)} \langle \gamma^0 \rangle : \langle \beta \rangle) &= t, \\ (e_\epsilon^{(a+1)} \langle \alpha \rangle : \langle \gamma^1 \rangle) &= s, & (f_\epsilon^{(a+1+c)} \langle \gamma^1 \rangle : \langle \beta \rangle) &= t, \end{aligned}$$

and we reach a contradiction as in the previous case.

$i_d(\alpha) = 0$ **and** $\beta \ominus_d \alpha > 0$: In this case, columns d and $d+1$ of α contain a spin-addable node and a spin-removable node (if neither column contains a spin-removable or spin-addable node, then necessarily $\beta \ominus_d \alpha = 0$), and the spin-addable node has been added in β . Then each equivalence class in Γ has the form $\{\gamma^1, \gamma^2\}$, where $\gamma^i \ominus_d \beta = -i$ for each i , and γ^2 is obtained from γ^1 by removing a node in column d or $d+1$. Now there are s, t such that

$$\begin{aligned} (e_\epsilon^{(a)} \langle \alpha \rangle : \langle \gamma^1 \rangle) &= s, & (f_\epsilon^{(a+c)} \langle \gamma^1 \rangle : \langle \beta \rangle) &= \sqrt{2}t, \\ (e_\epsilon^{(a+1)} \langle \alpha \rangle : \langle \gamma^2 \rangle) &= \sqrt{2}s, & (f_\epsilon^{(a+1+c)} \langle \gamma^2 \rangle : \langle \beta \rangle) &= t, \end{aligned}$$

and again the terms cancel. □

Given $\alpha \in \mathcal{D}$ and $\epsilon \in \{0, 1\}$, we construct a strict partition $\alpha^{\otimes \epsilon}$ as follows. Examine each pair of adjacent columns with spin-residue ϵ , indexed by d and $d+1$ for some non-negative $d \equiv 2\epsilon \pmod{4}$, in turn: if α has spin-addable nodes but not spin-removable nodes in columns d and $d+1$, then we add these spin-addable nodes; if α has spin-removable nodes but not spin-addable nodes in columns d and $d+1$, then we remove the spin-removable nodes; otherwise we do nothing in columns d and $d+1$. We can equivalently characterise $\alpha^{\otimes \epsilon}$ as the unique strict partition such that:

- ◇ $\alpha^{\otimes \epsilon}$ is obtained from α by adding and/or removing ϵ -spin-nodes, and
- ◇ $\alpha^{\otimes \epsilon} \ominus_d \alpha = i_d(\alpha)$ for every $d \equiv 2\epsilon \pmod{4}$.

Let $\dot{\Delta}_\epsilon \alpha$ be the number of ϵ -spin-addable nodes of α minus the number of ϵ -spin-removable nodes (that is, $\dot{\Delta}_\epsilon \alpha = \sum_{d \equiv 2\epsilon \pmod{4}} i_d(\alpha)$, and also $\dot{\Delta}_\epsilon \alpha = |\alpha^{\otimes \epsilon}| - |\alpha|$).

Proposition 6.10. Suppose $\alpha \in \mathcal{D}$ and $\epsilon \in \{0, 1\}$. Then $S_\epsilon^{(\dot{\Delta}_\epsilon \alpha)} \langle \alpha \rangle = \pm \langle \alpha^{\otimes \epsilon} \rangle$.

Proof. Suppose $\langle \beta \rangle$ appears with non-zero coefficient in $S_\epsilon^{(\dot{\Delta}_\epsilon \alpha)} \langle \alpha \rangle$. Lemma 6.9 gives $\beta \ominus_d \alpha \leq i_d(\alpha)$ for every $d \equiv 2\epsilon \pmod{4}$. But the sum over all d of $\beta \ominus_d \alpha$ is $|\beta| - |\alpha|$; this is the net number of nodes added by $S_\epsilon^{(\dot{\Delta}_\epsilon \alpha)}$, which is $\dot{\Delta}_\epsilon \alpha = \sum_{d \equiv 2\epsilon \pmod{4}} i_d(\alpha)$. Thus we have equality for every d , which forces $\beta = \alpha^{\otimes \epsilon}$.

It remains to determine the coefficient of $\langle \alpha^{\otimes \epsilon} \rangle$ in $S_\epsilon^{(\dot{\Delta}_\epsilon \alpha)} \langle \alpha \rangle$. To do this, let Γ be the set of $\gamma \in \mathcal{D}$ such that α and $\alpha^{\otimes \epsilon}$ can both be obtained by adding ϵ -spin-nodes to γ . The strict partition $\alpha \cap \alpha^{\otimes \epsilon}$, obtained from α by removing the pairs of ϵ -spin-removable nodes in adjacent columns of α (and the ϵ -spin-removable node in column 1, if it exists), is one such partition. Setting $a = |\alpha| - |\alpha \cap \alpha^{\otimes \epsilon}|$, we have from Theorem 2.6

$$(e_\epsilon^{(a)} \langle \alpha \rangle : \langle \alpha \cap \alpha^{\otimes \epsilon} \rangle) = (f_\epsilon^{(a+\dot{\Delta}_\epsilon \alpha)} \langle \alpha \cap \alpha^{\otimes \epsilon} \rangle : \langle \alpha^{\otimes \epsilon} \rangle) = 1.$$

Thus the contribution from $\alpha \cap \alpha^{\otimes \epsilon}$ to the coefficient of $\langle\langle \alpha^{\otimes \epsilon} \rangle\rangle$ is $(-1)^{a+\dot{\Delta}_\epsilon \alpha}$.

To describe Γ fully, let D be the set of all $d \equiv 2\epsilon \pmod{4}$ such that α has both a spin-removable node and a spin-addable node in columns $d, d+1$. For every $E \subseteq D$, define γ_E to be the unique strict partition such that

- ◇ γ_E agrees with $\alpha \cap \alpha^{\otimes \epsilon}$ except in columns $d, d+1$ for $d \in E$, and
- ◇ $\gamma_E \ominus_d \alpha = -1$ for all $d \in E$ (that is, the spin-removable node in column d or $d+1$ has been removed in γ_E for all $d \in E$).

Then $\Gamma = \{\gamma_E \mid E \subseteq D\}$, with $\gamma_\emptyset = \alpha \cap \alpha^{\otimes \epsilon}$ and $|\gamma_E| = |\alpha \cap \alpha^{\otimes \epsilon}| - |E|$ for each E . Furthermore, Theorem 2.6 gives

$$\left(e_\epsilon^{(a+|E|)} \langle\langle \alpha \rangle\rangle : \langle\langle \gamma_E \rangle\rangle\right) = \left(f_\epsilon^{(a+|E|+c)} \langle\langle \gamma_E \rangle\rangle : \langle\langle \alpha^{\otimes \epsilon} \rangle\rangle\right) = \sqrt{2}^{|E|}.$$

Summing over E , we obtain

$$\begin{aligned} \left(S_\epsilon^{(\dot{\Delta}_\epsilon \alpha)} \langle\langle \alpha \rangle\rangle : \langle\langle \alpha^{\otimes \epsilon} \rangle\rangle\right) &= (-1)^{a+\dot{\Delta}_\epsilon \alpha} \sum_{E \subseteq D} (-2)^{|E|} \\ &= (-1)^{a+\dot{\Delta}_\epsilon \alpha} \sum_{k=0}^{|D|} \binom{|D|}{k} (-2)^k \\ &= (-1)^{a+\dot{\Delta}_\epsilon \alpha} (1-2)^{|D|} \\ &= (-1)^{a+\dot{\Delta}_\epsilon \alpha+|D|}. \end{aligned}$$

□

In view of Proposition 6.10, we would like to understand the partition $\alpha^{\otimes \epsilon}$.

Lemma 6.11. *Suppose $\alpha \in \mathcal{D}$ and $\epsilon \in \{0, 1\}$. Then $\alpha^{\otimes \epsilon}$ is obtained from α by:*

- ◇ replacing each part $d \equiv 2\epsilon - 1 \pmod{4}$ with $d+2$,
- ◇ replacing each part $d > 1$ such that $d \equiv 2\epsilon + 1 \pmod{4}$ with $d-2$,
- ◇ inserting the part 1, if $\epsilon = 0$ and $1 \notin \alpha$,
- ◇ removing the part 1, if $\epsilon = 0$ and $1 \in \alpha$

and then reordering parts into decreasing order.

Proof. For each $j > 0$ with $j \equiv 2\epsilon \pmod{4}$ we can consider the two columns $j, j+1$ of spin-removable in isolation and verify the result; there are eight cases to check, depending on which of the integers $j-1, j, j+1$ are parts of α .

When $\epsilon = 0$, we additionally need to consider column 1, and again the result is easy to check. □

Remark 6.12. Lemma 6.11 can alternatively be described in terms of the 4-bar-abacus introduced by Bessenrodt and Olsson [BO97]: it says that the abacus display for $\alpha^{\otimes \epsilon}$ can be obtained from the abacus display for α by swapping runners 1 and 3. This gives further justification to the name “runner-swapping function”. We refer the reader to [BO97, §3] for more details on the 4-bar-abacus.

Now we can give an analogue of Lemma 6.5 for strict partitions.

Lemma 6.13.

(i) Let $a \geq 0$ and let $\epsilon \in \{0, 1\}$. Then

$$\mathring{\Delta}_\epsilon \bar{\kappa}_a = \begin{cases} a+1 & \text{if } \epsilon \equiv a \pmod{2}; \\ -a & \text{if } \epsilon \not\equiv a \pmod{2}, \end{cases} \quad \bar{\kappa}_a^{\otimes \epsilon} = \begin{cases} \bar{\kappa}_{a+1} & \text{if } \epsilon \equiv a \pmod{2}; \\ \bar{\kappa}_{a-1} & \text{if } \epsilon \not\equiv a \pmod{2}, \end{cases}$$

unless $a = 0$ and $\epsilon = 1$, in which case $\mathring{\Delta}_1 \bar{\kappa}_0 = 0$ still holds but $(\bar{\kappa}_0)^{\otimes 1} = \bar{\kappa}_0$.

(ii) Let $\gamma \in \mathcal{D}$ with all parts odd, let $\eta \in \mathcal{D}$ and let $\epsilon \in \{0, 1\}$. Then

$$(\gamma \sqcup 2\eta)^{\otimes \epsilon} = \gamma^{\otimes \epsilon} \sqcup 2\eta$$

and hence $\mathring{\Delta}_\epsilon(\gamma \sqcup 2\eta) = \mathring{\Delta}_\epsilon \gamma$.

Proof. Part (i) is routine, and part (ii) follows immediately from Lemma 6.11. □

Theorem 6.8 now follows by combining Proposition 6.10 and Lemma 6.13.

Remark 6.14. It is interesting to ask what we get when we apply the function $S_\epsilon^{(c)}$ for values of c other than $\Delta_\epsilon \lambda$ or $\mathring{\Delta}_\epsilon \alpha$.

First consider a partition λ . Lemma 6.2 (and the argument at the start of the proof of Proposition 6.3) shows that if $c > \Delta_\epsilon \lambda$, then $S_\epsilon^{(c)} \chi(\lambda) = 0$. On the other hand, if r_ϵ denotes the number of removable ϵ -nodes of λ , then clearly $S_\epsilon^{(c)} \chi(\lambda) = 0$ for $c < -r_\epsilon$, whilst $S_\epsilon^{(-r_\epsilon)} \chi(\lambda) = \pm \chi(\lambda^{-\epsilon})$, where $\lambda^{-\epsilon}$ is the partition obtained by removing all the removable ϵ -nodes from λ . Meanwhile if $-r_\epsilon < c < \Delta_\epsilon \lambda$, it is not hard to show that $S_\epsilon^{(c)} \chi(\lambda)$ is a linear combination of two or more different characters.

Now consider $\alpha \in \mathcal{D}$. If $c > \mathring{\Delta}_\epsilon \alpha$, then Lemma 6.9 shows that $S_\epsilon^{(c)} \langle\langle \alpha \rangle\rangle = 0$. On the other hand, if r_ϵ denotes the number of ϵ -spin-removable nodes of α , then $S_\epsilon^{(c)} \langle\langle \alpha \rangle\rangle = 0$ for $c < -r_\epsilon$, whilst $S_\epsilon^{(-r_\epsilon)} \langle\langle \alpha \rangle\rangle$ is a non-zero multiple of $\alpha^{-\epsilon}$, where $\alpha^{-\epsilon}$ is the partition obtained by removing all the ϵ -spin-removable nodes from α . However, this multiple need not be ± 1 ; for example, $S_1^{(-1)} \langle\langle (2) \rangle\rangle = \sqrt{2} \langle\langle (1) \rangle\rangle$. Meanwhile if $-r_\epsilon < c < \mathring{\Delta}_\epsilon \alpha$, then $S_\epsilon^{(c)} \langle\langle \alpha \rangle\rangle$ is a linear combination of two or more characters, except in some cases where $c = \mathring{\Delta}_\epsilon \alpha - 1$. We leave the reader to work out the details.

7 Quotient-redistributing functions

The purpose of this section is to describe the action of the quotient-redistributing function on certain RoCK characters and spin characters. We recall the definition of the function from §5.

Definition 5.5. For $d \in \mathbb{Z}$ and $\epsilon \in \{0, 1\}$, define the *quotient-redistributing function* to be

$$R_\epsilon^{(d)} = \sum_{a \geq \max\{0, -d\}} (-1)^{a+d} f_\epsilon^{(a+d)} f_{\bar{\epsilon}}^{(a+d)} e_{\bar{\epsilon}}^{(a)} e_{\epsilon}^{(a)}.$$

7.1 Action on characters

We will apply the quotient-redistributing function to characters in RoCK blocks. We say that a partition λ is RoCK if the character $\chi(\lambda)$ lies in a RoCK block; that is, if $w \leq c + 1$, where λ has 2-weight w and 2-core $(c, c - 1, \dots, 1)$. In fact the structure of RoCK partitions is quite easy to understand; they were first analysed by James and Mathas in [JM96] (where they were called “partitions with enormous 2-core”). Recall that if λ is a partition, then we write $\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)})$ for its 2-quotient.

If λ is RoCK, then λ is obtained from its 2-core by adding $\lambda_i^{(1)}$ horizontal dominoes (i.e. pairs of orthogonally adjacent nodes) in row i , and adding $(\lambda^{(0)})'_i$ vertical dominoes in column i , for each i . The RoCK condition ensures that the horizontal and vertical dominoes do not meet. This structure is illustrated in Figure 3.

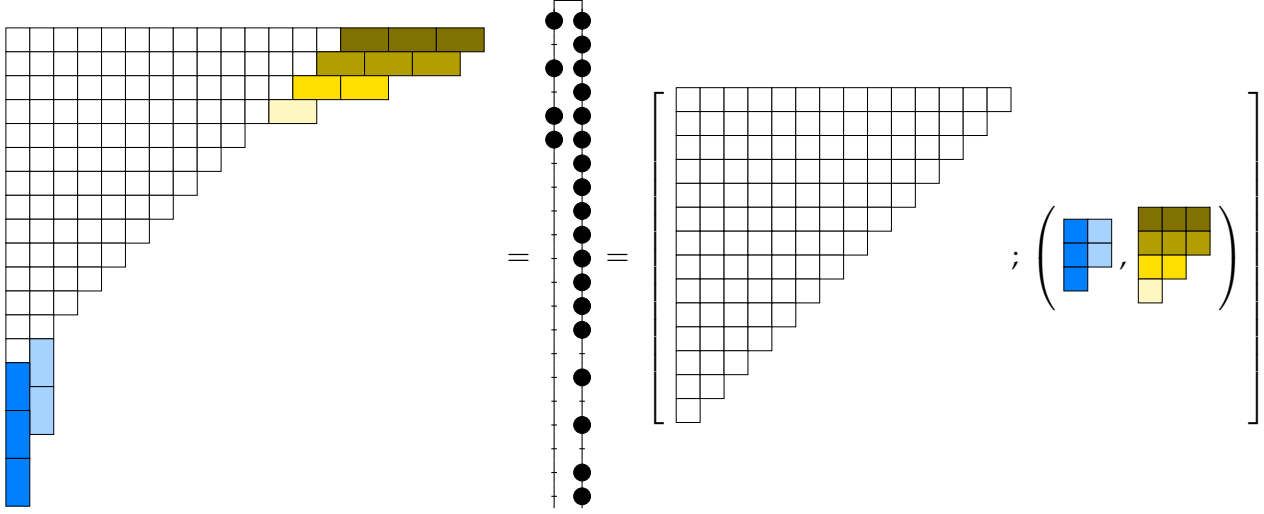


Figure 3: The RoCK partition $(20, 19, 16, 13, 10, 9, 8, 7, 6, 5, 4, 3, 2^5, 1^3)$, depicted as a Young diagram, on an abacus, and in terms of its 2-core and 2-quotient. Observe the correspondence between nodes in the components of the 2-quotient and “dominoes” in the Young diagram indicated by the colouring.

We introduce some notation to help us describe the action of the quotient-redistributing function on RoCK partitions.

Definition 7.1. Given bipartitions $\underline{\lambda}$ and $\underline{\mu}$, write $\underline{\lambda} \rightarrow \underline{\mu}$ if $\mu^{(0)}$ can be obtained from $\lambda^{(0)}$ by adding nodes in distinct columns, and $\mu^{(1)}$ can be obtained from $\lambda^{(1)}$ by adding nodes in distinct rows.

Now the following result comes from the branching rule (after some translation of notation, part (ii) is effectively a special case of [CT02, Lemma 3.1]).

Lemma 7.2. Let λ be a partition with 2-core ν and 2-weight w . Let $r \geq 0$. Let ϵ be the residue of $l(\nu) + 1$ modulo 2.

(i) Suppose $w \leq l(\nu) + 1$ (that is, that λ is RoCK).

$$\left(e_{\epsilon}^{(r)} e_{\epsilon}^{(r)} \chi(\lambda) : \chi(\mu) \right) = \begin{cases} 1 & \text{if } \mu \text{ is a partition with 2-core } \nu \text{ and 2-weight } w - r, \text{ and } \underline{\mu} \rightarrow \underline{\lambda}; \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Suppose $w + r \leq l(\nu) + 1$ (that is, that any partition obtained from λ by adding r dominoes is RoCK).

$$\left(f_{\epsilon}^{(r)} f_{\epsilon}^{(r)} \chi(\lambda) : \chi(\mu) \right) = \begin{cases} 1 & \text{if } \mu \text{ is a partition with 2-core } \nu \text{ and 2-weight } w + r, \text{ and } \underline{\mu} \leftarrow \underline{\lambda}; \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the functors $e_{\epsilon}^{(r)} e_{\epsilon}^{(r)}$ and $f_{\epsilon}^{(r)} f_{\epsilon}^{(r)}$ modify only the 2-quotient of the partition labelling the character they act on, and not its 2-core (in fact this is true even if the partition is not RoCK).

We deduce a description of the action of the quotient-redistributing functor.

Definition 7.3. Given bipartitions $\underline{\lambda}$ and $\underline{\mu}$, let

$$\mathcal{I}(\underline{\lambda}, \underline{\mu}) = \{ \underline{\nu} \mid \underline{\lambda} \leftarrow \underline{\nu} \rightarrow \underline{\mu} \}.$$

The following proposition follows immediately from the definitions and Lemma 7.2.

Proposition 7.4. Let λ be a partition with 2-core ν and 2-weight w . Let $d \in \mathbb{Z}$. Let ϵ be the residue of $l(\nu) + 1$ modulo 2. Suppose $\max\{w, w + d\} \leq l(\nu) + 1$ (that is, that λ , and any partition obtained from λ by adding d dominoes if $d > 0$, is RoCK). Then

$$R_{\epsilon}^{(d)} \chi(\lambda) = \sum_{\mu} \left(\sum_{\underline{\nu} \in \mathcal{I}(\underline{\lambda}, \underline{\mu})} (-1)^{|\underline{\mu}/\underline{\nu}|} \right) \chi(\mu)$$

where the sum is over all partitions μ with 2-core ν and 2-weight $w - d$.

It remains, then, to consider $\mathcal{I}(\underline{\lambda}, \underline{\mu})$ for the desired bipartitions. Given $r, s \geq 0$, we write $\underline{\kappa}_{r,s}$ for the bipartition (κ_r, κ_s) .

Lemma 7.5. Let $s > r \geq 0$. Let $\underline{\mu}$ be a bipartition of $|\underline{\kappa}_{r+1,s-1}|$. Then

$$\sum_{\underline{\nu} \in \mathcal{I}(\underline{\kappa}_{r,s}, \underline{\mu})} (-1)^{|\underline{\mu}/\underline{\nu}|} = \begin{cases} (-1)^{r+1} & \text{if } \underline{\mu} = \underline{\kappa}_{r+1,s-1} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\underline{\lambda}$ be any bipartition and let $\underline{\nu} \in \mathcal{I}(\underline{\lambda}, \underline{\mu})$. Suppose there exists a removable node x in $\underline{\lambda}$ which is also present (though not necessarily removable) in $\underline{\mu}$, and furthermore that if x lies in $\underline{\mu}^{(0)}$ then there is no node directly below it in $\underline{\mu}^{(0)}$, while if x lies in $\underline{\mu}^{(1)}$ then there is no node directly to its right in $\underline{\mu}^{(1)}$. Observe that if x is present in $\underline{\nu}$, then it is removable in $\underline{\nu}$, and furthermore the column or row (according to whether x lies in the 0th or 1st component) containing x is unchanged between $\underline{\lambda}$, $\underline{\nu}$ and $\underline{\mu}$, so that $\underline{\nu}/x$ also lies in $\mathcal{I}(\underline{\lambda}, \underline{\mu})$. On the other hand, if x is not present in $\underline{\nu}$, then x is addable to $\underline{\nu}$, and $\underline{\nu} + x$ lies in $\mathcal{I}(\underline{\lambda}, \underline{\mu})$. Thus we can define an involution on the set $\mathcal{I}(\underline{\lambda}, \underline{\mu})$ by adding or removing the node x . This involution also changes the number of nodes added to reach $\underline{\mu}$ from $\underline{\nu}$ by exactly 1, and therefore reverses the sign $(-1)^{|\underline{\mu}/\underline{\nu}|}$ in the sum $\sum_{\underline{\nu} \in \mathcal{I}(\underline{\lambda}, \underline{\mu})} (-1)^{|\underline{\mu}/\underline{\nu}|}$. Thus the contributions cancel out, and the sum vanishes.

Now consider $\underline{\lambda} = \underline{\kappa}_{r,s}$ which contains a removable node in every nonempty row and column. The previous paragraph shows that our sum vanishes unless $\underline{\mu}$ is obtained from $\underline{\kappa}_{r,s}$ in such a way that every nonempty column of κ_r and every nonempty row of κ_s has a node either added or removed. In this case, the number of added and removed nodes amongst the nonempty rows and columns is equal to $r + s$, and so the change in the number of nodes amongst these rows and columns is congruent to $r + s$ modulo 2. But the total change in the number of nodes going from $\underline{\kappa}_{r,s}$ to $\underline{\mu}$ is

$$|\underline{\mu}| - |\underline{\kappa}_{r,s}| = |\underline{\kappa}_{r+1,s-1}| - |\underline{\kappa}_{r,s}| = -(s - r - 1).$$

So there are more nodes added or removed than in just the nonempty rows or columns. The only possibilities are the addition of nodes in the first row (i.e. the empty columns) of κ_r and the addition of nodes in the first column (i.e. the empty rows) of κ_s .

If nodes are added in the first column of κ_s , this means the final nonempty row in κ_s cannot have a node removed, so a node must be added; then the row above that cannot have a node removed, and so on; thus every nonempty row of κ_s has a node added. Then the total change in number of nodes from $\underline{\kappa}_{r,s}$ to $\underline{\mu}$ is at least $(s + 1) - r > 0$, a net gain, contradicting the assumed size of $\underline{\mu}$. Thus nodes must be added in the first row of κ_r , and hence a node must be added in every nonempty

column of κ_r ; then for the total change in number of nodes to be $-(s - r - 1)$ there must be exactly 1 node added to the first row of κ_r and a node removed from every nonempty row of κ_s . This precisely yields $\kappa_{r+1,s-1}$.

It remains to see that $\sum_{\nu \in \mathcal{I}(\kappa_r, \kappa_{r+1,s-1})} (-1)^{|\kappa_{r+1,s-1}/\nu|}$ equals $(-1)^{r+1}$. Indeed, $\kappa_{r,s-1}$ is the unique element of $\mathcal{I}(\kappa_r, \kappa_{r+1,s-1})$, and $|\kappa_{r+1,s-1}/\kappa_{r,s-1}| = r + 1$. \square

Combining Proposition 7.4 and Lemma 7.5 allows us to deduce the action on the characters required in §5, restated below.

Proposition 5.10 (Action of quotient-redistributing function on certain RoCK characters). *Let $s > r \geq 0$ and let $a \geq \frac{1}{2}(r(r+1) + s(s+1)) - 1$. Let ϵ be the residue of $a + 1$ modulo 2. Then*

$$R_\epsilon^{(-(s-r-1))} \chi([\kappa_a; (\kappa_r, \kappa_s)]) = \pm \chi([\kappa_a; (\kappa_{r+1}, \kappa_{s-1})]).$$

7.2 Action on spin characters

Now we come to spin characters, where we also focus on RoCK blocks. We say that $\alpha \in \mathcal{D}$ is RoCK if the character $\langle\!\langle \alpha \rangle\!\rangle$ lies in a RoCK block.

We begin by looking at the structure of 4-semicongruent RoCK strict partitions. Part (i) of the following lemma is immediate; part (ii) is given in [Fay18, Corollaries 5.3 & 5.5].

Lemma 7.6. *Let $\alpha \in \mathcal{D}$ with 4-bar-core γ .*

- (i) *α is 4-semicongruent if and only if $\alpha = (\gamma + 4\sigma) \sqcup 2\eta$ for some $\sigma \in \mathcal{P}, \eta \in \mathcal{D}$.*
- (ii) *α is RoCK if and only if $\alpha = (\gamma + 4\sigma) \sqcup 2\eta$ for some $\sigma \in \mathcal{P}, \eta \in \mathcal{D}$ with $2|\sigma| + |\eta| \leq l(\text{dbl}(\gamma)) + 1$.*

It turns out that the action of the quotient-redistributing function can easily be described in terms of the decomposition given in Lemma 7.6.

We begin with a spin analogue of Lemma 7.2. We continue to use the notation \rightarrow from §7.1, and define the following parameter.

Definition 7.7. Given $\eta, \theta \in \mathcal{D}$ with $\eta \supseteq \theta$, define $k(\eta/\theta)$ to be the number of values $c \geq 1$ such that exactly one of columns c and $c + 1$ contains a node of η/θ .

We note that $k(\eta/\theta)$ can also be characterised as the number of integers which are parts of η or θ but not both.

Proposition 7.8 [Fay20, Proposition 5.6 and its Frobenius reciprocal]. *Suppose $\alpha \in \mathcal{D}$ with 4-bar-core γ and 4-bar-weight w . Write $\alpha = (\gamma + 4\sigma) \sqcup 2\eta$, where $\sigma \in \mathcal{P}, \eta \in \mathcal{D}$ are such that $2|\sigma| + |\eta| = w$. Let $r \geq 0$. Let ϵ be the residue of $l(\text{dbl}(\gamma))$ modulo 2.*

- (i) *Suppose $w \leq l(\text{dbl}(\gamma)) + 1$.*

$$\left(e_\epsilon^{(r)} e_\epsilon^{(r)} \langle\!\langle \alpha \rangle\!\rangle : \langle\!\langle \beta \rangle\!\rangle \right) = \begin{cases} \sqrt{2}^{k(\eta/\theta)} & \text{if } \beta \in \mathcal{D} \text{ with 4-bar-core } \gamma \text{ and 4-bar-weight } w - r, \\ & \text{expressed as } \beta = (\gamma + 4\tau) \sqcup 2\theta \text{ where } (\theta, \tau) \rightarrow (\eta, \sigma); \\ 0 & \text{otherwise.} \end{cases}$$

- (ii) *Suppose $w + r \leq l(\text{dbl}(\gamma)) + 1$.*

$$\left(f_\epsilon^{(r)} f_\epsilon^{(r)} \langle\!\langle \alpha \rangle\!\rangle : \langle\!\langle \beta \rangle\!\rangle \right) = \begin{cases} \sqrt{2}^{k(\eta/\theta)} & \text{if } \beta \in \mathcal{D} \text{ with 4-bar-core } \gamma \text{ and 4-bar-weight } w + r, \\ & \text{expressed as } \beta = (\gamma + 4\tau) \sqcup 2\theta \text{ where } (\theta, \tau) \leftarrow (\eta, \sigma); \\ 0 & \text{otherwise.} \end{cases}$$

In view of this, it is natural to make the following definitions.

Definition 7.9. Given $\eta, \theta \in \mathcal{D}$, let

$$\mathcal{I}_0(\eta, \theta) = \left\{ \zeta \in \mathcal{D} \mid \begin{array}{l} \text{both } \eta \text{ and } \theta \text{ can be obtained from } \zeta \\ \text{by adding nodes in distinct columns} \end{array} \right\}$$

and given $\sigma, \tau \in \mathcal{P}$, let

$$\mathcal{I}_1(\sigma, \tau) = \left\{ \varphi \in \mathcal{P} \mid \begin{array}{l} \text{both } \sigma \text{ and } \tau \text{ can be obtained from } \\ \varphi \text{ by adding nodes in distinct rows} \end{array} \right\}.$$

Let $A_{\sigma, \tau} = |\mathcal{I}_1(\sigma, \tau)|$.

Proposition 7.10. Suppose $\alpha \in \mathcal{D}$ with 4-bar-core γ and 4-bar-weight w , and $d \in \mathbb{Z}$. Let ϵ be the residue of $l(\text{dbl}(\gamma)) + 1$ modulo 2. Suppose $\max\{w, w + d\} \leq l(\text{dbl}(\gamma)) + 1$. Write $\alpha = (\gamma + 4\sigma) \sqcup 2\eta$, where $\eta \in \mathcal{D}$ and $\sigma \in \mathcal{P}$ with $2|\sigma| + |\eta| = w$. Then

$$R_\epsilon^{(d)} \langle\langle \alpha \rangle\rangle = \sum_{\beta} \left(A_{\sigma, \tau} \sum_{\zeta \in \mathcal{I}_0(\eta, \theta)} (-1)^{|\theta/\zeta|} \sqrt{2}^{k(\eta/\zeta) + k(\theta/\zeta)} \right) \langle\langle \beta \rangle\rangle$$

where the sum is over all $\beta \in \mathcal{D}$ with 4-bar-core γ and 4-bar-weight $w + d$ expressed as $\beta = (\gamma + 4\tau) \sqcup 2\theta$.

Proof. It follows from the definitions and Proposition 7.8 that

$$R_\epsilon^{(d)} \langle\langle \alpha \rangle\rangle = \sum_{\beta} \left(\sum_{(\zeta, \varphi) \in \mathcal{I}_0(\eta, \theta) \times \mathcal{I}_1(\sigma, \tau)} (-1)^{|\theta/\zeta| + 2|\tau/\varphi|} \sqrt{2}^{k(\eta/\zeta) + k(\theta/\zeta)} \right) \langle\langle \beta \rangle\rangle$$

where the sum is over all $\beta \in \mathcal{D}$ with 4-bar-core γ and 4-bar-weight $w + d$ expressed as $\beta = (\gamma + 4\tau) \sqcup 2\theta$. This simplifies to the expression in the proposition by noting that the summands of the inner sum do not depend on $\varphi \in \mathcal{I}_1(\sigma, \tau)$. \square

Our next task is to evaluate the sum over $\mathcal{I}_0(\eta, \theta)$ appearing in Proposition 7.10, which we denote

$$B_{\eta, \theta} = \sum_{\zeta \in \mathcal{I}_0(\eta, \theta)} (-1)^{|\theta/\zeta|} \sqrt{2}^{k(\eta/\zeta) + k(\theta/\zeta)}.$$

Proposition 7.11. Suppose $\eta, \theta \in \mathcal{D}$. Let $d = |\theta| - |\eta|$, and let r be the number of parts of η greater than $|d|$. Then

$$B_{\eta, \theta} = \begin{cases} (-1)^r \sqrt{2} & \text{if } \theta = \eta \setminus (-d); \\ (-1)^r & \text{if } \theta = \eta; \\ (-1)^{r+d} \sqrt{2} & \text{if } \theta = \eta \sqcup (d); \\ 0 & \text{otherwise.} \end{cases}$$

We prove Proposition 7.11 by iteratively removing columns from η and θ . Fix η and θ , and for $c \geq 1$, let $\Delta_c = \eta'_c - \theta'_c$ be the difference between the length of the c th columns in η and θ . For any $\zeta \in \mathcal{D}$, let $\tilde{\zeta}$ be the strict partition obtained by deleting the first column.

The iterative step hinges on the following lemma.

Lemma 7.12. Suppose $\eta, \theta \in \mathcal{D}$. Then

$$B_{\eta, \theta} = \begin{cases} B_{\check{\eta}, \check{\theta}} & \text{if } \Delta_1 = \Delta_2 = 1, \text{ or if } \Delta_1 = \Delta_2 = 0 \text{ and } \eta'_1 - \eta'_2 = 0; \\ -B_{\check{\eta}, \check{\theta}} & \text{if } \Delta_1 = \Delta_2 = -1, \text{ or if } \Delta_1 = \Delta_2 = 0 \text{ and } \eta'_1 - \eta'_2 = 1; \\ \sqrt{2} B_{\check{\eta}, \check{\theta}} & \text{if } \Delta_1 = 1 \text{ and } \Delta_2 = 0; \\ -\sqrt{2} B_{\check{\eta}, \check{\theta}} & \text{if } \Delta_1 = -1 \text{ and } \Delta_2 = 0; \\ 0 & \text{if } \Delta_1 = 0 \text{ and } \Delta_2 \neq 0, \text{ or if } |\Delta_c| > 1 \text{ for any } c. \end{cases}$$

Proof. Observe that for any $\zeta \in \mathcal{I}_0(\eta, \theta)$, we must have $\eta'_c, \theta'_c \in \{\zeta'_c, \zeta'_c + 1\}$ for all c . In particular, if $|\Delta_c| > 1$ for any c then $\mathcal{I}_0(\eta, \theta) = \emptyset$ and $B_{\eta, \theta} = 0$ as claimed. For the remainder of the proof, assume $|\Delta_c| \leq 1$ for all c .

For $\zeta \in \mathcal{I}_0(\eta, \theta)$, let $b_{\eta, \theta}^{\zeta} = (-1)^{|\theta/\zeta|} \sqrt{2}^{k(\eta/\zeta) + k(\theta/\zeta)}$ denote the contribution to $B_{\eta, \theta}$ from ζ . Our strategy is to compare $b_{\eta, \theta}^{\zeta}$ and $b_{\check{\eta}, \check{\theta}}^{\check{\zeta}}$. We claim that $\zeta \mapsto \check{\zeta}$ defines a surjective function from $\mathcal{I}_0(\eta, \theta)$ to $\mathcal{I}_0(\check{\eta}, \check{\theta})$, so that our strategy does consider all contributions to $B_{\check{\eta}, \check{\theta}}$. Indeed, it is clear that if $\zeta \in \mathcal{I}_0(\eta, \theta)$ then $\check{\zeta} \in \mathcal{I}_0(\check{\eta}, \check{\theta})$. For surjectivity, consider a strict partition $\delta \in \mathcal{I}_0(\check{\eta}, \check{\theta})$, and observe that $\delta'_1 \leq \eta'_2 \leq \eta'_1$ and also that $\eta'_1 \leq \eta'_2 + 1 \leq \delta'_1 + 2$, and likewise for θ , so that

$$\delta'_1 \leq \eta'_1, \theta'_1 \leq \delta'_1 + 2.$$

Since we assume $|\Delta_1| \leq 1$, we then have that either $\{\eta'_1, \theta'_1\} \subseteq \{\delta'_1, \delta'_1 + 1\}$ and hence adjoining a column of length δ'_1 to δ yields an element of $\mathcal{I}_0(\eta, \theta)$, or $\{\eta'_1, \theta'_1\} \subseteq \{\delta'_1 + 1, \delta'_1 + 2\}$ and hence adjoining a column of length $\delta'_1 + 1$ to δ yields an element of $\mathcal{I}_0(\eta, \theta)$. (Note also that these are the only possible column lengths that can be adjoined to δ to obtain a strict partition, and thus every element of $\mathcal{I}_0(\check{\eta}, \check{\theta})$ has a preimage of size 1 or 2.)

To compare the contributions $b_{\eta, \theta}^{\zeta}$ and $b_{\check{\eta}, \check{\theta}}^{\check{\zeta}}$, we break down into several cases.

$\Delta_1 = 1$: In this case $\zeta'_1 = \theta'_1 = \eta'_1 - 1$ for every $\zeta \in \mathcal{I}_0(\eta, \theta)$, and it follows that $\zeta \mapsto \check{\zeta}$ is a bijection. Clearly $|\theta/\zeta| = |\check{\theta}/\check{\zeta}|$ for every $\zeta \in \mathcal{I}_0(\eta, \theta)$. Note that column 1 contributes to $k(\eta/\zeta)$ if and only if $\eta'_2 = \zeta'_2$, while column 1 contributes to $k(\theta/\zeta)$ if and only if $\theta'_2 = \zeta'_2 + 1$, and thus

$$k(\eta/\zeta) = \begin{cases} k(\check{\eta}/\check{\zeta}) + 1 & \text{if } \eta'_2 = \zeta'_2 \\ k(\check{\eta}/\check{\zeta}) & \text{if } \eta'_2 = \zeta'_2 + 1, \end{cases} \quad k(\theta/\zeta) = \begin{cases} k(\check{\theta}/\check{\zeta}) & \text{if } \theta'_2 = \zeta'_2 \\ k(\check{\theta}/\check{\zeta}) + 1 & \text{if } \theta'_2 = \zeta'_2 + 1, \end{cases}$$

and hence

$$k(\eta/\zeta) + k(\theta/\zeta) = \begin{cases} k(\check{\eta}/\check{\zeta}) + k(\check{\theta}/\check{\zeta}) + 1 & \text{if } \Delta_2 = 0; \\ k(\check{\eta}/\check{\zeta}) + k(\check{\theta}/\check{\zeta}) & \text{if } \Delta_2 = 1. \end{cases}$$

Therefore

$$b_{\eta, \theta}^{\zeta} = \begin{cases} \sqrt{2} b_{\check{\eta}, \check{\theta}}^{\check{\zeta}} & \text{if } \Delta_2 = 0; \\ b_{\check{\eta}, \check{\theta}}^{\check{\zeta}} & \text{if } \Delta_2 = 1. \end{cases}$$

for every $\zeta \in \mathcal{I}_0(\eta, \theta)$. Summing over ζ gives the claimed values when $\Delta_1 = 1$.

$\Delta_1 = -1$: This is very similar to the previous case; now $|\theta/\zeta| = |\check{\theta}/\check{\zeta}| + 1$ for every ζ , and the formulae from the last case apply with η and θ interchanged and a change of sign in the last formula.

$\Delta_1 = 0$ and $\Delta_2 \neq 0$: We consider the case $\Delta_2 = 1$; the case $\Delta_2 = -1$ is obtained by interchanging η and θ in what follows. Having $\Delta_1 = 0$ and $\Delta_2 = 1$ implies $\zeta'_2 = \theta'_2 = \eta'_2 - 1$ for every $\zeta \in \mathcal{I}_0(\eta, \theta)$, while ζ'_1 can be either $\eta'_1 = \theta'_1$ or $\eta'_1 - 1 = \theta'_1 - 1$. So for each $\delta \in \mathcal{I}_0(\check{\eta}, \check{\theta})$ there are two partitions $\zeta^+, \zeta^- \in \mathcal{I}_0(\eta, \theta)$ satisfying $\check{\zeta}^+ = \check{\zeta}^- = \delta$: we obtain ζ^+ by adjoining a column of length η'_1 to δ , and ζ^- by adjoining a column of length $\eta'_1 - 1$. But now $|\theta/\zeta^-| - |\theta/\zeta^+| = 1$ (and also $|\eta/\zeta^-| - |\eta/\zeta^+| = 1$), while

$$k(\eta/\zeta^-) = k(\check{\eta}/\delta) + 1 = k(\eta/\zeta^+) + 1, \quad k(\theta/\zeta^+) = k(\check{\theta}/\delta) + 1 = k(\theta/\zeta^-) + 1,$$

so that $b_{\eta, \theta}^{\zeta^+} + b_{\eta, \theta}^{\zeta^-} = 0$. This applies for all $\delta \in \mathcal{I}_0(\check{\eta}, \check{\theta})$, giving $B_{\eta, \theta} = 0$, as required.

$\Delta_1 = \Delta_2 = 0$ and $\eta'_1 - \eta'_2 = 0$: For each $\delta \in \mathcal{I}_0(\check{\eta}, \check{\theta})$ with $\delta'_1 = \eta'_2$, there is a unique $\zeta \in \mathcal{I}_0(\eta, \theta)$ satisfying $\check{\zeta} = \delta$, obtained by adding a column of length η'_1 to δ . This ζ satisfies

$$|\theta/\zeta| = |\check{\theta}/\delta|, \quad k(\eta/\zeta) = k(\check{\eta}/\delta), \quad k(\theta/\zeta) = k(\check{\theta}/\delta)$$

so that $b_{\eta, \theta}^{\zeta} = b_{\check{\eta}, \check{\theta}}^{\delta}$.

For each $\delta \in \mathcal{I}_0(\check{\eta}, \check{\theta})$ with $\delta'_1 = \eta'_2 - 1$, there are two partitions $\zeta^+, \zeta^- \in \mathcal{I}_0(\eta, \theta)$ satisfying $\check{\zeta}^+ = \check{\zeta}^- = \delta$, obtained from δ by adding columns of length η'_1 and $\eta'_1 - 1$ respectively. These partitions satisfy

$$|\theta/\zeta^-| = |\check{\theta}/\delta| + 1 = |\theta/\zeta^+| + 1,$$

$$k(\eta/\zeta^+) = k(\check{\eta}/\delta) + 1 = k(\eta/\zeta^-) + 1, \quad k(\theta/\zeta^+) = k(\check{\theta}/\delta) + 1 = k(\theta/\zeta^-) + 1,$$

and therefore $b_{\eta, \theta}^{\zeta^+} = 2b_{\check{\eta}, \check{\theta}}^{\delta}$ and $b_{\eta, \theta}^{\zeta^-} = -b_{\check{\eta}, \check{\theta}}^{\delta}$. Hence

$$b_{\eta, \theta}^{\delta^+} + b_{\eta, \theta}^{\delta^-} = b_{\check{\eta}, \check{\theta}}^{\delta}.$$

Now summing over $\mathcal{I}_0(\eta, \theta)$ gives $B_{\eta, \theta} = B_{\check{\eta}, \check{\theta}}$, as required.

$\Delta_1 = \Delta_2 = 0$ and $\eta'_1 - \eta'_2 = 1$: This case is similar to the previous one. Now for each $\delta \in \mathcal{I}_0(\check{\eta}, \check{\theta})$ with $\delta'_1 = \eta'_2 - 1$ there is a unique $\zeta \in \mathcal{I}_0(\eta, \theta)$ such that $\check{\zeta} = \delta$, and this ζ satisfies $b_{\eta, \theta}^{\zeta} = -b_{\check{\eta}, \check{\theta}}^{\delta}$. For each δ with $\delta'_1 = \eta'_2$, there are two partitions $\zeta^+, \zeta^- \in \mathcal{I}_0(\eta, \theta)$ such that $\check{\zeta}^+ = \check{\zeta}^- = \delta$, and these partitions satisfy $b_{\eta, \theta}^{\zeta^+} = b_{\check{\eta}, \check{\theta}}^{\delta}$ and $b_{\eta, \theta}^{\zeta^-} = -2b_{\check{\eta}, \check{\theta}}^{\delta}$. Summing over $\mathcal{I}_0(\eta, \theta)$ then gives $B_{\eta, \theta} = -B_{\check{\eta}, \check{\theta}}$. This case is illustrated Figure 4. \square

Proof of Proposition 7.11. Let $\Delta = (\Delta_1, \Delta_2, \dots)$ be the sequence of column differences between η and θ . Observe that $\eta = \theta \sqcup (-d)$ if and only if $\Delta = (1, \dots, 1, 0, \dots, 0)$ where the number of 1s is $-d$, that $\theta = \eta \sqcup (d)$ if and only if $\Delta = (-1, \dots, -1, 0, \dots, 0)$, where the number of -1 s is d , and trivially $\eta = \theta$ if and only if $\Delta = (0, \dots, 0)$. So we are required to show $B_{\eta, \theta}$ takes the claimed values when Δ is of one of these forms, and is zero otherwise.

We do so using Lemma 7.12 iteratively. The fifth case of Lemma 7.12 tells us that if Δ has any terms strictly greater than 1 in absolute value, or if Δ has any non-zero term following a 0, then $B_{\eta, \theta} = 0$. Meanwhile Δ cannot contain a 1 followed by a -1 , for if $\Delta_c = 1$ then $\eta'_{c+1} \geq \eta'_c - 1 = \theta'_c \geq \theta'_{c+1}$ and hence $\Delta_{c+1} \geq 0$, and similarly Δ cannot contain a -1 followed by a 1. Thus if $B_{\eta, \theta} \neq 0$, then Δ is of the form $(1, \dots, 1, 0, \dots, 0)$, $(-1, \dots, -1, 0, \dots, 0)$ or $(0, \dots, 0)$, where the number of non-zero entries is $|d|$.

If $\Delta = (0, \dots, 0)$, the first and second cases of Lemma 7.12 tell us that, as we remove columns from η and θ , we pick up a sign every time a column length decreases (by 1, the only possibly amount by

ζ		$ \theta/\zeta $	$k(\eta/\zeta)$	$k(\theta/\zeta)$	$b_{\eta,\theta}^\zeta$
(6,5,2,1)		1	2	2	-4
(6,5,2)		2	3	3	8
(6,5,1)		3	3	1	-4
(6,4,2,1)		2	2	4	8
(6,4,2)		3	3	5	-16
(6,4,1)		4	3	3	8
(6,3,2,1)		3	2	2	-4
(6,3,2)		4	3	3	8
(6,3,1)		5	3	1	-4

$\check{\zeta}$		$ \check{\theta}/\check{\zeta} $	$k(\check{\eta}/\check{\zeta})$	$k(\check{\theta}/\check{\zeta})$	$b_{\check{\eta},\check{\theta}}^{\check{\zeta}}$
(5,4,1)		1	2	2	-4
(5,4)		2	3	1	4
(5,3,1)		2	2	4	8
(5,3)		3	3	3	-8
(5,2,1)		3	2	2	-4
(5,2)		4	3	1	4

Figure 4: A comparison of the calculations of $B_{\eta,\theta}$ and $B_{\check{\eta},\check{\theta}}$ when $\eta = (7,6,2,1)$ and $\theta = (6,5,3,1)$ (and hence $\check{\eta} = (6,5,1)$ and $\check{\theta} = (5,4,2)$). This example falls into the final case ($\Delta_1 = \Delta_2 = 0$ and $\eta'_1 - \eta'_2 = 1$) in the proof of Lemma 7.12. On the left are the partitions in $\mathcal{I}_0(\eta, \theta)$, on the right the partitions in $\mathcal{I}_0(\check{\eta}, \check{\theta})$, matched up according to the function $\zeta \mapsto \check{\zeta}$. The Young diagrams of the partitions are drawn, with the nodes which must be added to form η or $\check{\eta}$ shaded blue and the nodes which must be added to form θ or $\check{\theta}$ shaded yellow (and the nodes which must be added in either case shaded green).

which a column length can decrease in a strict partition). Thus $B_{\eta,\theta} = (-1)^{l(\eta)} = (-1)^r$ in this case, as required.

If $\Delta = (1, \dots, 1, 0, \dots, 0)$, the first case of Lemma 7.12 tells us that removing the columns which differ by 1 do not alter the value of $B_{\eta,\theta}$ until the last, when the third case tells us that we pick up a factor of $\sqrt{2}$. Then the remaining columns are those meeting parts greater than $|d|$, so arguing as in the previous paragraph gives $B_{\eta,\theta} = (-1)^r \sqrt{2}$ as required. If $\Delta = (-1, \dots, -1, 0, \dots, 0)$, a similar argument applies but using the second and fourth cases of Lemma 7.12, from which we pick up extra signs for each of the d differing columns, yielding $B_{\eta,\theta} = (-1)^{r+d} \sqrt{2}$ as required. \square

Combining Propositions 7.10 and 7.11 then yields the following. (Recall that given partitions σ and τ , we write $A_{\sigma,\tau} = |\mathcal{I}_1(\sigma, \tau)|$.)

Theorem 7.13. *Suppose $\alpha \in \mathcal{D}$ has 4-bar-core γ and 4-bar-weight w . Let $d \in \mathbb{Z}$. Let ϵ be the residue of $l(\text{dbl}(\gamma)) + 1$ modulo 2. Suppose $\max\{w, w + d\} \leq l(\text{dbl}(\gamma)) + 1$. Write $\alpha = (\gamma + 4\sigma) \sqcup 2\eta$, where*

$\eta \in \mathcal{D}$ and $\sigma \in \mathcal{P}$ with $2|\sigma| + |\eta| = w$. Let r be the number of parts of η greater than $|d|$. Then

$$R_\epsilon^{(d)} \langle\langle \alpha \rangle\rangle = \begin{cases} (-1)^r \sqrt{2} \sum_{\tau} A_{\sigma, \tau} \langle\langle (\gamma + 4\tau) \sqcup 2(\eta \setminus (-d)) \rangle\rangle & \text{if } d < 0 \text{ and } -d \in \eta; \\ (-1)^r \sum_{\tau} A_{\sigma, \tau} \langle\langle (\gamma + 4\tau) \sqcup 2\eta \rangle\rangle & \text{if } d = 0; \\ (-1)^{r+d} \sqrt{2} \sum_{\tau} A_{\sigma, \tau} \langle\langle (\gamma + 4\tau) \sqcup 2(\eta \sqcup (d)) \rangle\rangle & \text{if } d > 0 \text{ and } d \notin \eta; \\ 0 & \text{otherwise;} \end{cases}$$

where in each case the sum is over partitions τ with $|\tau| = |\sigma|$.

Unfortunately it does not seem to be possible to give a simpler expression for $A_{\sigma, \tau}$. Nevertheless, we can deduce the result we need.

Corollary 7.14. Suppose $\alpha \in \mathcal{D}$ with 4-bar-core γ and suppose $\alpha = \gamma \sqcup 2\eta$ for some $\eta \in \mathcal{D}$. Let $d \in \mathbb{Z}$. Let ϵ be the residue of $l(\text{dbl}(\gamma)) + 1$ modulo 2. Suppose $\max\{|\eta|, |\eta| + d\} \leq l(\text{dbl}(\gamma)) + 1$. Then

$$R_\epsilon^{(d)} \langle\langle \alpha \rangle\rangle = \begin{cases} (-1)^r \sqrt{2} \langle\langle \gamma \sqcup 2(\eta \setminus (-d)) \rangle\rangle & \text{if } d < 0 \text{ and } -d \in \eta; \\ (-1)^r \langle\langle \gamma \sqcup 2\eta \rangle\rangle & \text{if } d = 0; \\ (-1)^{r+d} \sqrt{2} \langle\langle \gamma \sqcup 2(\eta \sqcup (d)) \rangle\rangle & \text{if } d > 0 \text{ and } d \notin \eta; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Set $\sigma = \emptyset$ in Theorem 7.13, which forces $\tau = \emptyset$ and hence $A_{\sigma, \tau} = 1$. □

We can now deduce the case required in our application, restated below.

Proposition 5.12 (Action of quotient-redistributing function on certain RoCK spin characters). Let $s > r \geq 0$ and let $a \geq \frac{1}{2}(r(r+1) + s(s+1)) - 1$. Let ϵ be the residue of $a + 1$ modulo 2. Then

$$R_\epsilon^{(-(s-r-1))} \langle\langle \bar{\kappa}_a \sqcup 2(\kappa_r + \kappa_s) \rangle\rangle = \pm \sqrt{2} \langle\langle \bar{\kappa}_a \sqcup 2(\kappa_{r+1} + \kappa_{s-1}) \rangle\rangle.$$

Proof. This follows from Corollary 7.14 by observing that $(\kappa_r + \kappa_s) \setminus (-(s-r-1)) = \kappa_{r+1} + \kappa_{s-1}$. □

The results required in §5 are now all proved, and thus our main theorem is established.

References

- [BO97] C. Bessenrodt & J. B. Olsson, ‘The 2-blocks of the covering groups of the symmetric groups’, *Adv. Math.* **129**(2) (1997), 261–300. [cited on pp. 10, 11, 36]
- [CR08] J. Chuang & R. Rouquier, ‘Derived equivalences for symmetric groups and \mathfrak{sl}_2 -categorification’, *Ann. of Math.* **167** (2008), 245–298. [3, 32]
- [CT02] J. Chuang & K. M. Tan, ‘Some canonical basis vectors in the basic $U_q(\widehat{\mathfrak{sl}}_n)$ -module’, *J. Algebra* **248** (2002), 765–779. [38]
- [Fay14] M. Feyers, ‘A generalisation of core partitions’, *J. Combin. Theory Ser. A* **127** (2014), 58–84. [3]
- [Fay16] M. Feyers, ‘The irreducible representations of the alternating group which remain irreducible in characteristic p' ’, *Trans. Amer. Math. Soc* **368** (2016), 5807–5855. [12, 32]

- [Fay18] M. Fayers, 'Irreducible projective representations of the symmetric group which remain irreducible in characteristic 2', *Proc. London Math. Soc.* **116** (2018), 878–928. [12, 40]
- [Fay20] M. Fayers, 'Irreducible projective representations of the alternating group which remain irreducible in characteristic 2', *Adv. Math.* **374** (2020), 107340. [9, 12, 13, 26, 29, 40]
- [HH92] P. Hoffman & J. Humphreys, *Projective representations of the symmetric groups*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 1992. [8, 9, 15]
- [Jam76] G. D. James, 'On the decomposition matrices of the symmetric groups II', *J. Algebra* **43** (1976), 45–54. [11]
- [Jam78a] G. D. James, *The representation theory of the symmetric groups*, Lecture notes in mathematics **682**, Springer-Verlag, New York/Berlin, 1978. [8, 9]
- [Jam78b] G. D. James, 'On a conjecture of Carter concerning irreducible Specht modules', *Math. Proc. Cambridge Philos. Soc.* **83** (1978), 11–17. [26]
- [JK81] G. James & A. Kerber, *The representation theory of the symmetric group*, Encyclopædia of Mathematics and its Applications **16**, Addison–Wesley, 1981. [12, 15]
- [JM96] G. D. James & A. Mathas, 'Hecke algebras of type A with $q = -1$ ', *J. Algebra* **184** (1996), 102–158. [37]
- [JM99] G. D. James & A. Mathas, 'The Irreducible Specht Modules in Characteristic 2', *Bull. London Math. Soc.* **31** (4), 47–462. [26]
- [Kle05] A. Kleshchev, 'Linear and projective representations of symmetric groups', *Cambridge Tracts in Mathematics* **163** (Cambridge University Press, Cambridge), 2005. [8]
- [Lit51] D. E. Littlewood, 'Modular representations of symmetric groups', *Proc. R. Soc. Lond. A* **209** (1951), 333–353. [11]
- [Mac95] I. Macdonald, *Symmetric functions and Hall polynomials, 2nd Edition*, Oxford University Press, 1995. [29]
- [McD24] E. McDowell, 'Characters and spin characters of alternating and symmetric groups determined by values on l' -classes', *Ark. Mat.* **62(1)** (2024), 127–146. [3, 4]
- [Naz90] M. Nazarov, 'Young's orthogonal form of projective representations of the symmetric group', *J. Lond. Math. Soc.* **42** (1990), 437–451. [8]
- [Sch11] I. Schur, 'Über die Darstellung der symmetrischen und der alternierende Gruppen durch gebrochene lineare Substitutionen', *J. Reine angew. Math.* **139** (1911), 155–250. [8, 9]
- [Sco91] J. Scopes, 'Cartan matrices and Morita equivalence for blocks of the symmetric groups', *J. Algebra* **142** (1991), 441–455. [32]
- [Wil08] M. Wildon, 'Character values and decomposition matrices of symmetric groups', *J. Algebra* **319** (2008), 3382–3397. [3, 4, 14]