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# On the structure of Specht modules

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## Contents



## 1 Introduction

Most of our notation is taken from James's book [7], where further details of the representation theory of the symmetric groups may be found; note, however, that we write functions on the left.

Let $n$ be a non-negative integer, and $\lambda$ a partition of $n$. Say that two $\lambda$-tableaux are *row equivalent* if one can be obtained from the other by permuting the entries within each row, and define column equivalence similarly. Let $\sim_{\mathrm{row}}$ and $\sim_{\mathrm{col}}$ denote these relations.

Given a tableau $s$, define the tabloid $\{s\}$ to be the $\sim_{\mathrm{row}}$-equivalence class containing $s$, and define the subgroups $R_s$ and $C_s$ of $\mathfrak{S}_n$ to be the row and column stabilisers of $s$. For any ring $R$, we define $M^\lambda_R$ to be the $R$-span of the $\lambda$-tabloids, and we define an inner product $\langle,\rangle$ on $M^\lambda_R$ by

$$\langle\{s\},\{t\}\rangle = \begin{cases} 1 & \text{if } \{s\} = \{t\} \\ 0 & \text{if } \{s\} \neq \{t\}. \end{cases}$$

For a $\lambda$-tableau $s$, we also define the elements

$$\rho_s = \sum_{\sigma\in R_s} \sigma, \qquad \kappa_s = \sum_{\sigma\in C_s} (-1)^\sigma \sigma$$

of the group algebra $R\mathfrak{S}_n$, and we define the *polytabloid* $e_s$ to be $\kappa_s\{s\}$. We define the *Specht module* $S^\lambda_R \subseteq M^\lambda_R$ to be the $R$-span of the $\lambda$-polytabloids.

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For any tableaux $s, u$, we define $\pi_{su}$ to be the element of $\mathfrak{S}_n$ taking $s$ to $u$.

Let $p$ be a prime. Schaper's formula (best described in English in [1]) gives information about the filtration

$$S^\lambda_{\mathbb{F}_p} = \overline{S^\lambda_{(0)}} \geqslant \overline{S^\lambda_{(1)}} \geqslant \overline{S^\lambda_{(2)}} \geqslant \dots,$$

where $\overline{S^\lambda_{(i)}}$ is the mod $p$ reduction of the submodule

$$S^\lambda_{(i)} = \{x \in S^\lambda \mid p^i | \langle x, y \rangle \ \forall\ y \in S^\lambda\}$$

of the integral Specht module $S^\lambda_{\mathbb{Z}}$, with the usual inner product $\langle , \rangle$. We define $L_i$ to be the $i$th layer of this filtration:

$$L_i = \overline{S^\lambda_{(i)}} / \overline{S^\lambda_{(i+1)}}.$$

In this paper, we discover some properties of these 'Schaper Layers'. We begin in Section 2 by examining two examples, namely the Specht modules corresponding to hook partitions and to two-part partitions. In Section 3, we address the question of which is the first non-zero layer $L_i$; it is well known that $L_0$ is zero if and only if $\lambda$ is $p$-singular. We prove some general results, and find a necessary and sufficient condition for $L_1$ to be the first non-zero layer.

In Section 4, we examine how the Schaper layers behave under the isomorphism

$$S^{\lambda'} \cong (S^\lambda \otimes \mathrm{sgn})^*,$$

where $\lambda'$ is the partition conjugate to $\lambda$. It turns out that there is a nice description of the behaviour in terms of the product of the hook lengths for $\lambda$.

It is hoped that these results could be extended further (for example, to determining completely which is the first non-zero layer $L_i$), and that they could be applied in conjunction with Schaper's formula to the decomposition number problem.

The topic of this paper is also relevant to the structure of Weyl modules for general linear groups; these have layers defined in an entirely analogous way, and these layers are preserved under the Schur functor. However, we reserve further discussion for a later paper.

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## 2 Two examples

### 2.1 Hook partitions

Suppose that $\lambda = (n-y, 1^y)$. Suppose also that $1 \leqslant y < n-1$ (the corresponding results are slightly different, but trivial, if $\lambda = (n)$ or $\lambda = (1^n)$). The following result is due to Peel [9].

**Lemma 2.1.** *Suppose $p$ is odd. Then the Specht module $S^\lambda_{\mathbb{F}_p}$ has two composition factors if $p$ divides $n$, and is irreducible otherwise.*

The implications for the Schaper layers are clear. Specht modules are indecomposable when $p$ is odd, so $S^{\lambda}_{\mathbb{F}_p}$ has exactly two non-zero layers $L_i$ if $p$ divides $n$, and one otherwise. We proceed to determine which layers these are, as well as showing that this is also true for $p = 2$.

Given $2 \leqslant a_1 < \ldots < a_y \leqslant n$, let $t$ be the standard $\lambda$-tableau whose column entries are $1, a_1, \ldots, a_y$, and let $e_{(a_1,\ldots,a_y)}$ be the corresponding standard polytabloid. For $x \in S^{\lambda}_{\mathbb{Z}}$, let $N(a_1, \ldots, a_y)$ be the coefficient of this polytabloid in $x$. For ease of notation, we extend the definition of $N(a_1, \ldots, a_y)$ to any $a_1, \ldots, a_y \in \{2, \ldots, n\}$ by setting $N(a_{\sigma(1)}, \ldots, a_{\sigma(y)}) = (-1)^{\sigma} N(a_1, \ldots, a_y)$ for $\sigma \in \mathfrak{S}_y$, and defining $N(a_1, \ldots, a_y)$ to be zero if some two $a_i$ are equal. For brevity, we write $N(\mathbf{a})$ for $N(a_1, \ldots, a_y)$, and $N(\mathbf{a}|a_i \to e)$ for $N(a_1, \ldots, a_{i-1}, e, a_{i+1}, \ldots, a_y)$.

For $2 \leqslant a_1 < \ldots < a_y \leqslant n$ and $2 \leqslant b_1 < \ldots < b_y \leqslant n$, we observe that

$$\langle e_{(a_1,\ldots,a_y)}, e_{(b_1,\ldots,b_y)} \rangle = \begin{cases} (y+1)! & \text{if } a_i = b_i \text{ for all } i \\ (-1)^{i-j} y! & \text{if } \{a_1, \ldots, \hat{a_i}, \ldots, a_y\} = \{b_1, \ldots, \hat{b_j}, \ldots, b_y\} \text{ and } a_i \neq b_j \\ 0 & \text{if } |\{a_1, \ldots, a_y\} \cap \{b_1, \ldots, b_y\}| \leqslant y - 2. \end{cases}$$

Hence we have

$$\langle e_{(a_1,\ldots,a_y)}, x \rangle = (y+1)! N(\mathbf{a}) + l! \sum_{i=1}^{y} \sum_{e \neq a_i} N(\mathbf{a}|a_i \to e),$$

so $x$ lies in $S^{\lambda}_{(r)}$ if and only if this is divisible by $p^r$ for all $a_1, \ldots, a_y$.

We begin by providing a bound for the bottom layer. Let $\overline{x} \in S^{\lambda}_{\mathbb{F}_p}$ denote the modular reduction of $x$.

**Lemma 2.2.** *If $r > \nu_p(y!n)$ and $x \in S^{\lambda}_{(r)}$, then $\overline{x} = 0$.*

**Proof.** We have

$$p^r \mid (y+1)! N(\mathbf{a}) + y! \sum_{i=1}^{y} \sum_{e \neq a_i} N(\mathbf{a}|a_i \to e)$$

for all $a_1 < \ldots < a_y$, and in fact it is easily seen that this holds for all $a_1, \ldots, a_y$. We re-write this as

$$p^r y! \mid N(\mathbf{a}) + \sum_{i=1}^{y} \sum_{e=2}^{n} N(\mathbf{a}|a_i \to e);$$

writing this with $(\mathbf{a}|a_i \to e)$ in place of $\mathbf{a}$ gives

$$p^r y! \mid N(\mathbf{a}|a_i \to e) + \sum_{j \neq i}^{y} \sum_{f=2}^{n} N(\mathbf{a}|a_i \to e, a_j \to f) + \sum_{f} N(\mathbf{a}|a_i \to f).$$

We substitute this last expression into the preceding one to give

$$p^r y! \mid N(\mathbf{a}) - \sum_{i=1}^{y} \sum_{j \neq i} \sum_{e=2}^{n} \sum_{f=2}^{n} N(\mathbf{a}|a_i \to e, a_j \to f) - \sum_{e=2}^{n} \sum_{f=2}^{n} \sum_{i=1}^{y} N(\mathbf{a}|a_i \to f).$$

Now for $i \neq j$, $N(\mathbf{a}|a_i \to e, a_j \to f) = -N(\mathbf{a}|a_j \to e, a_i \to f)$, so the $i \neq j$ part of the sum vanishes to give

$$\begin{aligned} p^r y!) \mid N(\mathbf{a}) - (n-1)\sum_{f=2}^{n}\sum_{i=1}^{y} N(\mathbf{a}|a_i \to f) \\ \equiv n.N(\mathbf{a}). \end{aligned}$$

Since $r > \nu_p(y!n)$, we must have $p \mid N(\mathbf{a})$ for all $a_1, \dots, a_y$, so that $\overline{x} = 0$. □

Next we show that there are no non-empty layers $L_i$ for $i$ between $\nu_p(y!)$ and $\nu_p(y!n)$.

**Lemma 2.3.** *If there exists a set of integers*

$$\{m(b_1, \dots, b_{y-1}) \mid 2 \leqslant b_1 < \dots < b_{y-1} \leqslant n\}$$

*such that*

$$N(\mathbf{a}) \equiv \sum_{i=1}^{y}(-1)^i m(a_1, \dots, \hat{a}_i, \dots, a_y) \pmod p$$

*for all* $2 \leqslant a_1 < \dots < a_y \leqslant n$*, then* $\overline{x} \in \overline{S^\lambda_{(\nu_p(y!n))}}$.

*Conversely, if* $x \in S^\lambda_{(\nu_p y!+1)}$*, then such a set of integers* $m(b_1, \dots, b_{y-1})$ *exists.*

**Proof.** Suppose that the integers $m(b_1, \dots, b_{y-1})$ exist as stated. For ease of notation, we define $m(b_1, \dots, b_{y-1})$ for arbitrary $b_1, \dots, b_{y-1} \in \{2, \dots, n\}$ exactly as for the $N(\mathbf{a})$. Since we are only concerned with the reduction modulo $p$ of $x$, we may assume that

$$N(\mathbf{a}) = \sum_{i=1}^{y}(-1)^i m(a_1, \dots, \hat{a}_i, \dots, a_y)$$

for $a_1 < \dots < a_y$. We then have

$$\begin{aligned} \langle e_{(a_1,\dots,a_y)}, x\rangle &= y!(N(\mathbf{a}) + \sum_{i=1}^{y}\sum_{e=2}^{n} N(\mathbf{a}|a_i \to e)) \\ &= y!(N(\mathbf{a}) + \sum_{i=1}^{y}\sum_{j\neq i}\sum_{e=2}^{n}(-1)^j m(a_1, \dots, \hat{a}_j, \dots, a_y|a_i \to e) \\ &\qquad + \sum_{i=1}^{y}\sum_{e=2}^{n}(-1)^i m(a_1, \dots, \hat{a}_i, \dots, a_y)). \end{aligned}$$

But for $i \neq j$ we have

$$(-1)^j m(a_1, \dots, \hat{a}_j, \dots, a_y|a_i \to e) + (-1)^i m(a_1, \dots, \hat{a}_i, \dots, a_y|a_j \to e) = 0,$$

so the $j \neq i$ part of the above sum vanishes to give

$$\langle e_{(a_1,\dots,a_y)}, x\rangle = y!(N(\mathbf{a}) + (n-1)\sum_{i}(-1)^i m(a_1, \dots, \hat{a}_i, \dots, a_y)),$$

which equals

$$y!(n.N(\mathbf{a}));$$

this is clearly divisible by $p^{\nu_p(y!n)}$, which is what we want.

For the second part of the lemma, we have

$$p \mid N(\mathbf{a}) + \sum_{i=1}^{y} \sum_{e=2}^{n} N(\mathbf{a}|a_i \to e);$$

putting

$$m(b_1, \dots, b_{y-1}) = -\sum_{e=2}^{n} N(e, b_1, \dots, b_{y-1})$$

is then sufficient, since $N(\mathbf{a}|a_i \to e) = (-1)^{i-1} N(e, a_1, \dots, \hat{a}_i, \dots, a_y)$. □

From Lemma 2.3, we see that the only layers $L_i$ which can possibly be non-zero are those corresponding to $i = \nu_p(y!)$ and (if $p \mid n$) $i = \nu_p(y!n)$, since obviously $S^\lambda_{\mathbb{Z}} = S^\lambda_{(\nu_p(y!))}$. It only remains to show that these two layers are in fact non-zero. But this is trivial: for $i = \nu_p(y!)$, we let $N(2, 3, \dots, y+1)$ equal one and all other $N(\mathbf{a})$ equal zero. And for $i = \nu_p(y!n)$, we choose integers $m(b_1, \dots, b_{y-1})$ as in Lemma 2.3, say by letting $m(2, 3, \dots, y)$ equal one and all other $m(b_1, \dots, b_{y-1})$ equal zero. That this gives a non-zero element of $S^\lambda_{\mathbb{F}_p}$ follows, since we shall have $N(2, 3, \dots, y+1) = (-1)^y$.

Hence we have proved the following theorem.

**Theorem 2.4.** *Let $\lambda = (n - y, 1^y)$, with $1 \leqslant y < n - 1$. For a prime $p$, the Schaper layer*

$$L_i = \overline{S^\lambda_{(i)}} / \overline{S^\lambda_{(i+1)}}$$

*is non-zero if and only if $i = \nu_p(y!)$ or $i = \nu_p(y!n)$.*

## 2.2 Two-part partitions

The decomposition numbers $[S^\lambda_{\mathbb{F}_p} : D^\mu_{\mathbb{F}_p}]$, where $\lambda$ and $\mu$ are both two-part partitions, are well known; furthermore, each decomposition number is either zero or one, and so we may apply Schaper's formula directly in order to find the Schaper layers. We begin with a statement of the decomposition numbers; for ease of notation, we refer to the Specht module $S^{(n-a,a)}$ as $S(n - 2a + 1)$, and similarly for the simple module $D^{(n-a,a)}$. The following result is due to James [4, 5].

**Theorem 2.5.** *The decomposition number $[S(t) : D(r)]$ is one if there exist integers*

$$1 - p \leqslant t_i \leqslant p - 1$$

*for $i = 0, 1, 2, \dots$ such that*

$$t = \sum t_i p^i$$

*and*

$$r = \sum |t_i| p^i,$$

*and zero otherwise.*

In order to find the layer in which each composition factor lies, we need to find the bound provided by Schaper's formula for its composition multiplicity. By applying Schaper's formula, we find the following.

**Lemma 2.6.** *Take $t > s \geqslant 0$ of the same parity. The coefficient of the Specht module $S(t)$ in the expression for* $\mathrm{rad}(S(s))$ *provided by Schaper's formula is*

$$\nu_p\left(\frac{t+s}{2}\right) - \nu_p\left(\frac{t-s}{2}\right).$$

To find the layers in which the composition factors lie, then, we simply need to multiply the 'Schaper matrix' (defined by Lemma 2.6) by the decomposition matrix. So, for $r > s$, the layer in which the composition factor $D(r)$ of $S(s)$ lies is the sum, over all $t > s$ with $[S(t) : D(r)] = 1$, of $\nu_p\left(\frac{t+s}{2}\right) - \nu_p\left(\frac{t-s}{2}\right)$.

Let $T$ be the set of such $t$, and for each $t \in T$ define $t_0, t_1, \dots$ as in Theorem 2.5; these are clearly unique. Since we are assuming $[S(s) : D(r)] = 1$, we can define $s_0, s_1, \dots$ similarly. The condition $t > s$ then simply means that the largest $i$ for which $t_i \neq s_i$ has $t_i > s_i$. We also have

$$\nu_p\left(\frac{t+s}{2}\right) = \min\{i \mid t_i = s_i \neq 0\}$$

and

$$\nu_p\left(\frac{t-s}{2}\right) = \min\{i \mid t_i = -s_i \neq 0\}.$$

We begin by showing that we may disregard most of the values of $t \in T$. Given $t$, let $\hat{\imath}(t) = \min\{i \mid t_i = s_i \neq 0\}$ and $\check{\imath}(t) = \min\{i \mid t_i = -s_i \neq 0\}$, and put $i(t) = \max(\hat{\imath}, \check{\imath})$. Now we define a new number $f(t) = \sum f(t)_j p^j$, where

$$f(t)_j = \begin{cases} -t_j & (j \leqslant i(t)) \\ t_j & (j > i(t)). \end{cases}$$

Let $i_1 < \ldots < i_l$ be the values of $i$ for which $s_i \neq 0$. We find that $f(t) \in T$ unless $f(t) \leqslant s$, which happens only in the following specific cases:

1. $s_{i_j} > 0$, $s_{i_{j-1}} < 0$, $t_{i_k} = s_{i_k}$ for $k \geqslant j$ and $t_{i_k} = -s_{i_k}$ for $k < j$;
2. $s_{i_j} < 0$, $s_{i_{j-1}} > 0$, $t_{i_k} = s_{i_k}$ for $k \neq j$ and $t_{i_j} = -s_{i_j}$.

Let the set of $t$ so described be denoted $T_0$; then $f$ is an involution (with no fixed points) on $T \setminus T_0$, and furthermore, for $t \in T \setminus T_0$,

$$\nu_p\left(\frac{f(t) \pm s}{2}\right) = \nu_p\left(\frac{t \mp s}{2}\right);$$

so the sum over $T \setminus T_0$ of

$$\nu_p\left(\frac{t+s}{2}\right) - \nu_p\left(\frac{t-s}{2}\right)$$

is zero, and we need only sum over $T_0$.

In case (1) above, we have

$$\nu_p\left(\frac{t+s}{2}\right) - \nu_p\left(\frac{t-s}{2}\right) = i_j - i_1,$$

while in case (2) we have

$$\nu_p\left(\frac{t+s}{2}\right) - \nu_p\left(\frac{t-s}{2}\right) = i_1 - i_j.$$

Summing this over all $j \geqslant 2$ for which $s_{i_j}$ and $s_{i_{j-1}}$ have different signs, and noting that $s_{i_l}$ must be positive, we find that we get the sum, over all $j$ such that $s_{i_j} < 0$, of $i_j - i_{j+1}$. We summarise this in the following theorem.

**Theorem 2.7.** *Suppose that $i_1 < \ldots < i_l$, and that $r = \sum_j r_{i_j} p^{i_j}$, with $0 < r_{i_j} < p$. Suppose that $s = \sum_j s_{i_j} p^{i_j}$, with $s_{i_j} = \pm r_{i_j}$ for each $j$. Then $D(r)$ is a composition factor of $S(s)$, and lies in the Schaper layer $L_i$, where*

$$i = \sum_{j \mid s_{i_j} < 0} (i_j - i_{j+1}).$$

## 3 The 'top' layer of $S^\lambda$

In this section we turn to the problem of finding the highest non-zero layer $L_i$; we define $\nu_p(\lambda)$ to be $\min\{i \mid \overline{S^\lambda_{(i)}} \neq \overline{S^\lambda_{(i+1)}}\}$. The following result tells us that we can work over the integral Specht module.

**Lemma 3.1.** *For any partition $\lambda$ and any $i \geqslant 0$,*

$$\overline{S^\lambda_{(0)}} = \overline{S^\lambda_{(i)}}$$

*if and only if*

$$S^\lambda_{(0)} = S^\lambda_{(i)}.$$

This requires a preliminary observation.

**Lemma 3.2.** *Let $x$ be an integral combination of $\lambda$-tabloids, with all coefficients divisible by $p$, and suppose that $x \in S^\lambda_{\mathbb{Z}}$. Then $\frac{x}{p} \in S^\lambda_{\mathbb{Z}}$.*

**Proof.** By [7, Corollary 8.12], there is a $\mathbb{Z}$-basis $\{e_1, \ldots, e_r\}$ of $S^\lambda$ such that each basis element involves a unique standard tabloid, and involves this tabloid with coefficient 1. Putting $x = \sum_1^r \mu_i e_i$ and examining the coefficient of each standard tabloid in $x$, we find that each $\mu_i$ is divisible by $p$. Hence

$$\frac{x}{p} = \sum_1^r \frac{\mu_i}{p} e_i \in S^\lambda_{\mathbb{Z}}. \qquad \square$$

**Proof of Lemma 3.1.** The 'if' part is trivial. Let $i$ be maximal such that $S^\lambda_{(0)} = S^\lambda_{(i)}$, let $j$ be maximal such that $\overline{S^\lambda_{(0)}} = \overline{S^\lambda_{(j)}}$ and suppose for a contradiction that $i < j$.

Take $x \in S^\lambda_{(i)} \setminus S^\lambda_{(i+1)}$. By assumption $\overline{x} \in \overline{S^\lambda_{(j)}}$, i.e. there exist $z \in S^\lambda$ and $y \in S^\lambda_{(j)}$ with $x = y + z$ and $\overline{z} = 0$.

By Lemma 3.2, $z$ equals $pw$ for some $w \in S^\lambda$. Since $S^\lambda = S^\lambda_{(i)}$, we have $p^i | \langle w, v \rangle$ for all $v \in S^\lambda$, whence $z \in S^\lambda_{(i+1)}$. But $y \in S^\lambda_{(i+1)}$, so $x = y + z \in S^\lambda_{(i+1)}$ as well; contradiction. $\square$

So in fact $\nu_p(\lambda)$ is the maximum value $i$ such that $S^\lambda_{(0)} = S^\lambda_{(i)}$, i.e. the maximum value of $i$ such that $p^i$ divides $\langle e_s, e_t \rangle$ for all polytabloids $e_s, e_t$. In [7], James finds those $\lambda$ for which $\nu_p(\lambda) = 0$.

**Lemma 3.3. [7, Lemma 10.4]**

*Suppose $\lambda$ has $z_j$ parts equal to $j$. Then*

$$\nu_p\Big(\prod_1^\infty z_j!\Big) \leqslant \nu_p(\lambda) \leqslant \nu_p\Big(\prod_1^\infty (z_j!)^j\Big).$$

**Corollary 3.4. [7, Theorem 10.5]**
$\nu_p(\lambda) = 0$ *if and only if* $\lambda$ *is p-regular.*

We shall prove some general results concerning $\nu_p(\lambda)$, and then use these to give necessary and sufficient conditions for $\nu_p(\lambda)$ to equal 1. It is hoped that these techniques could be developed to find $\nu_p(\lambda)$ for all $\lambda$.

We begin by proving a result similar to Donkin's generalisation [3] of James's Principle of Row Removal [8].

**Theorem 3.5.** *Let $\lambda$, $\mu$ be any partitions of positive integers $n_1$, $n_2$ respectively with $n_1 + n_2 = n$, and let $\lambda * \mu$ be the partition obtained by arranging all the parts of $\lambda$ and $\mu$ in descending order. Then*

$$\nu_p(\lambda * \mu) \geqslant \nu_p(\lambda) + \nu_p(\mu).$$

**Proof.** We partition the set of rows of the Young diagram for $\lambda * \mu$ into a $\lambda$ part and a $\mu$ part, i.e. so that the rows in the $\lambda$ part have lengths equal to the parts of $\lambda$. Then, for any $\lambda * \mu$-tableau $u$, we define $C_1(u) \subseteq \{1, \dots, n\}$ to be the set of values which appear in the $\lambda$ part of the tableau, and $C_2(u)$ similarly. Then $u$ defines a $\lambda$-tableau $u_1$ with entries in $C_1(u)$ and a $\mu$-tableau $u_2$ with entries in $C_2(u)$.

We have

$$\langle e_s, e_t \rangle = \sum_{x,w} (-1)^{\pi_{sx}} (-1)^{\pi_{wt}},$$

summing over $w$ and $x$ with $s \sim_{\text{col}} w \sim_{\text{row}} x \sim_{\text{col}} t$. If we take a tableau $u$ with $s \sim_{\text{col}} u$ and sum only over those $w$ and $x$ for which $C_i(w) = C_i(x) = C_i(u)$, then we get

$$(-1)^{\pi_{su}\pi_{tv}} \langle e_{u_1}, e_{v_1} \rangle \langle e_{u_2}, e_{v_2} \rangle$$

(with the usual inner products on $S^\lambda$, $S^\mu$) if there exists $v$ with $u \sim_{\text{row}} v \sim_{\text{col}} t$, and zero otherwise. In either case this is divisible by $p^{\nu_p(\lambda)}.p^{\nu_p(\mu)}$. Summing over all possible sets $C_1(u)$ gives $\langle e_s, e_t \rangle$ divisible by $p^{\nu_p(\lambda)}.p^{\nu_p(\mu)}$ as well. □

In order to prove further results, we adopt a graph-theoretic approach. Given $\lambda$-tableaux $s$ and $t$ with $s \sim_{\text{row}} t$, we define the multi-graph $G(s, t)$ as follows. Take labelled vertices $s_1, s_2, s_3, \dots$ and $t_1, t_2, t_3, \dots$. Then draw $n$ labelled edges $e_1, \dots, e_n$, with $e_i$ joining $s_j$ and $t_k$, where $i$ appears in the $j$th column of $s$ and the $k$th column of $t$.

Now, for any tableaux $u$ and $v$ with $s \sim_{\text{col}} u \sim_{\text{row}} v \sim_{\text{col}} t$, we colour the edges of $G(s, t)$ with colours $c_1, c_2, \dots$: colour edge $e_i$ with colour $c_l$, where the number $i$ appears in the $l$th row of $u$ (and of $v$). Note that for each $l$, colour $c_l$ appears exactly once at each of the vertices $s_1, \dots, s_{\lambda'_l}, t_1, \dots, t_{\lambda'_l}$; call such a colouring *admissible*, and let $A(G)$ denote the set of admissible colourings of $G$. An admissible colouring induces a permutation of the numbers $\{1, \dots, \lambda'_l\}$; this permutation is exactly the permutation of row $l$ which is needed to get from $u$ to $v$. So if we take the signatures of these permutations for all $l$ and multiply them, we get $(-1)^{\pi_{uv}} = (-1)^{\pi_{st}} (-1)^{\pi_{su}} (-1)^{\pi_{tu}}$. We define the signature $(-1)^C$ of any admissible colouring $C$ to be this product of signatures. Every admissible colouring of $G(s, t)$ defines a pair of tableaux $(u, v)$ as above, and so we have the following.

**Proposition 3.6.**

$$\sum_{C \in A(G)} (-1)^C = (-1)^{\pi_{st}} \langle e_s, e_t \rangle.$$

We use this interpretation to prove the following theorem, reminiscent of James's Principle of Column Removal [8].

**Theorem 3.7.** *Let $\hat{\lambda}$ be the partition whose Young diagram is obtained by removing the first column of the Young diagram for $\lambda$. Then $\nu_p(\lambda) \geqslant \nu_p(\hat{\lambda})$.*

**Proof.** Given $\lambda$-tableaux $s$ and $t$, construct the graph $G = G(s,t)$ as above. Let $e_{i_1}, \dots, e_{i_m}$ be the edges of $G$ which meet $s_1$ but not $t_1$; suppose that $e_{i_k}$ also meets vertex $t_{f(k)}$. Similarly, let $e_{j_1}, \dots, e_{j_m}$ be the edges which meet $t_1$ but not $s_1$, and suppose that $e_{j_k}$ also meets $s_{g(k)}$.

Now, given any $\sigma \in \mathfrak{S}_m$, we form a new graph $G_\sigma$. We delete vertices $s_1$ and $t_1$ and all edges meeting them, and add new edges $e'_1, \dots, e'_m$, where $e'_k$ joins vertices $t_{f(k)}$ and $s_{g(\sigma k)}$.

After a re-numbering of vertices and edges, we may regard $G_\sigma$ as the graph $G(s_\sigma, t_\sigma)$ for some $\hat{\lambda}$-tableaux $s_\sigma, t_\sigma$; choose such a pair $(s_\sigma, t_\sigma)$ for each $\sigma$ arbitrarily.

Now we look at the relationship between colourings of $G$ and of $G_\sigma$. We say that an admissible colouring of $G_\sigma$ is *respectable* if it colours edges $e'_1, \dots, e'_m$ with different colours, and we let $R(G_\sigma)$ denote the set of respectable colourings of $G_\sigma$. We then have the following.

1. A respectable colouring $C'$ of $G_\sigma$ gives rise to $(\lambda'_1 - m)!$ different admissible colourings of $G$. Begin by colouring all edges not meeting $s_1$ or $t_1$ as in $C'$; then colour edges $e_{i_k}$ and $e_{j_{\sigma k}}$ the same colour as $e'_k$. Finally colour the $(\lambda'_1 - m)$ edges from $s_1$ to $t_1$ with the colours not used for $e'_1, \dots, e'_m$, in any order. This gives an admissible colouring $C$ of $G$; an examination of the permutations induced by the colourings shows that
$$(-1)^C = (-1)^m (-1)^{C'}.$$

2. An admissible colouring $C$ of $G$ gives a respectable colouring of $G_\sigma$ for some $\sigma \in \mathfrak{S}_m$. The edges $e_{i_1}, \dots, e_{i_m}$ have the same colours as $e_{j_1}, \dots, e_{j_m}$ in some order; let $\sigma$ be such that $e_{i_k}$ and $e_{j_{\sigma k}}$ have the same colour. Now colour $G_\sigma$ by giving $e'_k$ the same colour as $e_{i_k}$ for each $k$, and letting each other edge have the same colour as in $C$. This gives a respectable colouring $C'$ of $G_\sigma$, and the relationship between $(-1)^C$ and $(-1)^{C'}$ is as in (1).

The procedures described in (1) and (2) above are mutually inverse, and so we get
$$(-1)^{\pi_{st}} \langle e_s, e_t \rangle = (-1)^m (\lambda'_1 - m)! \sum_{\sigma \in \mathfrak{S}_m} \sum_{C \in R(G_\sigma)} (-1)^C.$$

We now show that considering only respectable colourings of $G_\sigma$ is sufficient. Given any admissible colouring $C$ of $G_\sigma$, define, for each $l$,
$$C_l = |\{k \mid e'_k \text{ has colour } c_l\}|,$$

so that $C$ is respectable if and only if each $C_l$ is at most one. Now, for any $d_1, d_2, \dots$, define $\mathbf{C}(d_1, d_2, \dots)$ to be the set of pairs $(\sigma, C)$, where $\sigma \in \mathfrak{S}_m$ and $C$ is an admissible colouring of $G_\sigma$ with $C_l = d_l$ for all $l$. The group $\mathfrak{S}_{d_1} \times \mathfrak{S}_{d_2} \times \dots$ acts on $\mathbf{C}(d_1, d_2, \dots)$ as follows: a permutation in $\mathfrak{S}_{d_l}$ permutes the endpoints (that is, the $t_{f(k)}$s) of those edges coloured with colour $c_l$. Moreover, this action is with signature in the sense that
$$(-1)^D = (-1)^\rho (-1)^C$$

where $\rho(\sigma, C) = (\tau, D)$ for $\rho \in \mathfrak{S}_{d_1} \times \mathfrak{S}_{d_2} \times \dots$. Now, provided some $d_l \geqslant 2$, we can find $\rho$ with $(-1)^\rho = -1$; summing over $\mathbf{C}(d_1, d_2, \dots)$, we obtain

$$\sum_{(\sigma,C)\in\mathbf{C}(d_1,\dots)} \delta(\sigma)(-1)^C = -\sum_{(\sigma,C)\in\mathbf{C}(d_1,\dots)} \delta(\sigma)(-1)^C,$$

which therefore equals zero.

Hence we have

$$\sum_{\sigma\in\mathfrak{S}_m} \sum_{C\in A(G_\sigma)} (-1)^C = \sum_{\sigma\in\mathfrak{S}_m} \sum_{C\in R(G_\sigma)} (-1)^C,$$

whence

$$(-1)^{\pi_{st}} \langle e_s, e_t \rangle = (-1)^m (\lambda_1' - m)! \sum_{\sigma\in\mathfrak{S}_m} (-1)^{\pi_{s_\sigma t_\sigma}} \langle e_{s_\sigma}, e_{t_\sigma} \rangle,$$

which is divisible by $p^{\nu_p(\hat{\lambda})}$. □

Using these general results, we proceed to determine exactly those $\lambda$ for which $\nu_p(\lambda) = 1$. Our main result is as follows.

**Theorem 3.8.** *Let $\lambda$ be a partition of n. $\nu_p(\lambda)$ is greater than 1 if and only if one of the following holds:*

1. *$\lambda$ is doubly p-singular, i.e. there exist i, j with $j \geqslant i + p$ and $\lambda_i = \lambda_{i+p-1}$ and $\lambda_j = \lambda_{j+p-1} > 0$;*
2. *there exists i such that $\lambda_i \leqslant \lambda_{i+2p-2} + 1$ and $\lambda_{i+p-1} \geqslant 2$.*

We begin with the 'only if' part. Suppose that $\lambda$ does not satisfy either of the criteria in Theorem 3.8. By Corollary 3.4, we may assume that $\lambda$ is $p$-singular; suppose $\lambda_{i-1} > \lambda_i = \lambda_j > \lambda_{j+1}$ with $j - i \geqslant p - 1$ (the first inequality is to be ignored if $i = 1$).

The case where $\lambda_i = 1$ is dealt with by Lemma 3.3, since by assumption $z_1 = j - i + 1 < 2p$. So assume $\lambda_i \geqslant 2$. Our assumption that condition (2) does not hold then guarantees that $\lambda_{j-2p+2} \geqslant \lambda_i + 2$ (or that $j < 2p - 2$) and also that $\lambda_{i+2p-2} \leqslant \lambda_i - 2$. We define the partition $\mu$ by

$$\mu_k = \begin{cases} \lambda_k + 1 & (i \leqslant k \leqslant j - p + 1) \\ \lambda_k - 1 & (i + p - 1 \leqslant k \leqslant j) \\ \lambda_k & \text{otherwise.} \end{cases}$$

The construction of $\mu$ guarantees the following.

**Lemma 3.9.** *$\mu$ is p-regular, and the simple module $D^\mu_{\mathbb{F}_p}$ occurs as a composition factor of $\overline{S^\lambda_{(1)}}/\overline{S^\lambda_{(2)}}$. In particular, $\nu_p(\lambda) \leqslant 1$.*

**Proof.** The coefficient of $S^\mu$ in the Schaper expression for $S^\lambda_{\mathbb{F}_p}$ is +1, and no other Specht module $S^\nu$ with $\mu \trianglerighteq \nu \trianglerighteq \lambda$ occurs. Hence $D^\mu$ occurs as a composition factor of $\mathrm{rad}(S^\lambda_{\mathbb{F}_p})$, with the bound for its composition multiplicity being 1. □

We proceed with the 'if' part.

**Lemma 3.10.** *Take $0 \leqslant r < p$, and put $\lambda = (2^{2p-1-r}, 1^r)$. Then $\nu_p(\lambda) \geqslant 2$.*

**Proof.** Take two $\lambda$-tableaux $s$ and $t$ with $s \sim_{\mathrm{row}} t$, and draw the graph $G(s,t)$ as described above. Since there are $2p-1$ edges meeting the vertex $s_1$, there must either be $p$ edges from $s_1$ to $t_1$, or $p$ edges from $s_1$ to $t_2$; call these edges $e_{i_1},\dots,e_{i_p}$.

Now the group $\mathfrak{S}_p$ acts faithfully on the set of admissible colourings of $G(s,t)$ in two different ways: by permuting the colours $c_1,\dots,c_p$, and by permuting the colours assigned to the edges $e_{i_1},\dots,e_{i_p}$. These actions commute, and both preserve the signatures of colourings. Hence we have a faithful signature-preserving action of $\mathfrak{S}_p \times \mathfrak{S}_p$ on the set of admissible colourings, and

$$\sum_{C\in A(G(s,t))} (-1)^C$$

is divisible by $p^2$. □

**Remark.** For an alternative proof of Lemma 3.10, we may combine the results of Sections 2.2 and 4, which do not depend on the present section.

**Lemma 3.11.** *Take* $0 \leqslant r < p$*, and put* $\lambda = (3^r, 2^{2p-1-r})$*. Then* $\nu_p(\lambda) \geqslant 2$*.*

**Proof.** We proceed by induction on $r$, with the case $r = 0$ being the case $r = 0$ of Lemma 3.10. For $r \geqslant 1$, take $\lambda$-tableaux $s$ and $t$ with $s \sim_{\mathrm{row}} t$, and draw the graph $G = G(s,t)$.

Suppose first of all that there is at least one edge $e$ from $s_3$ to $t_3$ in $G$. There is a faithful signature-preserving action of $\mathfrak{S}_r$ on the set of admissible colourings of $G$, given by permuting the colours $c_1,\dots,c_r$. So if we sum the signatures of all admissible colourings of $G$ in which $e$ has colour $c_r$, we will get $\frac{\langle e_s,e_t\rangle}{r!}$; since $r < p$, this is divisible by $p^2$ if and only if $\langle e_s,e_t\rangle$ is.

Deleting $e$ gives the graph $G(s',t')$ for some $(3^{r-1}, 2^{2p-2-r})$-tableaux $s'$ and $t'$. Furthermore, there is a one-to-one correspondence between admissible colourings of $G(s',t')$ and admissible colourings of $G$ in which $e$ has colour $c_r$. This preserves the signature, and so we find that the sum of $(-1)^C$ over all admissible colourings $C$ of $G$ in which $e$ has colour $c_r$ equals $\langle e_{s'},e_{t'}\rangle$. This is divisible by $p^2$, by induction.

Now we assume that there are no edges from $s_3$ to $t_3$ in $G(s,t)$. For this case we adopt a method similar to that used in the proof of Theorem 3.7. Let $e_{i_1},\dots,e_{i_r}$ be the edges meeting $s_3$ (and suppose $e_{i_k}$ also meets $t_{f(k)}$), and similarly define $e_{j_1},\dots,e_{j_r}$ and $s_{g(1)},\dots,s_{g(r)}$.

Given $\sigma \in \mathfrak{S}_r$, we form the graph $G_\sigma$ by deleting $s_3$ and $t_3$ and the edges $e_{i_k}$, $e_{j_k}$, and adding edges $e'_k$ joining $t_{f(k)}$ and $s_{g(\sigma k)}$, for $k = 1,\dots,r$. As in Theorem 3.7, we find that $G_\sigma = G(s_\sigma,t_\sigma)$ for some $(2^{2p-1})$-tableaux $s_\sigma$, $t_\sigma$.

There is an obvious one-to-one correspondence between colourings $C \in A(G)$ and pairs $(\sigma, C')$, where $\sigma \in \mathfrak{S}_r$ and $C'$ is an admissible colouring of $G_\sigma$ in which edges $e'_1,\dots,e'_r$ have colours $c_1,\dots,c_r$ in some order. An examination of the permutations induced by the colourings shows that

$$(-1)^C = (-1)^r(-1)^{C'}.$$

Thus we have

$$(-1)^{\pi_{st}}\langle e_s,e_t\rangle = (-1)^r \sum_{\sigma\in\mathfrak{S}_r}\sum_{C}(-1)^C,$$

summing over all $C \in A(G_\sigma)$ in which $e'_1,\dots,e'_r$ have colours $c_1,\dots,c_r$.

As before, we define $R(G_\sigma)$ to be the set of colourings of $G_\sigma$ in which $e'_1, \dots, e'_r$ have different colours. There is a faithful signature-preserving action of $\mathfrak{S}_{2p-1}$ on $R(G_\sigma)$ given by permuting the colours, so we get

$$(-1)^{\pi_{st}}\langle e_s, e_t\rangle = \frac{1}{\binom{2p-1}{r}} \sum_{\sigma\in\mathfrak{S}_r} \sum_{C\in R(G_\sigma)} (-1)^C.$$

Exactly as in the proof of Theorem 3.7, we may replace the sum over $R(G_\sigma)$ with the sum over $A(G_\sigma)$ to get

$$(-1)^{\pi_{st}}\langle e_s, e_t\rangle = \frac{1}{\binom{2p-1}{r}} \sum_{\sigma\in\mathfrak{S}_r} (-1)^{\pi_{s_\sigma t_\sigma}}\langle e_{s_\sigma}, e_{t_\sigma}\rangle;$$

since $\binom{2p-1}{r}$ is not divisible by $p$, the result follows. □

**Proof of Theorem 3.8.** The 'only if' part follows from Corollary 3.4 and Lemma 3.9. If $\lambda$ is doubly $p$-singular, the result follows from Lemma 3.3. Otherwise, we may use Theorems 3.5 and 3.7, and assume that $\lambda$ equals $(2^{2p-1-r}, 1^r)$ or $(3^r, 2^{2p-1-r})$ for some $r < p$; these cases are dealt with in Lemmata 3.10 and 3.11. □

## 4 The Specht module corresponding to the conjugate partition

Fix a $\lambda$-tableau $t$. Let $H$ be the product of the hook lengths in the Young diagram for $\lambda$. Fix a prime $p$, and define $h = \nu_p(H)$ (for $\lambda$ in a $p$-block of abelian defect, $h$ is then equal to the defect). We quote the following from [6, p. 13].

**Lemma 4.1.** *In the Specht module $S^\lambda$ over any ring,*

$$\kappa_t\rho_t\kappa_t\{t\} = H.\kappa_t\{t\}.$$

We deduce the following (where $\mathbb{I}(S)$ denotes the indicator function of statement $S$).

**Corollary 4.2.**

1.
$$\sum_{\substack{\kappa_1,\kappa_2\in C_t\\ \rho_1,\rho_2\in R_t}} (-1)^{\kappa_1\kappa_2}\mathbb{I}(\kappa_1\rho_1\kappa_2\rho_2 = 1) = H.$$

2.
$$\sum_{\substack{\kappa_1,\kappa_2,\kappa_3,\kappa_4\in C_t\\ \rho_1,\rho_2,\rho_3,\rho_4\in R_t}} (-1)^{\kappa_1\kappa_2\kappa_3\kappa_4}\mathbb{I}(\kappa_1\rho_1\kappa_2\rho_2\kappa_3\rho_3\kappa_4\rho_4 = 1) = H^3.$$

**Proof.**

1. Compare coefficients of $\{t\}$ in Lemma 4.1.

2. Apply Lemma 4.1 three times to get

$$\kappa_t\rho_t\kappa_t\rho_t\kappa_t\rho_t\kappa_t\{t\} = H^3\kappa_t\{t\},$$

and compare coefficients of $\{t\}$.

□

Now let $\lambda'$ be the partition conjugate to $\lambda$, and for any $\lambda$-tableau $s$, let $s'$ be the corresponding $\lambda'$-tableau. The main object of attention in this section is the following theorem.

**Theorem 4.3. [7, Theorem 8.15]**
*Over any field k,*

$$S_k^\lambda \cong (S_k^{\lambda'} \otimes \mathrm{sgn})^*.$$

We proceed to construct this isomorphism explicitly; our construction will work over any field. We begin by constructing a form

$$[,] : M_{\mathbb{Z}}^\lambda \otimes M_{\mathbb{Z}}^{\lambda'} \longrightarrow \mathbb{Z};$$

given a $\lambda$-tableau $s$ and a $\lambda'$-tableau $u'$, define $[\{s\},\{u'\}]$ as follows. If there are two numbers in the same row of $s$ and the same row of $u'$, put $[\{s\},\{u'\}] = 0$. Otherwise, there exists a unique $\lambda$-tableau $v$ such that $s \sim_{\text{row}} v$ and $u' \sim_{\text{row}} v'$. In this case, define $[\{s\},\{u'\}] = (-1)^{\pi_{tv}}$. Extend $[,]$ bilinearly.

The following crucial property of $[,]$ follows from the construction.

**Lemma 4.4.** $[,]$ *defines a module homomorphism from* $M_{\mathbb{Z}}^\lambda \otimes M_{\mathbb{Z}}^{\lambda'}$ *to the signature representation, i.e. for all* $x \in M^\lambda$, $y \in M^{\lambda'}$, $\sigma \in \mathfrak{S}_n$,

$$[\sigma x, \sigma y] = (-1)^\sigma [x, y].$$

Let us examine the restriction of $[,]$ to $S_{\mathbb{Z}}^\lambda \otimes S_{\mathbb{Z}}^{\lambda'}$. Recall the dominance order $\trianglerighteq$ on $\lambda$-tabloids.

**Lemma 4.5.** *For any $\lambda$-tableaux s,u:*

1. $[e_s, e_{u'}] = 0$ *or* $[e_s, e_{u'}] = \pm[e_t, e_{t'}]$;
2. $[e_{\sigma t}, e_{\sigma t'}] = (-1)^\sigma [e_t, e_{t'}]$;
3. *if s and u are standard,* $[e_s, e_{u'}] = 0$ *unless* $\{s\} \trianglerighteq \{u\}$.

**Proof.** If there are two numbers, $a$ and $b$ say, in the same column of $s$ and the same column of $u'$, then we have

$$\begin{aligned}[e_s, e_{u'}] &= -[(ab)e_s, (ab)e_{u'}] \\ &= -[-e_s, -e_{u'}] \\ &= 0.\end{aligned}$$

If not, then there exists a unique tableau $v$ such that $s \sim_{\text{col}} v$ and $u' \sim_{\text{col}} v'$. This gives

$$\begin{aligned}[e_s, e_{u'}] &= \pm[e_v, e_{v'}] \\ &= \pm(-1)^{\pi_{tv}}[e_t, e_{t'}],\end{aligned}$$

so (1) holds. (2) is just a special case of Lemma 4.4. Now suppose that $s$ and $u$ are standard and that $v$ exists as above; we need to show that $\{s\} \trianglerighteq \{u\}$. But since $s$ is standard and $v$ is obtained from $s$ by a column permutation, we have $\{s\} \triangleright \{v\}$; of course, $\{v\} = \{u\}$, which gives (3). □

Lemma 4.5 shows that the homomorphic property defines $[,]$ uniquely on $S^\lambda \otimes S^{\lambda'}$, up to a scalar. We need to know that $[e_t, e_{t'}]$ is non-zero; in fact, we can find it exactly.

**Lemma 4.6.**

$$[e_t, e_{t'}] = H.$$

**Proof.** We have

$$[e_t, e_{t'}] = [\kappa_t\{t\}, \kappa_{t'}\{t'\}];$$

each term $[(-1)^\kappa\kappa\{t\}, (-1)^\rho\rho\{t'\}]$ for $\kappa \in C_t$, $\rho \in R_t$ contributes a factor $(-1)^\kappa(-1)^\rho(-1)^{\xi\rho}$ if there exist $\pi \in R_{\kappa t}$, $\xi \in C_{\rho t}$ with

$$\pi\kappa t = \xi\rho t,$$

and zero otherwise. But $R_{\kappa t} = \kappa R_t \kappa^{-1}$, and $C_{\rho t} = \rho C_t \rho^{-1}$, so we seek the sum over all $\kappa, \kappa_2 \in C_t$, $\rho, \rho_2 \in R_t$ with $\kappa\rho_2 = \rho\kappa_2$ of $(-1)^{\kappa_2\kappa}$. By Corollary 4.2 (1), this is $H$. □

Given this, we can define $\theta$.

**Definition.** Define the form

$$(,) : S^\lambda_\mathbb{Z} \otimes S^{\lambda'}_\mathbb{Z} \longrightarrow \mathbb{Z}$$

by

$$(x, y) = \frac{[x, y]}{H},$$

and let $\theta : S^\lambda_\mathbb{Z} \to (S^{\lambda'}_\mathbb{Z} \otimes \mathrm{sgn})^*$ be given by

$$\theta(x)(y \otimes 1) = [x, y].$$

**Proposition 4.7.** *$\theta$ is an isomorphism of $\mathbb{Z}\mathfrak{S}_n$-modules.*

**Proof.** That $\theta$ is a homomorphism follows from Lemma 4.4. Lemma 4.5 guarantees that the matrix of $(,)$ with respect to the standard bases of $S^\lambda_\mathbb{Z}$, $S^{\lambda'}_\mathbb{Z}$ (the polytabloids corresponding to standard tableaux) is upper triangular with integer entries and diagonal entries all $\pm1$. So the matrix has determinant $\pm1$, and is invertible. □

By extending scalars or by modular reduction, we can easily define $\theta$ over the fields $\mathbb{Q}$ and $\mathbb{F}_p$.

Our aim is to see how the submodules $\overline{S^\lambda_{(i)}}$, $\overline{S^{\lambda'}_{(j)}}$ correspond under $\theta$. Our main theorem is as follows.

**Theorem 4.8.** *For any prime $p$, and any $i$,*

$$\overline{S^\lambda_{(i)}}/\overline{S^\lambda_{(i+1)}} \cong \left(\overline{S^{\lambda'}_{(h-i)}}/\overline{S^{\lambda'}_{(h-i+1)}}\right) \otimes \mathrm{sgn}.$$

This follows from the following.

**Theorem 4.9.** *If $i + j > h$ and $x \in S^\lambda_{(i)}$, $y \in S^{\lambda'}_{(j)}$, then $p$ divides $[x, y]$. On the other hand, if $i + j \leqslant h$ and $x \in S^\lambda_\mathbb{Z}$ is such that $x + pz$ is never in $S^\lambda_{(i+1)}$ for $z \in S^\lambda_\mathbb{Z}$, then there exists $y \in S^{\lambda'}_{(j)}$ such that $p$ does not divide $[x, y]$.*

Given Theorem 4.9, we see that over a field of characteristic $p$, the image under $\theta$ of $\overline{S^\lambda_{(i)}}$ is precisely the annihilator in $(S^{\lambda'} \otimes \mathrm{sgn})^*$ of $\overline{S^{\lambda'}_{(h-i+1)}} \otimes \mathrm{sgn}$. Hence we have

$$\overline{S^\lambda_{(i)}}/\overline{S^\lambda_{(i+1)}} \cong (\overline{S^{\lambda'}_{(h-i+1)}} \otimes \mathrm{sgn})^\circ/(\overline{S^{\lambda'}_{(h-i)}} \otimes \mathrm{sgn})^\circ$$

which is naturally isomorphic to

$$\left(\overline{S^{\lambda'}_{(h-i)}}\otimes \mathrm{sgn}\,/\,\overline{S^{\lambda'}_{(h-i+1)}}\otimes \mathrm{sgn}\right)^{*};$$

since the layers $L_i$ are known to be self-dual, Theorem 4.8 follows.

**Example.** In characteristic two , the Specht module $S^{(7,1)}$ has composition factors $D^{(7,1)}$ and $D^{(8)}$, lying in layers $L_0$ and $L_3$ respectively. The product of the hook lengths for the partition $(7, 1)$ is $5760 = 2^7.45$, so the Specht module $S^{(2,1^6)}$ corresponding to the conjugate partition has these composition factors lying in layers $L_7$ and $L_4$ respectively.

We proceed to prove Theorem 4.9. To avoid tensoring with the signature representation, we define

$$\alpha : S^{\lambda}_{\mathbb{Z}} \longrightarrow S^{\lambda'*}_{\mathbb{Z}}$$

to be the map induced by $(,)$ (of course $\alpha$ isn't a homomorphism, but that doesn't matter). Also define

$$\beta : S^{\lambda}_{\mathbb{Z}} \longrightarrow S^{\lambda*}_{\mathbb{Z}}$$

to be the map induced by $\langle,\rangle$, and

$$\gamma : S^{\lambda'}_{\mathbb{Z}} \longrightarrow S^{\lambda'*}_{\mathbb{Z}}$$

that induced by the corresponding inner product on $M^{\lambda'}_{\mathbb{Z}}$. A crucial result is then the following.

**Proposition 4.10.** *As maps from $S^{\lambda'}$ to $S^{\lambda*}$,*

$$\beta\alpha^{-1}\gamma = H\alpha^{*}.$$

We start by proving this in a special case.

**Lemma 4.11.**

$$\beta\alpha^{-1}\gamma(e_{t'})(e_t) = H.$$

**Proof.** For $\lambda$-tableaux $s$ and $u$, define

$$\Gamma(s,u) = \begin{cases} (-1)^{\pi_{tv}} & \text{if there exists } v \text{ such that } s \sim_{\text{row}} v \sim_{\text{col}} u \\ 0 & \text{otherwise.} \end{cases}$$

Let $T$ be the set of standard $\lambda$-tableaux. Then by the definition of $\alpha$, we have, for $f \in S^{\lambda'*}_{\mathbb{Z}}$,

$$\alpha^{-1}(f) = \sum_{s\in T} \mu_s e_s,$$

where

$$\mu_s = \sum_{u\in T} f(e_{u'})\Gamma(s,u).$$

Of course, a similar expression is valid for any other bases of $S^{\lambda}_{\mathbb{Z}}$, $S^{\lambda'}_{\mathbb{Z}}$; in particular, we may replace the above sums with $\sum_{s\in\tau T}$ and $\sum_{u\in v T}$ for any $\tau, v \in \mathfrak{S}_n$. If we do this and sum over all $\tau$ and $v$, then we sum over every pair $(s, u)$ of tableaux $\dim S^{\lambda} \times \dim S^{\lambda'}$ times; by the Hook Length Formula,

$$\dim S^{\lambda} = \dim S^{\lambda'} = \frac{n!}{H},$$

so we obtain

$$(n!)^2\alpha^{-1}(f) = \left(\frac{n!}{H}\right)^2 \sum_s \mu_s e_s,$$

with

$$\mu_s = \sum_u f(e_{u'})\Gamma(s,u).$$

Now we look at $\beta$. For $\lambda$-tableaux $s,u$, $\beta(e_s)(e_u)$ is simply $\langle e_s, e_u\rangle$. This is the sum over all pairs of tableaux $a$,$b$ with $s \sim_{\text{col}} a \sim_{\text{row}} b \sim_{\text{col}} u$ of $(-1)^{\pi_{sa}}(-1)^{\pi_{bu}}$. A similar expression holds for $\gamma(e_{s'})(e_{u'})$, and we combine these expressions for $\gamma$, $\alpha^{-1}$ and $\beta$ to find that

$$\beta\alpha^{-1}\gamma(e_{t'})(e_t) = \frac{1}{H^2}\sum_{c,d,u,x,s,a,b}(-1)^{\pi_{tc}}(-1)^{\pi_{du}}(-1)^{\pi_{tx}}(-1)^{\pi_{sa}}(-1)^{\pi_{bt}},$$

the sum being over all tableaux $c,d,u,x,s,a,b$ with

$$t \sim_{\text{row}} c \sim_{\text{col}} d \sim_{\text{row}} u \sim_{\text{col}} x \sim_{\text{row}} s \sim_{\text{col}} a \sim_{\text{row}} b \sim_{\text{col}} t.$$

The summand $(-1)^{\pi_{tx}}(-1)^{\pi_{sa}}(-1)^{\pi_{bt}}$ is the same as $(-1)^{\pi_{cd}\pi_{ux}\pi_{sa}\pi_{bt}}$, and the result now follows from (2) in Lemma 4.2. □

**Proof of Proposition 4.10.** $\alpha$ is a homomorphism twisted by the signature representation; that is,

$$\alpha(\sigma x) = (-1)^\sigma\sigma(\alpha(x));$$

in fact, it follows from Lemma 4.5 that this property defines $\alpha$ up to a scalar; of course, a similar statement holds for $\alpha^*$. Now $\beta$ and $\gamma$ are homomorphisms and $\alpha^{-1}$ is a twisted homomorphism, so $\beta\alpha^{-1}\gamma$ is a twisted homomorphism, and so equals a scalar multiple of $\alpha^*$. Lemma 4.11 gives the scalar, and the result follows. □

Now we let $A$, $B$ and $C$ be the matrices of $\alpha, \beta, \gamma$ respectively with respect to the standard bases of $S^\lambda$ and $S^{\lambda'}$ and their dual bases. We then have

$$BA^{-1}C = H.A^{\mathrm{T}}.$$

**Lemma 4.12.** *Suppose that $B$, $D$ are $d$ by $d$ matrices with integer entries such that $BD = H.I$. Suppose also that $x \in \mathbb{Z}^d$ has the property that, for any $z \in \mathbb{Z}^d$, some component of $(x + pz)^{\mathrm{T}}B$ is not divisible by $p^{i+1}$. Then there exists $w \in \mathbb{Z}^d$ such that*

- *$p^{h-i}$ divides every component of $Dw$, and*
- *$p$ does not divide $x^{\mathrm{T}}w$.*

**Proof.** Using Smith's Normal Form [2, p. 322], we may find invertible matrices $M, N$ over $\mathbb{Z}$ such that $MBN$ is diagonal, with diagonal entries $b_1, \dots, b_d$ say. Of course, $N^{-1}DM^{-1}$ is then also diagonal, with diagonal entries $d_j = \frac{H}{b_j}$. If $x \in \mathbb{Z}^d$ has the stated property, then, putting $\xi = (M^{-1})^{\mathrm{T}}x$, we have that, for every $z \in \mathbb{Z}^d$, some component of $\xi^{\mathrm{T}}MB$ is not divisible by $p^{i+1}$. Since $N$ is invertible, the same holds for $\xi^{\mathrm{T}}MBN$. But $MBN$ is diagonal, so it is easily seen that this property is equivalent to:

*for some $j$, $\nu_p(b_j) < i$ and $p$ does not divide $\xi_j$.*

By letting $\omega$ be the vector with a one in the $j$th position and zeroes elsewhere, we then find that, since $b_j d_j = H$, $p^{h-i}$ divides every component of $N^{-1}DM^{-1}\omega$, but $p$ does not divide $\xi^{\mathrm{T}}\omega$. Putting $w = M^{-1}\omega$ completes the proof. □

**Proof of Theorem 4.9.** Using the standard bases, we may regard $x$ and $y$ as elements of $\mathbb{Z}^{\dim(S^\lambda)}$; the condition $x \in S^\lambda_{(i)}$ then simply means that $p^i$ divides every component of $x^{\mathrm{T}}B$, and similarly for $y \in S^{\lambda'}_{(j)}$. So if $i + j > h$ and $x \in S^\lambda_{(i)}$, $y \in S^{\lambda'}_{(j)}$, we have

$$\begin{aligned}[x, y] &= x^{\mathrm{T}}A^{\mathrm{T}}y \\ &= \frac{x^{\mathrm{T}}BA^{-1}Cy}{H},\end{aligned}$$

which is divisible by $p$. Now suppose $x$ is such that, for all $z$, some component of $(x+pz)^{\mathrm{T}}B$ is not divisible by $p^{i+1}$. Putting $D = A^{-1}C(A^{-1})^{\mathrm{T}}$, we have $BD = H.I$, and so by Lemma 4.12 there exists $w \in \mathbb{Z}^{\dim(S^\lambda)}$ such that $p^{h-i}$ divides every component of $Dw$ but $p$ does not divide $x^{\mathrm{T}}w$. Putting $y = (A^{-1})^{\mathrm{T}}w$ then tell us that $p^{h-i}$ divides every component of $Cy$, but $p$ does not divide $x^{\mathrm{T}}A^{\mathrm{T}}y$, which is what we want. □

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