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# On the blocks of $\mathfrak{S}_{13}$ over a field of characteristic three

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## 1 Introduction

In [16] and [17], Tan looks at the principal blocks of the symmetric groups $\mathfrak{S}_9$, $\mathfrak{S}_{10}$, $\mathfrak{S}_{11}$, $\mathfrak{S}_{12}$ over a field of characteristic three, and highlights the contrast with blocks of small abelian defect. Here we continue by determining the Ext-quivers of the blocks of $\mathfrak{S}_{13}$ over the same field; this in particular provides examples of a [4 : 1]-pair and a [3 : 2]-pair with non-abelian defect.

For an account of the basic facts of modular representation theory, see Alperin's book [1]. We use the following notation (some of it in common with Alperin).

- $k$ will be an algebraically closed field of characteristic $p$; beyond this introduction, we shall take $p = 3$.
- $M\uparrow_A^B$ (resp. $M\downarrow_B^A$) will denote the module $M$ induced (resp. restricted) from the block $A$ to the block $B$. The $A$ may be omitted if it is clear.
- $P(M)$ will denote the projective cover of a module $M$, and $\Omega(M)$ the submodule of $P(M)$ with $P(M)/\Omega(M) \cong M$.
- For any module $M$ and any simple module $S$, $[M : S]$ will denote the multiplicity of $S$ as a composition factor of $M$.
- If a module $M$ has composition factors $S_1, \ldots, S_r$, we write

$$M \sim S_1 + \ldots + S_r.$$

- We write

$$M \sim \begin{matrix} N_1 \\ \vdots \\ N_r \end{matrix}$$

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to indicate that $M$ has a filtration

$$0 = M_0 \leqslant \ldots \leqslant M_r = M$$

with $M_i/M_{i-1} \cong N_i$ for all $i$.

- If $S$ and $T$ are simple modules and $M_1, \ldots, M_r$ any modules, we write

$$\begin{matrix} S \\ M_1 \ldots M_r \\ T \end{matrix}$$

to indicate a module with simple cosocle isomorphic to $S$, simple socle isomorphic to $T$ and heart isomorphic to $M_1 \oplus \ldots \oplus M_r$.

- The *Ext-quiver* (or *ordinary quiver*) of a symmetric group block $B$ is a graph with vertices $v_S$ indexed by simple modules $S$, and with the number of arrows from $v_S$ to $v_T$ being $\dim_k \mathrm{Ext}^1_B(S, T)$.

## 1.1 The representation theory of the symmetric group

For a detailed account of the representation theory of the symmetric group, the reader is urged to consult the books of James [6] and James & Kerber [7]; we summarise the salient points as well as new results not found in [6] or [7].

In characteristic zero, the *Specht modules* $S^\lambda$ give a complete set of mutually non-isomorphic simple modules for $\mathfrak{S}_n$ as $\lambda$ runs through the set of partitions of $n$. In positive characteristic $p$, the Specht module $S^\lambda$ has a simple cosocle $D^\lambda$ provided $\lambda$ is $p$-regular, and the set of such give a complete set of irreducible modules. If we let $\trianglerighteqslant$ denote the dominance order on partitions, then all composition factors $D^\mu$ of $\mathrm{rad}(S^\lambda)$ satisfy $\mu \rhd \lambda$ if $\lambda$ is $p$-regular, while all composition factors $D^\mu$ of $S^\lambda$ satisfy $\mu \rhd \lambda$ if $\lambda$ is $p$-singular. We use James's abacus notation [7] for partitions of $n$.

The block structure of $k\mathfrak{S}_n$ is given by Nakayama's 'Conjecture'. In terms of the abacus notation, this states that the Specht modules $S^\lambda$ and $S^\mu$ for $k\mathfrak{S}_n$ lie in the same block of $k\mathfrak{S}_n$ if and only if $\lambda$ and $\mu$ can be displayed on abacuses with the same number of beads on runner $i$, for each $i$. The partition whose abacus display is obtained from that of $\lambda$ by moving all the beads on each runner as far up as they will go we call the *$p$-core* of $\lambda$. If the $p$-core of $\lambda$ is a partition of $n - \omega p$, we say that the block containing $S^\lambda$ has *weight* $\omega$.

In addition to the Branching Rule for Specht modules [6, Theorem 9.3], we use results of Kleshchev concerning induction and restriction of simple $\mathfrak{S}_n$-modules $D^\lambda$. He derives necessary and sufficient conditions for $D^\lambda\!\uparrow^{k\mathfrak{S}_{n+1}}$ and $D^\lambda\!\downarrow_{k\mathfrak{S}_{n-1}}$ to be semi-simple, and describes the socle and cosocle of each. In [4] this is broken down further, and the same result is achieved for the induction (resp. restriction) of $D^\lambda$ to each block of $k\mathfrak{S}_{n+1}$ (resp. $k\mathfrak{S}_{n-1}$); we use these stronger results, which we now state.

Let $D^\lambda$ be a simple module lying in a block $B$ of $k\mathfrak{S}_n$, and take an abacus display for $\lambda$. Say that a bead $b$ on runner $i$ and in row $r$ of the display is:

- *normal* if there is no bead immediately to the left of $b$ and if for every $j \geqslant 1$ the number of beads on runner $i$ in rows $r+1, \ldots, r+j$ is at least the number of beads on runner $i-1$ in rows $r+1, \ldots, r+j$;

- *good* if $b$ is the highest normal bead on runner $i$;
- *conormal* if there is no bead immediately to the right of $b$ and if for every $j \geqslant 1$ the number of beads on runner $i$ in rows $r-1, \ldots, r-j$ is at least the number of beads on runner $i+1$ in rows $r-1, \ldots, r-j$;
- *cogood* if $b$ is the lowest conormal bead on runner $i$.

Let $B^+$ be the block of $k\mathfrak{S}_{n+1}$ whose abacus is obtained by moving a bead from runner $i$ to runner $i+1$, and let $B^-$ be the block of $k\mathfrak{S}_{n-1}$ whose abacus is obtained by moving a bead from runner $i$ to runner $i-1$. If $b$ is normal, let $\lambda_b$ be the partition obtained by moving $b$ one place to its left, and if $b$ is conormal, let $\lambda^b$ be the partition obtained by moving $b$ one place to its right. With these definitions, the following holds.

**Theorem 1.1.**

1. *$D^\lambda \downarrow_{B^-}^{B} = 0$ if there are no normal beads on runner i. Otherwise $D^\lambda \downarrow_{B^-}^{B}$ is an indecomposable module with simple cosocle and socle both isomorphic to $D^{\lambda_b}$, where b is the unique good bead on runner i; $D^\lambda \downarrow_{B^-}^{B}$ is simple if and only if b is the only normal bead on runner i.*
2. *$D^\lambda \uparrow_{B}^{B^+} = 0$ if there are no conormal beads on runner i. Otherwise $D^\lambda \uparrow_{B}^{B^+}$ is an indecomposable module with simple cosocle and socle both isomorphic to $D^{\lambda^b}$, where b is the unique cogood bead on runner i; $D^\lambda \uparrow_{B}^{B^+}$ is simple if and only if b is the only conormal bead on runner i.*

In [12], Mullineux gave an algorithm which constructs a bijection $f$ from the set of $p$-regular partitions of $n$ to itself, and conjectured that

$$D^\lambda \otimes \mathrm{sgn} = D^{f(\lambda)} \tag{$*$}$$

for all $p$-regular $\lambda$; here sgn denotes the alternating representation of $\mathfrak{S}_n$. This conjecture was finally verified by Ford and Kleshchev [5], by using the equivalent algorithm given by Kleshchev in [9]. We now describe the algorithm. For each $p$-regular partition $\lambda$ we construct a *Mullineux symbol* (this term is due to Bessenrodt & Olsson [2]) by removing rim $p$-hooks from the Young diagram of $\lambda$; since we use the abacus notation extensively, we shall describe the process in terms of the abacus. We form a sequence of partitions $\lambda = \lambda^0, \ldots, \lambda^u = (0)$, where $\lambda^i$ is a partition of some $n_i < n$, and $\lambda^{i+1}$ is obtained from $\lambda^i$ by the following algorithm.

1. Let $x$ be the greatest occupied position in the abacus display of $\lambda^i$.
2. If there is no unoccupied position less than $x$ in the display, then stop. Otherwise, let $y$ be
   - the greatest unoccupied position less than $x$ on the same runner as $x$, if there are any, or
   - the least unoccupied position in the display, if not.

   Move the bead at position $x$ to position $y$.
3. Let $x$ be the greatest occupied position less than $y$ in the abacus, and return to step 2.

It is clear that this procedure will eventually produce the partition (0). Given the partitions $\lambda^0, \dots, \lambda^u$, define the Mullineux symbol to be the pair of vectors $(r_1, \dots, r_u)$, $(s_1, \dots, s_u)$ by

$$r_i = \text{the number of non-zero parts of } \lambda^{i-1},$$

$$s_i = n_{i-1} - n_i.$$

Mullineux shows that a given Mullineux symbol corresponds to at most one partition, i.e. that a partition can be reconstructed from its Mullineux symbol. We construct a bijection between Mullineux symbols as follows: let $((r_1, \dots, r_u), (s_1, \dots, s_u))$ correspond to $((r'_1, \dots, r'_u), (s_1, \dots, s_u))$, with

$$r'_i = \begin{cases} s_i - r_i & (p \mid s_i) \\ s_i - r_i + 1 & (p \nmid s_i); \end{cases}$$

this function is evidently self-inverse. It turns out that if $((r_1, \dots, r_u), (s_1, \dots, s_u))$ corresponds to a $p$-regular partition $\lambda$ of $n$, then $((r'_1, \dots, r'_u), (s_1, \dots, s_u))$ also corresponds to a $p$-regular partition of $n$; call this $f(\lambda)$. $(*)$ then holds.

### 1.2 The blocks of $k\mathfrak{S}_{13}$

From now on, we let $p = 3$. By Nakayama's Conjecture, $k\mathfrak{S}_{13}$ has five blocks. Two of these have defect one, and so are well understood, so we consider the others, namely the principal block with core (1) and weight four, and the weight three blocks with cores $(3, 1)$ and $(2, 1^2)$. These last two are conjugate, so we need only consider one of them.

## 2 The principal block of $k\mathfrak{S}_{13}$

Let $B$ denote the principal block of $k\mathfrak{S}_{13}$, and $\tilde{B}$ the principal block of $k\mathfrak{S}_{12}$; these blocks form a $[4:1]$-pair in the sense of Scopes. We shall use the $\langle 4,5,4\rangle$-notation to denote partitions of $B$ as follows: form the display of a partition $\lambda$ on an abacus with four beads on the first runner, five on the second and four on the third, and then denote $\lambda$ by

- $\langle i\rangle$ if the display has a bead of weight four on runner $i$;
- $\langle \underline{i}, j\rangle$ if the display has a bead of weight three on runner $i$ and a bead of weight one on runner $j$;
- $\langle i, j\rangle$ if the display has beads of weight two on runners $i$ and $j$;
- $\langle i, j, k\rangle$ if the display has a bead of weight two on runner $i$ and beads of weight one on runners $j$ and $k$;
- $\langle i, j, k, l\rangle$ if the display has beads of weight one on runners $i$, $j$, $k$ and $l$.

We denote the partitions of $\tilde{B}$ in a similar way, using $\langle 5,4,4\rangle$-notation.

By Theorem 1.1, we find that fifteen of the simple modules of $B$ restrict to simple modules in $\tilde{B}$, while five do not; call these modules *non-exceptional* and *exceptional* modules respectively. The restriction of non-exceptional modules to $\tilde{B}$ (and induction from $\tilde{B}$) is given by Theorem 1.1, and is as follows.

$$
\begin{array}{rr}
D^{\langle 2\rangle}{\downarrow}^{B}_{\tilde{B}} \cong D^{\langle 1\rangle}; & D^{\langle 1\rangle}{\uparrow}^{B}_{\tilde{B}} \cong D^{\langle 2\rangle}; \\
D^{\langle 2,3\rangle}{\downarrow}^{B}_{\tilde{B}} \cong D^{\langle 1,3\rangle}; & D^{\langle 1,3\rangle}{\uparrow}^{B}_{\tilde{B}} \cong D^{\langle 2,3\rangle}; \\
D^{\langle \underline{2},3\rangle}{\downarrow}^{B}_{\tilde{B}} \cong D^{\langle \underline{1},3\rangle}; & D^{\langle \underline{1},3\rangle}{\uparrow}^{B}_{\tilde{B}} \cong D^{\langle \underline{2},3\rangle}; \\
D^{\langle 3,1,2\rangle}{\downarrow}^{B}_{\tilde{B}} \cong D^{\langle 3,1,1\rangle}; & D^{\langle 3,1,1\rangle}{\uparrow}^{B}_{\tilde{B}} \cong D^{\langle 3,1,2\rangle}; \\
D^{\langle 2,1,3\rangle}{\downarrow}^{B}_{\tilde{B}} \cong D^{\langle 1,1,3\rangle}; & D^{\langle 1,1,3\rangle}{\uparrow}^{B}_{\tilde{B}} \cong D^{\langle 2,1,3\rangle}; \\
D^{\langle 3\rangle}{\downarrow}^{B}_{\tilde{B}} \cong D^{\langle 3\rangle}; & D^{\langle 3\rangle}{\uparrow}^{B}_{\tilde{B}} \cong D^{\langle 3\rangle}; \\
D^{\langle \underline{3},2\rangle}{\downarrow}^{B}_{\tilde{B}} \cong D^{\langle \underline{3},1\rangle}; & D^{\langle \underline{3},1\rangle}{\uparrow}^{B}_{\tilde{B}} \cong D^{\langle \underline{3},2\rangle}; \\
D^{\langle \underline{3},1\rangle}{\downarrow}^{B}_{\tilde{B}} \cong D^{\langle \underline{3},2\rangle}; & D^{\langle \underline{3},2\rangle}{\uparrow}^{B}_{\tilde{B}} \cong D^{\langle \underline{3},1\rangle}; \\
D^{\langle 1,3\rangle}{\downarrow}^{B}_{\tilde{B}} \cong D^{\langle 3,1,2\rangle}; & D^{\langle 3,1,2\rangle}{\uparrow}^{B}_{\tilde{B}} \cong D^{\langle 1,3\rangle}; \\
D^{\langle 3,3\rangle}{\downarrow}^{B}_{\tilde{B}} \cong D^{\langle 3,3\rangle}; & D^{\langle 3,3\rangle}{\uparrow}^{B}_{\tilde{B}} \cong D^{\langle 3,3\rangle}; \\
D^{\langle \underline{1},2\rangle}{\downarrow}^{B}_{\tilde{B}} \cong D^{\langle 1,2\rangle}; & D^{\langle 1,2\rangle}{\uparrow}^{B}_{\tilde{B}} \cong D^{\langle \underline{1},2\rangle}; \\
D^{\langle 1,2,2\rangle}{\downarrow}^{B}_{\tilde{B}} \cong D^{\langle 2,1,1\rangle}; & D^{\langle 2,1,1\rangle}{\uparrow}^{B}_{\tilde{B}} \cong D^{\langle 1,2,2\rangle}; \\
D^{\langle \underline{1},3\rangle}{\downarrow}^{B}_{\tilde{B}} \cong D^{\langle 1,2,3\rangle}; & D^{\langle 1,2,3\rangle}{\uparrow}^{B}_{\tilde{B}} \cong D^{\langle \underline{1},3\rangle}; \\
D^{\langle 1\rangle}{\downarrow}^{B}_{\tilde{B}} \cong D^{\langle \underline{1},2\rangle}; & D^{\langle \underline{1},2\rangle}{\uparrow}^{B}_{\tilde{B}} \cong D^{\langle 1\rangle}; \\
D^{\langle 1,2\rangle}{\downarrow}^{B}_{\tilde{B}} \cong D^{\langle 1,1\rangle}; & D^{\langle 1,1\rangle}{\uparrow}^{B}_{\tilde{B}} \cong D^{\langle 1,2\rangle}.
\end{array}
$$

Thus the $\mathrm{Ext}^1$-space between any two of the non-exceptional simple modules of $B$ may be determined using the Eckmann-Shapiro relations and the Ext-quiver of $\tilde{B}$. So we turn our attention to the exceptional simple modules of $B$, namely $D^{\langle \underline{2},2\rangle}$, $D^{\langle 3,2,2\rangle}$, $D^{\langle 2,2\rangle}$, $D^{\langle 2,2,3,3\rangle}$ and $D^{\langle 2,2,3\rangle}$. First we determine the extensions between an exceptional and a non-exceptional module, and then between exceptional modules.

## 2.1 The exceptional simple modules

Since the abacus display for each of the corresponding partitions has two beads which may be moved from the second to the first runner, we consider restriction to the block $\check{B}$ of $k\mathfrak{S}_{11}$ with core $(3, 1^2)$. For ease of notation, we label partitions of $B$, $\tilde{B}$ and $\check{B}$ as follows.

- In $B$: $\alpha_1 = \langle \underline{2}, 2\rangle, \alpha_2 = \langle 3, 2, 2\rangle, \alpha_3 = \langle 2, 2\rangle, \alpha_4 = \langle 2, 2, 3, 3\rangle, \epsilon = \langle 2, 2, 3\rangle$.
- In $\tilde{B}$: $\tilde{\alpha}_1 = \langle 2\rangle, \tilde{\alpha}_2 = \langle 2, 3\rangle, \tilde{\alpha}_3 = \langle \underline{2}, 1\rangle, \tilde{\alpha}_4 = \langle 2, 3, 3\rangle, \tilde{\epsilon} = \langle \underline{2}, 3\rangle$.
- In $\check{B}$: $\check{\alpha}_1 = (9, 1^2), \check{\alpha}_2 = (5, 4, 2), \check{\alpha}_3 = (6, 4, 1), \check{\alpha}_4 = (3^2, 2^2, 1), \check{\epsilon} = (6, 3, 2)$.

We reproduce the Ext-quiver of $\check{B}$ from [17].

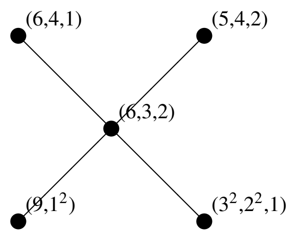



Figure 1: The Ext-quiver of $\check{B}$

Theorem 1.1 implies that:

- $D^{\tilde{\alpha}_j}\downarrow_{\check{B}} \cong D^{\check{\alpha}_j}$ for $j = 1, 2, 3, 4$;
- $D^{\tilde{\epsilon}}\downarrow_{\check{B}} \cong D^{\check{\epsilon}}$;
- $D^{\tilde{\lambda}}\downarrow_{\check{B}} = 0$ for other simple modules $D^{\tilde{\lambda}}$ in $\tilde{B}$.

We consider the relationships in $\tilde{B}$ between modules restricted from $B$ and modules induced from $\check{B}$.

**Lemma 2.1.**

$$\mathrm{cosoc}(\Omega(D^{\check{\alpha}_j}\uparrow^{\tilde{B}})) \cong D^{\tilde{\epsilon}};$$
$$\mathrm{cosoc}(\Omega(D^{\check{\epsilon}}\uparrow^{\tilde{B}})) \cong D^{\tilde{\alpha}_1} \oplus D^{\tilde{\alpha}_2} \oplus D^{\tilde{\alpha}_3} \oplus D^{\tilde{\alpha}_4}.$$

**Proof.** By inducing the short exact sequence

$$0 \longrightarrow \Omega(D^{\check{\alpha}_j}) \longrightarrow P(D^{\check{\alpha}_j}) \longrightarrow D^{\check{\alpha}_j} \longrightarrow 0$$

to $\tilde{B}$, we obtain

$$0 \longrightarrow \Omega(D^{\check{\alpha}_j})\!\uparrow^{\tilde{B}} \longrightarrow P(D^{\tilde{\alpha}_j}) \longrightarrow D^{\check{\alpha}_j}\!\uparrow^{\tilde{B}} \longrightarrow 0,$$

the middle term following by Frobenius reciprocity. This implies

$$\Omega(D^{\check{\alpha}_j}\!\uparrow^{\tilde{B}}) \cong \Omega(D^{\check{\alpha}_j})\!\uparrow^{\tilde{B}};$$

the latter module has cosocle $D^{\tilde{\epsilon}}$, by Frobenius reciprocity.

The second part is proved similarly. □

$D^{\check{\alpha}_j}\!\uparrow^{\tilde{B}}$ turns out to be a rather large module; in particular, it contains $D^{\alpha_j}\!\downarrow_{\tilde{B}}$, as we shall now see.

**Proposition 2.2.** *If $M$ is a $\tilde{B}$-module with simple cosocle isomorphic to $D^{\tilde{\alpha}_j}$, and $D^{\tilde{\epsilon}}$ does not occur as a composition factor of $M$, then $M$ is a quotient of $D^{\check{\alpha}_j}\!\uparrow^{\tilde{B}}$.*

*If $M$ is a $\tilde{B}$-module with simple cosocle isomorphic to $D^{\tilde{\epsilon}}$, and none of $D^{\tilde{\alpha}_1}$, $D^{\tilde{\alpha}_2}$, $D^{\tilde{\alpha}_3}$, $D^{\tilde{\alpha}_4}$ occurs as a composition factor of $M$, then $M$ is a quotient of $D^{\check{\epsilon}}\!\uparrow^{\tilde{B}}$.*

**Proof.** For the first assertion, it suffices to show that $\Omega(M)$ contains $\Omega(D^{\check{\alpha}_j}\!\uparrow^{\tilde{B}})$ as a submodule (viewing both as submodules of $P(D^{\tilde{\alpha}_j})$). Suppose the contrary; then

$$\Omega(M) \cap \Omega(D^{\check{\alpha}_j}\!\uparrow^{\tilde{B}}) \lneq \Omega(D^{\check{\alpha}_j})\!\uparrow^{\tilde{B}},$$

which implies that $D^{\tilde{\epsilon}}$ occurs as a composition factor of

$$\frac{\Omega(D^{\check{\alpha}_j}\!\uparrow^{\tilde{B}})}{\Omega(M) \cap \Omega(D^{\check{\alpha}_j}\!\uparrow^{\tilde{B}})} \cong \frac{\Omega(M) + \Omega(D^{\check{\alpha}_j}\!\uparrow^{\tilde{B}})}{\Omega(M)}$$

which is isomorphic to a submodule of $M$; contradiction.

The second statement is proved similarly. □

In order to apply Proposition 2.2, we determine explicitly the module structures of the restrictions of the exceptional simple modules of $B$ to $\tilde{B}$.

**Lemma 2.3.**

$$D^{\alpha_1}\!\downarrow_{\tilde{B}} \cong \begin{array}{c} D^{\langle 2\rangle} \\ D^{\langle 3\rangle} D^{\langle \underline{1},2\rangle} \\ D^{\langle 2\rangle} \end{array};$$

$$D^{\alpha_2}\!\downarrow_{\tilde{B}} \cong \begin{array}{c} D^{\langle 2,3\rangle} \\ D^{\langle 1,2\rangle} D^{\langle 3,1,2\rangle} \\ D^{\langle 2,3\rangle} \end{array};$$

$$D^{\alpha_3}\!\downarrow_{\tilde{B}} \cong \begin{array}{c} D^{\langle \underline{2},1\rangle} \\ D^{\langle \underline{1},2\rangle} D^{\langle \underline{3},1\rangle} D^{\langle 1,2\rangle} \\ D^{\langle \underline{2},1\rangle} \end{array};$$

$$D^{\alpha_4}\!\downarrow_{\tilde{B}} \cong \begin{array}{c} D^{\langle 2,3,3\rangle} \\ D^{\langle 3\rangle} D^{\langle 3,1,2\rangle} D^{\langle 2,1,1\rangle} \\ D^{\langle 2,3,3\rangle} \end{array};$$

$$D^{\epsilon}{\downarrow_{\tilde{B}}} \cong \begin{array}{c} D^{\langle \underline{2},3\rangle} \\ D^{\langle \underline{3},2\rangle} D^{\langle 1,2,3\rangle} \\ D^{\langle \underline{2},3\rangle} \end{array}.$$

**Proof.** We may obtain the composition factors of each of these restricted modules using the Branching Rule. First we find the composition factors of $S^{\alpha_i}$ and of $S^{\alpha_i}{\downarrow_{\tilde{B}}}$ by using the Branching Rule and the decomposition matrices for $B$ and $\tilde{B}$. Then for each composition factor $D^{\mu}$ of $\mathrm{rad}(S^{\alpha_i})$, we delete from $S^{\alpha_i}{\downarrow_{\tilde{B}}}$ the composition factor(s) of $D^{\mu}{\downarrow_{\tilde{B}}}$. Given that $D^{\alpha_i}{\downarrow_{\tilde{B}}}$ must be self-dual with cosocle and socle both isomorphic to $D^{\tilde{\alpha}_i}$, this is enough to determine the module structure completely. □

**Corollary 2.4.**

1. *$D^{\alpha_j}{\downarrow_{\tilde{B}}}$ is a quotient of $D^{\check{\alpha}_j}{\uparrow^{\tilde{B}}}$ for $j = 1, 2, 3, 4$;*
2. *$D^{\epsilon}{\downarrow_{\tilde{B}}}$ is a quotient of $D^{\check{\epsilon}}{\uparrow^{\tilde{B}}}$.*

**Proof.** This follows immediately from Proposition 2.2 and Lemma 2.3. □

Given this last result, we find that almost all simple modules of $\tilde{B}$ which extend $D^{\tilde{\alpha}_j}$ (resp. $D^{\tilde{\epsilon}}$) appear as factors of $D^{\check{\alpha}_j}{\uparrow^{\tilde{B}}}$ (resp. $D^{\check{\epsilon}}{\uparrow^{\tilde{B}}}$).

**Proposition 2.5.**

1. *Suppose $M$ is a submodule of $D^{\check{\alpha}_j}{\uparrow^{\tilde{B}}}$ such that $D^{\alpha_j}{\downarrow_{\tilde{B}}} \cong D^{\check{\alpha}_j}{\uparrow^{\tilde{B}}}/M$. Suppose also that $D^{\tilde{\lambda}}$ is a simple module in $\tilde{B}$ not isomorphic to $D^{\tilde{\epsilon}}$. Then*

$$\dim \mathrm{Ext}^1_{\tilde{B}}(D^{\alpha_j}{\downarrow_{\tilde{B}}}, D^{\tilde{\lambda}}) = [\mathrm{cosoc}(M) : D^{\tilde{\lambda}}].$$

2. *Suppose $M$ is a submodule of $D^{\check{\epsilon}}{\uparrow^{\tilde{B}}}$ such that $D^{\epsilon}{\downarrow_{\tilde{B}}} \cong D^{\check{\epsilon}}{\uparrow^{\tilde{B}}}/M$. Suppose also that $D^{\tilde{\lambda}}$ is a simple module in $\tilde{B}$ not isomorphic to any $D^{\tilde{\alpha}_j}$. Then*

$$\dim \mathrm{Ext}^1_{\tilde{B}}(D^{\epsilon}{\downarrow_{\tilde{B}}}, D^{\tilde{\lambda}}) = [\mathrm{cosoc}(M) : D^{\tilde{\lambda}}].$$

**Proof.** $P(D^{\tilde{\alpha}_j})$ has a filtration of the form

$$\begin{array}{c} D^{\alpha_j}{\downarrow_{\tilde{B}}} \\ M \\ \Omega(D^{\check{\alpha}_j}{\uparrow^{\tilde{B}}}) \end{array},$$

and so the result follows. Similarly for $P(D^{\tilde{\epsilon}})$. □

Before proceeding, we state a very general lemma, which will be useful later and whose proof is obvious.

**Lemma 2.6.** *Suppose $B$ is a block of $k\mathfrak{S}_n$, where $k$ is a field of any characteristic, and $\tilde{B}$ is a block of $k\mathfrak{S}_{n-1}$. Suppose that $D^{\lambda}$ and $D^{\mu}$ are simple modules of $B$, with $\mathrm{Ext}^1_B(D^{\lambda}, D^{\mu}) \neq 0$. Suppose further that $\mathrm{soc}(D^{\lambda}{\downarrow_{\tilde{B}}})$ is a simple module $D^{\tilde{\lambda}}$. Then either*

1. $\mathrm{Ext}^1_B(D^{\tilde\lambda}\uparrow^B, D^\mu) = 0$, *in which case* $\dim \mathrm{Ext}^1_B(D^\lambda, D^\mu)$ *is the composition multiplicity of* $D^\mu$ *in the second Loewy layer of* $D^{\tilde\lambda}\uparrow^B$*, or*
2. $\mathrm{Ext}^1_{\tilde B}(D^{\tilde\lambda}, D^\mu\downarrow_{\tilde B}) \neq 0$*; in particular,* $D^\mu\downarrow_{\tilde B}\neq 0$.

## 2.2 Extensions of exceptional simple modules

Now we are able to determine the extensions between exceptional and non-exceptional simple modules of $B$. Let $D^\lambda$ be a non-exceptional simple module, and let $D^{\tilde\lambda}$ be the simple module of $\tilde B$ such that

$$D^\lambda\downarrow_{\tilde B}\cong D^{\tilde\lambda}, \qquad D^{\tilde\lambda}\uparrow^B\cong D^\lambda.$$

Then

$$\mathrm{Ext}^1_B(D^{\alpha_j}, D^\lambda) \cong \mathrm{Ext}^1_{\tilde B}(D^{\alpha_j}\downarrow_{\tilde B}, D^{\tilde\lambda})$$

and

$$\mathrm{Ext}^1_B(D^{\epsilon}, D^\lambda) \cong \mathrm{Ext}^1_{\tilde B}(D^{\epsilon}\downarrow_{\tilde B}, D^{\tilde\lambda}).$$

In view of Corollary 2.4, we look at the structures of the induced modules $D^{\check\alpha_j}\uparrow^{\tilde B}$, $D^{\check\epsilon}\uparrow^{\tilde B}$, which we obtain using the Branching Rule.

### 2.2.1 Extensions of $D^{\alpha_1}$ and $D^{\alpha_2}$

From the decomposition matrix of $\check B$ we see that $D^{(9,1^2)} \cong S^{(9,1^2)}$. By the Branching Rule we find that $D^{(9,1^2)}\uparrow^{\tilde B}$ has a filtration

$$\begin{matrix} S^{\langle 2\rangle} \\ S^{\langle \underline{1},2\rangle} \\ S^{\langle \underline{1},1\rangle} \end{matrix}.$$

Looking at the decomposition matrix for $\tilde B$, we see that

$$\begin{aligned} S^{\langle 2\rangle} &\sim D^{\langle 2\rangle} + D^{\langle 3\rangle};\\ S^{\langle \underline{1},2\rangle} &\sim D^{\langle \underline{1},2\rangle} + D^{\langle 2\rangle} + D^{\langle \underline{1},3\rangle} + D^{\langle 1\rangle} + D^{\langle 1\rangle} + D^{\langle 3\rangle};\\ S^{\langle \underline{1},1\rangle} &\sim D^{\langle \underline{1},2\rangle} + D^{\langle 2\rangle}. \end{aligned}$$

**Lemma 2.7.**

$$\mathrm{soc}(S^{\langle \underline{1},2\rangle}) \cong D^{\langle 3\rangle}.$$

**Proof.** From the Ext-quiver of $\tilde B$ we see that the copy of $D^{\langle \underline{1},3\rangle}$ in $S^{\langle \underline{1},2\rangle}$ must lie between the copies of $D^{\langle 1\rangle}$, and so the only possible factors of $\mathrm{soc}(S^{\langle \underline{1},2\rangle})$ are $D^{\langle 3\rangle}$, $D^{\langle 2\rangle}$ and one of the copies of $D^{\langle 1\rangle}$.

Consider the block $C$ of $k\mathfrak{S}_{11}$ with 3-core $(1^2)$. Using the decomposition matrices of $C$ and $\tilde B$ we find that

$$D^{(9,2)}\uparrow_C^{\tilde B}\sim D^{\langle \underline{1},3\rangle}\times 2 + D^{\langle 1\rangle}\times 2 + D^{\langle \underline{1},2\rangle}.$$

By Theorem 1.1 we have that $\mathrm{cosoc}(D^{(9,2)}\uparrow_C^{\tilde B}) \cong D^{\langle \underline{1},3\rangle}$, and so $D^{(9,2)}\uparrow_C^{\tilde B}$ has a quotient isomorphic to the unique non-split extension of $D^{\langle \underline{1},3\rangle}$ by $D^{\langle 1\rangle}$. Hence if $D^{\langle 1\rangle}$ appears in the socle of $S^{\langle \underline{1},2\rangle}$ we must have

$$0 \neq \mathrm{Hom}(D^{(9,2)}\uparrow_C^{\tilde B}, S^{\langle \underline{1},2\rangle})$$

$$\begin{aligned} &\cong \mathrm{Hom}(D^{(9,2)}, S^{\langle\underline{1},2\rangle}\downarrow_C^{\tilde{B}}) \\ &\cong \mathrm{Hom}(D^{(9,2)}, S^{(9,2)}) \end{aligned}$$

by the Branching rule. But $S^{(9,2)}$ is not simple. Contradiction.

Next we show that $D^{\langle 2\rangle}$ does not lie in the socle of $S^{\langle\underline{1},2\rangle}$. $D^{(9,1^2)}\uparrow^{\tilde{B}}$ has a submodule isomorphic to

$$D^{\alpha_1}\downarrow_{\tilde{B}} \cong \begin{array}{c} D^{\langle 2\rangle} \\ D^{\langle 3\rangle} D^{\langle\underline{1},2\rangle} \\ D^{\langle 2\rangle} \end{array};$$

from the above Specht filtration and decompositions we see that the top copy of $D^{\langle 2\rangle}$ and the copy of $D^{\langle 3\rangle}$ in this submodule are factors of $S^{\langle\underline{1},2\rangle}$. Hence $D^{\langle 2\rangle}$ does not lie in the socle of $S^{\langle\underline{1},2\rangle}$. Thus the only factor of $\mathrm{soc}(S^{\langle\underline{1},2\rangle})$ is $D^{\langle 3\rangle}$. □

Thus we can determine the structure of $D^{(9,1^2)}\uparrow^{\tilde{B}}$. In particular, we have the following.

**Lemma 2.8.** *Let $M$ be the submodule of $D^{(9,1^2)}\uparrow^{\tilde{B}}$ with*

$$\frac{D^{(9,1^2)}\uparrow^{\tilde{B}}}{M} \cong D^{\alpha_1}\downarrow_{\tilde{B}}.$$

*Then $M$ has (simple) cosocle $D^{\langle 1\rangle}$.*

**Proof.** The composition factors of $M$ are $D^{\langle\underline{1},2\rangle}$, $D^{\langle 2\rangle}$, $D^{\langle\underline{1},3\rangle}$, $D^{\langle 1\rangle}$ (twice) and $D^{\langle 3\rangle}$. As in the proof of Lemma 2.7, the copy of $D^{\langle\underline{1},3\rangle}$ must lie between the two copies of $D^{\langle 1\rangle}$; these copies of $D^{\langle 1\rangle}$ must extend $D^{\langle\underline{1},2\rangle}$ and $D^{\langle 3\rangle}$ from what we know about the structure of $S^{\langle\underline{1},2\rangle}$ and since $D^{(9,1^2)}\uparrow^{\tilde{B}}$ is self-dual. The copy of $D^{\langle 2\rangle}$ constitutes the socle of $M$. □

Given this last result, we can apply Proposition 2.5. We find that the only non-exceptional simple module of $B$ extending $D^{\alpha_1}$ is $D^{\langle 1\rangle}\uparrow^B = D^{\langle 2\rangle}$, and

$$\mathrm{Ext}^1_B(D^{\alpha_1}, D^{\langle 2\rangle}) \cong k.$$

To find the extensions of $D^{\alpha_2}$, we use Mullineux's algorithm, which tells us that

$$D^{\alpha_2} \cong D^{\alpha_1} \otimes \mathrm{sgn}.$$

It is clear that for any $k\mathfrak{S}_n$-modules $M$, $N$,

$$\mathrm{Ext}^1_{k\mathfrak{S}_n}(M \otimes \mathrm{sgn}, N \otimes \mathrm{sgn}) \cong \mathrm{Ext}^1_{k\mathfrak{S}_n}(M, N);$$

and so the only non-exceptional simple module of $B$ extending $D^{\alpha_2}$ is $D^{\langle 2\rangle} \otimes \mathrm{sgn} \cong D^{\langle 2,3\rangle}$, and the corresponding Ext-space is one-dimensional.

### 2.2.2 Extensions of $D^{\alpha_3}$ and $D^{\alpha_4}$

Now we look at $D^{\alpha_3}$; the corresponding results for $D^{\alpha_4}$ will follow, since $D^{\alpha_4} \cong D^{\alpha_3} \otimes \mathrm{sgn}$. We have

$$D^{\check{\alpha}_3} = D^{(6,4,1)} \cong S^{(6,4,1)},$$

and the Branching Rule gives a filtration

$$D^{(6,4,1)}\uparrow^{\tilde{B}} \cong \begin{array}{c} S^{\langle \underline{2},1\rangle} \\ S^{\langle 1,2\rangle} \\ S^{\langle 1,1\rangle} \end{array}.$$

Looking at the decomposition matrix for $\tilde{B}$, we see

$$\begin{aligned} S^{\langle \underline{2},1\rangle} &\sim D^{\langle \underline{2},1\rangle} + D^{\langle \underline{1},3\rangle} + D^{\langle \underline{1},2\rangle} + D^{\langle \underline{3},1\rangle} + D^{\langle 1\rangle}; \\ S^{\langle 1,2\rangle} &\sim D^{\langle 1,2\rangle} + D^{\langle \underline{2},1\rangle} + D^{\langle 1,3\rangle} + D^{\langle \underline{3},1\rangle}; \\ S^{\langle 1,1\rangle} &\sim D^{\langle 1,1\rangle} + D^{\langle 1,2\rangle} + D^{\langle \underline{2},1\rangle} + D^{\langle \underline{1},2\rangle} + D^{\langle 1\rangle}. \end{aligned}$$

**Lemma 2.9.**

$$\begin{aligned} \mathrm{soc}(S^{\langle \underline{2},1\rangle}) &\cong D^{\langle \underline{1},3\rangle}; \\ \mathrm{soc}(S^{\langle 1,2\rangle}) &\cong D^{\langle \underline{3},1\rangle}. \end{aligned}$$

**Proof.** Again we consider the block $C$ of $\mathfrak{S}_{11}$ with core $(1^2)$. By using the Branching Rule and the decomposition matrices we find that

$$\begin{aligned} D^{\langle 1\rangle}\downarrow^{\tilde{B}}_{C} &= 0, \\ D^{\langle \underline{1},2\rangle}\downarrow^{\tilde{B}}_{C} &= 0, \\ D^{\langle \underline{2},1\rangle}\downarrow^{\tilde{B}}_{C} &= 0, \\ D^{\langle \underline{1},3\rangle}\downarrow^{\tilde{B}}_{C} &\cong D^{(9,2)}, \\ D^{\langle \underline{3},1\rangle}\downarrow^{\tilde{B}}_{C} &\cong D^{(7,4)}, \\ D^{\langle 1,3\rangle}\downarrow^{\tilde{B}}_{C} &\cong D^{(6,5)}. \end{aligned}$$

The Branching Rule also gives filtrations

$$S^{(7,4)}\uparrow^{\tilde{B}} \sim \begin{array}{c} S^{\langle \underline{3},1\rangle} \\ S^{\langle \underline{2},1\rangle} \end{array}, \qquad S^{(6,5)}\uparrow^{\tilde{B}} \sim \begin{array}{c} S^{\langle 1,3\rangle} \\ S^{\langle 1,2\rangle} \end{array};$$

so for a simple module $D^{\tilde{\lambda}}$ in $\tilde{B}$,

$$\begin{aligned} \mathrm{Hom}_{\tilde{B}}(D^{\tilde{\lambda}}, S^{\langle \underline{2},1\rangle}) &\leqslant \mathrm{Hom}_{\tilde{B}}(D^{\tilde{\lambda}}, S^{(7,4)}\uparrow^{\tilde{B}}) \\ &\cong \mathrm{Hom}_{C}(D^{\tilde{\lambda}}\downarrow_{C}, S^{(7,4)}) \end{aligned}$$

and

$$\mathrm{Hom}_{\tilde{B}}(D^{\tilde{\lambda}}, S^{\langle 1,2\rangle}) \leqslant \mathrm{Hom}_{\tilde{B}}(D^{\tilde{\lambda}}, S^{(6,5)}\uparrow^{\tilde{B}})$$

$$\cong \mathrm{Hom}_C(D^{\tilde{\lambda}}\downarrow_C, S^{(6,5)}).$$

The result now follows, given the module structures

$$S^{(7,4)} \cong \begin{array}{c} D^{(7,4)} \\ D^{(9,2)} \end{array} \qquad S^{(6,5)} \cong \begin{array}{c} D^{(6,5)} \\ D^{(7,4)} \end{array}$$

and the above decompositions of $S^{\langle \underline{2},1\rangle}$ and $S^{\langle 1,2\rangle}$. □

**Lemma 2.10.** *Let $M$ be the submodule of $D^{(6,4,1)}\uparrow^{\tilde{B}}$ with*

$$\frac{D^{(6,4,1)}\uparrow^{\tilde{B}}}{M} \cong D^{\alpha_3}\downarrow_{\tilde{B}}.$$

*Then* $\mathrm{cosoc}(M)$ *has factors $D^{\langle 1\rangle}$ and possibly $D^{\langle 1,3\rangle}$.*

**Proof.** We know the composition factors of $M$ from above, and we know that $D^{\langle \underline{2},1\rangle}\uparrow^{\tilde{B}}$ has both a submodule and a quotient isomorphic to

$$D^{\alpha_3}\downarrow_{\tilde{B}} \cong \begin{array}{c} D^{\langle \underline{2},1\rangle} \\ D^{\langle \underline{1},2\rangle} D^{\langle \underline{3},1\rangle} D^{\langle 1,2\rangle} \\ D^{\langle \underline{2},1\rangle} \end{array}.$$

The copy of $D^{\langle \underline{1},3\rangle}$ in $M$ is the socle of $S^{\langle \underline{2},1\rangle}$, so lies below one of the copies of $D^{\langle 1\rangle}$, and hence above the other, since $D^{\langle \underline{2},1\rangle}\uparrow^{\tilde{B}}$ is self-dual.

$D^{\langle 1,1\rangle}$ lies above the lower copy of $D^{\langle 1\rangle}$, and hence below the upper copy; all other factors of $S^{\langle 1,1\rangle}$ lie below this.

$D^{\langle \underline{3},1\rangle}$ lies below $D^{\langle 1,3\rangle}$, being the socle of $S^{\langle 1,2\rangle}$, and so the only factors which can possibly lie in the cosocle of $M$ are the upper copy of $D^{\langle 1\rangle}$ and the copy of $D^{\langle 1,3\rangle}$. The former cannot lie below the latter since $D^{\langle \underline{2},1\rangle}\uparrow^{\tilde{B}}$ is self-dual. □

Now we can apply Proposition 2.5.

**Corollary 2.11.**

$$\mathrm{Ext}^1_B(D^{\alpha_3}, D^{\langle 2\rangle}) \cong k;$$
$$\mathrm{Ext}^1_B(D^{\alpha_3}, D^{\langle 2,3\rangle}) \cong 0 \text{ or } k;$$

$\mathrm{Ext}^1_B(D^{\alpha_3}, D^{\lambda}) = 0$ *for any other non-exceptional simple module $D^{\lambda}$ of $B$.*

**Proof.** From Proposition 2.5 and Lemma 2.10. □

We still have one Ext-space undetermined. To find this, we shall use Lemma 2.6, with $\lambda = \alpha_3$ and $\mu = \langle 2,3\rangle$. Now $\mathrm{Ext}^1_{\tilde{B}}(D^{\langle \underline{2},1\rangle}, D^{\langle 1,3\rangle}) = 0$, so the dimension of the unknown Ext-space is the number of copies of $D^{\langle 2,3\rangle}$ lying in the second Loewy layer of $D^{\langle \underline{2},1\rangle}\uparrow^{B}$.

Using the Branching Rule and the decomposition matrices, we find that

$$D^{\langle \underline{2},1\rangle}{\uparrow}^{B} \sim D^{\langle 2,2\rangle} \times 2 + D^{\langle 2\rangle} \times 2 + D^{\langle 2,3\rangle} + D^{\langle \underline{2},3\rangle} + D^{\langle 1,2\rangle}.$$

In addition,

$$\mathrm{Ext}^1_B(D^{\langle 2\rangle}, D^{\langle 2,3\rangle}) \cong \mathrm{Ext}^1_{\tilde{B}}(D^{\langle 1\rangle}, D^{\langle 1,3\rangle}) = 0$$

and $D^{\langle \underline{2},1\rangle}{\uparrow}^{B}$ is self-dual, so there is exactly one copy of $D^{\langle 2,3\rangle}$ lying in its second Loewy layer. Thus

$$\mathrm{Ext}^1_B(D^{\alpha_3}, D^{\langle 2,3\rangle}) \cong k.$$

### 2.2.3 Extensions of $D^{\epsilon}$

Finally we look at $D^{\epsilon}$. Here we have

$$D^{\check{\epsilon}} \cong D^{(6,3,2)} \cong S^{(6,1^5)},$$

and from the Branching Rule we get

$$D^{(6,3,2)}{\uparrow}^{\tilde{B}} \cong \begin{array}{c} S^{\langle \underline{2},2\rangle} \\ S^{\langle 1,2,2\rangle} \\ S^{\langle 1,1,1\rangle} \end{array}.$$

The decomposition matrices give

$$\begin{aligned} S^{\langle \underline{2},2\rangle} &\sim D^{\langle \underline{2},3\rangle} + D^{\langle \underline{3},2\rangle}, \\ S^{\langle 1,2,2\rangle} &\sim D^{\langle \underline{2},3\rangle} + D^{\langle \underline{3},2\rangle} + D^{\langle 1,2,3\rangle} + D^{\langle 1,1,3\rangle}, \\ S^{\langle 1,1,1\rangle} &\sim D^{\langle 1,2,3\rangle} + D^{\langle \underline{2},3\rangle}. \end{aligned}$$

Thus we may deduce the following.

**Lemma 2.12.** *Let $M$ be the submodule of $D^{(6,3,2)}{\uparrow}^{\tilde{B}}$ with*

$$\frac{D^{(6,3,2)}{\uparrow}^{\tilde{B}}}{M} \cong D^{\epsilon}{\downarrow}_{\tilde{B}}\ .$$

*Then* $\mathrm{cosoc}(M)$ *has factors $D^{\langle 1,1,3\rangle}$ and possibly one of $D^{\langle \underline{3},2\rangle}$ or $D^{\langle 1,2,3\rangle}$.*

**Proof.** $D^{(6,3,2)}{\uparrow}^{\tilde{B}}$ has the above listed factors and is self-dual, with both a submodule and a quotient isomorphic to

$$D^{\epsilon}{\downarrow}_{\tilde{B}} \cong \begin{array}{c} D^{\langle \underline{2},3\rangle} \\ D^{\langle \underline{3},2\rangle} D^{\langle 1,2,3\rangle} \\ D^{\langle \underline{2},3\rangle} \end{array}.$$

The result follows; note that $D^{\langle \underline{3},2\rangle}$ and $D^{\langle 1,2,3\rangle}$ cannot both occur in the cosocle of $M$. □

We apply Proposition 2.5 once more.

**Corollary 2.13.**

$$\mathrm{Ext}^1_B(D^\epsilon, D^{\langle 2,1,3\rangle}) \cong k;$$

$\mathrm{Ext}^1_B(D^\epsilon, D^\lambda) = 0$ *for any other non-exceptional simple module* $D^\lambda$ *of* $B$.

**Proof.** From Proposition 2.5 and Lemma 2.12, we get:

- $\mathrm{Ext}^1_B(D^\epsilon, D^{\langle 2,1,3\rangle}) \cong k$;
- $\mathrm{Ext}^1_B(D^\epsilon, D^{\langle \underline{3},1\rangle}) \cong 0$ or $k$;
- $\mathrm{Ext}^1_B(D^\epsilon, D^{\langle \underline{1},3\rangle}) \cong 0$ or $k$;
- $\mathrm{Ext}^1_B(D^\epsilon, D^\lambda) = 0$ for any other non-exceptional simple module $D^\lambda$ of $B$.

But by Mullineux's algorithm we have

$$D^\epsilon \otimes \mathrm{sgn} \cong D^\epsilon$$

and

$$D^{\langle \underline{3},1\rangle} \otimes \mathrm{sgn} \cong D^{\langle \underline{1},3\rangle},$$

whence

$$\mathrm{Ext}^1_B(D^\epsilon, D^{\langle \underline{3},1\rangle}) \cong \mathrm{Ext}^1_B(D^\epsilon, D^{\langle \underline{1},3\rangle}).$$

But from the last statement of the proof of Lemma 2.12, these spaces cannot both be non-zero; hence they are both zero, and the result follows. □

Thus we have a complete list of non-zero Ext-spaces between exceptional and non-exceptional simple modules of $B$ (all one-dimensional).

| Exceptional Module | Non-exceptional Modules |
|---|---|
| $D^{\langle \underline{2},2\rangle}$ | $D^{\langle 2\rangle}$ |
| $D^{\langle 3,2,2\rangle}$ | $D^{\langle 2,3\rangle}$ |
| $D^{\langle 2,2\rangle}$ | $D^{\langle 2\rangle}$, $D^{\langle 2,3\rangle}$ |
| $D^{\langle 2,2,3,3\rangle}$ | $D^{\langle 2\rangle}$, $D^{\langle 2,3\rangle}$ |
| $D^{\langle 2,2,3\rangle}$ | $D^{\langle 2,1,3\rangle}$ |

## 2.3 Extensions between exceptional modules

In order to determine the Ext-quiver of $B$, it remains to determine the extensions between the exceptional simple modules of $B$. These will turn out to follow exactly the extensions between the simple modules of $\check{B}$.

**Proposition 2.14.**

1. $D^{\alpha_i}{\downarrow_{\tilde{B}}}$ *does not extend* $D^{\tilde{\alpha}_j}$ *for any* $i,\ j$.
2. $D^\epsilon{\downarrow_{\tilde{B}}}$ *does not extend* $D^{\tilde{\epsilon}}$.

**Proof.** This follows from Proposition 2.5, given what we have seen about the structures of $D^{\check{\alpha}_j}\uparrow^{\tilde{B}}$ and $D^{\check{\epsilon}}\uparrow^{\tilde{B}}$. □

Using this we can find all the zero Ext-spaces between the exceptional simple modules; in particular, we find that they do not self-extend. We use the technique outlined in Lemma 2.3 to find composition factors of induced modules.

**Corollary 2.15.**

1. *$D^{\alpha_i}$ does not extend $D^{\alpha_j}$ for any i, j.*

2. *$D^{\epsilon}$ does not self-extend.*

**Proof.** From Lemma 2.6 and Proposition 2.14, we need only show that $D^{\tilde{\alpha}_i}\uparrow^B$ does not contain any $D^{\alpha_j}$ in its second Loewy layer, and that $D^{\tilde{\epsilon}}\uparrow^B$ does not contain $D^{\epsilon}$ in its second Loewy layer. But in fact by using the Branching rule and the decomposition matrices together with Theorem 1.1, we find that $D^{\tilde{\alpha}_i}\uparrow^B$ (respectively $D^{\tilde{\epsilon}}\uparrow^B$) with socle and cosocle both isomorphic to $D^{\alpha_i}$ (respectively $D^{\epsilon}$) and no factor isomorphic to $D^{\alpha_j}$ (respectively $D^{\epsilon}$) in its (non-zero) heart. □

Finally we find the non-zero Ext-spaces between exceptional simple modules.

**Lemma 2.16.**

$$\mathrm{Ext}^1_B(D^{\alpha_j}, D^{\epsilon}) \cong k$$

*for* $j = 1, 2, 3, 4$.

**Proof.** First we claim that $\mathrm{Ext}^1_{\tilde{B}}(D^{\alpha_j}\downarrow_{\tilde{B}}, D^{\tilde{\epsilon}}) \cong k$. Certainly $\mathrm{Ext}^1_{\tilde{B}}(D^{\tilde{\alpha_j}}, D^{\tilde{\epsilon}}) \neq 0$, and $D^{\alpha_j}\downarrow_{\tilde{B}}$ has simple head $D^{\tilde{\alpha_j}}$ and does not include $D^{\tilde{\epsilon}}$ as a composition factor; so

$$0 \neq \mathrm{Ext}^1_{\tilde{B}}(D^{\tilde{\alpha_j}}, D^{\tilde{\epsilon}}) \subseteq \mathrm{Ext}^1_{\tilde{B}}(D^{\alpha_j}\downarrow_{\tilde{B}}, D^{\tilde{\epsilon}}).$$

Now from the filtration given in the proof of Proposition 2.5 and the fact that $D^{\tilde{\epsilon}}$ does not feature as a composition factor of $D^{\check{\alpha}_j}\uparrow^{\tilde{B}}$, we have $\mathrm{Ext}^1_{\tilde{B}}(D^{\alpha_j}\downarrow_{\tilde{B}}, D^{\tilde{\epsilon}}) \cong k$. By the Eckmann-Shapiro relations we then have

$$k \cong \mathrm{Ext}^1_B(D^{\alpha_j}, D^{\tilde{\epsilon}}\uparrow^B).$$

Now

$$D^{\tilde{\epsilon}}\uparrow^B \cong \begin{matrix} D^{\epsilon} \\ D^{\langle 2,1,3\rangle} \\ D^{\epsilon} \end{matrix}$$

and no $D^{\alpha_j}$ extends $D^{\langle 2,1,3\rangle}$, so each $D^{\alpha_j}$ must extend $D^{\epsilon}$. $\mathrm{Ext}^1_B(D^{\alpha_j}, D^{\epsilon})$ is a vector subspace of $\mathrm{Ext}^1_B(D^{\alpha_j}, D^{\tilde{\epsilon}}\uparrow^B)$, and so is one-dimensional. □

## 2.4 The Ext-quiver of $B$

We have now determined the Ext-quiver of $B$. In order to illustrate the comparison with the principal block $\tilde{B}$ of $k\mathfrak{S}_{12}$, we give the Ext-quiver of this also (note that the vertex labelling is not the same as in [17], since Tan uses $\langle 4^3 \rangle$-notation for $\tilde{B}$), and highlight the vertices corresponding to exceptional simple modules.

In each quiver, the automorphism corresponding to the Mullineux map is achieved by rotating the entire diagram (except for the two central vertices) through 180° about its centre.

In [17], Tan notes that the Ext-quiver of $\tilde{B}$ is not bipartite; this holds for the Ext-quiver of $B$ also. Note, however, that both Ext-quivers are *subdivisions* of bipartite graphs (as defined by Bollobás [3, p. 16]): the Ext-quiver of $\tilde{B}$ has four vertices of valency two; if we delete each of these and join its neighbours with a single edge, we obtain a bipartite graph. Similarly we may delete four of the two-valent vertices of the Ext-quiver of $B$ and replace each with an edge joining its neighbours to obtain a bipartite graph (note that the Ext-quiver of $B$ has six two-valent vertices; we must be careful which ones we delete).

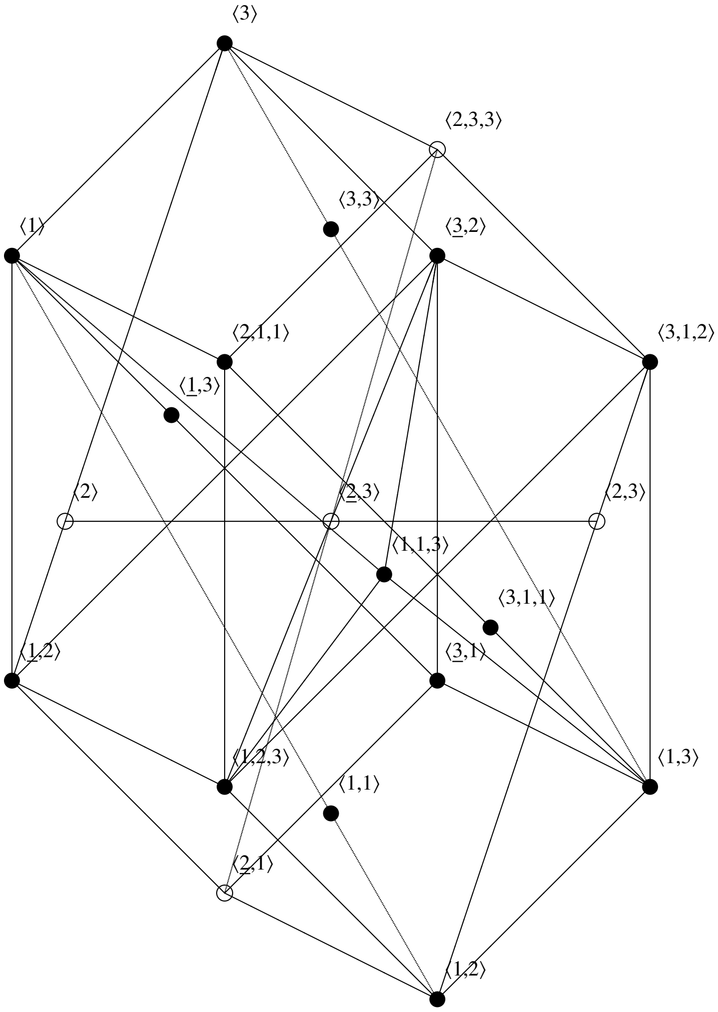



Figure 2: The Ext-quiver of $\tilde{B}$

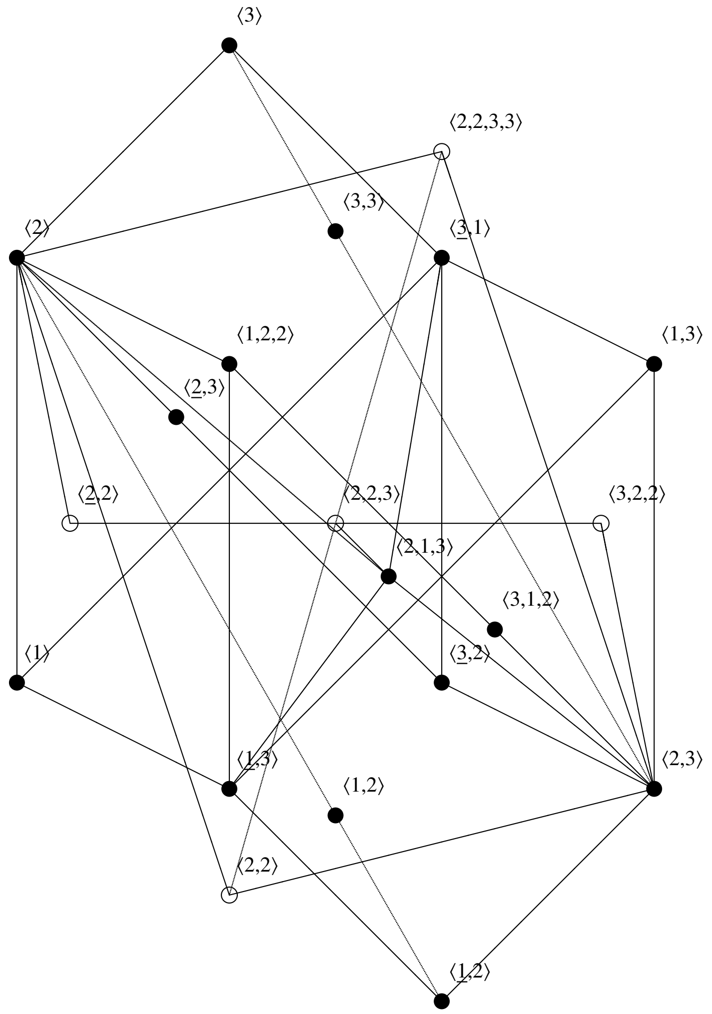



Figure 3: The Ext-quiver of $B$

# 3 The block of $k\mathfrak{S}_{13}$ with 3-core $(3, 1)$

In order to determine completely the Ext-quiver of $k\mathfrak{S}_{13}$, we must look at the block $B_1$ with 3-core $(3, 1)$. This has weight three (and hence defect four), and forms a $[3 : 2]$-pair with the principal block $\tilde{B}_1$ of $k\mathfrak{S}_{11}$. We shall use this fact to determine the extensions of all but one simple module of $B_1$; the remaining extensions we determine using a range of standard elementary techniques.

We shall use the $\langle 3, 5, 3\rangle$-notation to denote partitions of $B_1$ as follows: form the display of a partition $\lambda$ on an abacus with three beads on the first runner, five on the second and three on the third, and then denote $\lambda$ by

- $\langle i\rangle$ if the display has a bead of weight three on runner $i$;
- $\langle i, j\rangle$ if the display has a bead of weight two on runners $i$ and a bead of weight one on runner $j$;
- $\langle i, j, k\rangle$ if the display has beads of weight one on runners $i$, $j$ and $k$.

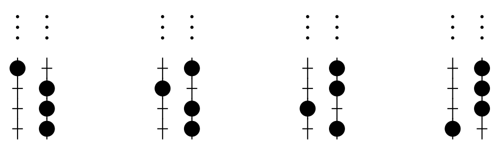
We denote the partitions of $\tilde{B}_1$ in a similar way, using $\langle 5, 3, 3\rangle$-notation.

## 3.1 $[3 : 2]$-pairs

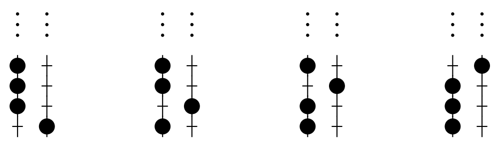
In [11], Martin and Russell discuss $[3 : 2]$-pairs. Although they consider the case of non-abelian defect, i.e. where the base field has characteristic at least five, many of their results hold with the same proofs in characteristic three. Let $C$ and $\tilde{C}$ be a pair of weight three blocks of symmetric groups over a field of characteristic $p$ such that the abacus display of the core of $C$ has two more beads on the $i$th runner than the $(i - 1)$th, and that the abacus display of the core of $\tilde{C}$ is the same as that of $C$ but with runners $i$ and $i - 1$ interchanged.

We call a partition $\lambda$ of $C$ *exceptional* if in the abacus display of $\lambda$ there are more than two beads which may be moved from runner $i$ to runner $i - 1$, and *non-exceptional* otherwise. Correspondingly, call a partition $\tilde{\lambda}$ of $\tilde{C}$ exceptional if there are three or more beads in its abacus display which may be moved from runner $i - 1$ to runner $i$, and non-exceptional otherwise.

There are four exceptional partitions of $C$, which we denote $\alpha, \beta, \gamma$ and $\delta$, and four of $\tilde{C}$, denoted $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ and $\tilde{\delta}$. The $(i - 1)$th and $i$th runners of their abacus displays are as follows.



$\alpha$ $\beta$ $\gamma$ $\delta$



$\tilde{\alpha}$ $\tilde{\beta}$ $\tilde{\gamma}$ $\tilde{\delta}$

Martin and Russell prove the following results concerning the induction and restriction of exceptional simple modules; we shall prove using the decomposition matrices that the same results hold for the blocks in which we are interested.

**Proposition 3.1.**

- *$\alpha$ and $\tilde{\alpha}$ are always p-regular.*
- *$\beta$ is p-regular if and only if $\tilde{\delta}$ is p-regular, and in this case*

$$D^{\beta}\downarrow_{\tilde{C}} \cong D^{\tilde{\delta}} \oplus D^{\tilde{\delta}}, \qquad D^{\tilde{\delta}}\uparrow^{C} \cong D^{\beta} \oplus D^{\beta}.$$

- *$\gamma$ is p-regular if and only if $\tilde{\gamma}$ is p-regular, and in this case*

$$D^{\gamma}\downarrow_{\tilde{C}} \cong D^{\tilde{\gamma}} \oplus D^{\tilde{\gamma}}, \qquad D^{\tilde{\gamma}}\uparrow^{C} \cong D^{\gamma} \oplus D^{\gamma}.$$

- *$\delta$ is p-regular if and only if $\tilde{\beta}$ is p-regular, and in this case*

$$D^{\delta}\downarrow_{\tilde{C}} \cong D^{\tilde{\beta}} \oplus D^{\tilde{\beta}}, \qquad D^{\tilde{\beta}}\uparrow^{C} \cong D^{\delta} \oplus D^{\delta}.$$

Applying the above notation to the block $B_1$, we find that $\gamma = \langle 1, 2\rangle$ and $\delta = \langle 1\rangle$ are 3-regular, while $\beta = \langle 2, 1\rangle$ is not. We have the following result concerning induction and restriction between $B_1$ and $\tilde{B}_1$.

**Proposition 3.2.**

$$\begin{aligned}
D^{\langle 2\rangle}\downarrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 1\rangle} \oplus D^{\langle 1\rangle}; & D^{\langle 1\rangle}\uparrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 2\rangle} \oplus D^{\langle 2\rangle};\\
D^{\langle 2,2\rangle}\downarrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 1,1\rangle} \oplus D^{\langle 1,1\rangle}; & D^{\langle 1,1\rangle}\uparrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 2,2\rangle} \oplus D^{\langle 2,2\rangle};\\
D^{\langle 2,3\rangle}\downarrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 1,3\rangle} \oplus D^{\langle 1,3\rangle}; & D^{\langle 1,3\rangle}\uparrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 2,3\rangle} \oplus D^{\langle 2,3\rangle};\\
D^{\langle 3\rangle}\downarrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 3\rangle} \oplus D^{\langle 3\rangle}; & D^{\langle 3\rangle}\uparrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 3\rangle} \oplus D^{\langle 3\rangle};\\
D^{\langle 3,2\rangle}\downarrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 3,1\rangle} \oplus D^{\langle 3,1\rangle}; & D^{\langle 3,1\rangle}\uparrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 3,2\rangle} \oplus D^{\langle 3,2\rangle};\\
D^{\langle 2,2,3\rangle}\downarrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 1,1,3\rangle} \oplus D^{\langle 1,1,3\rangle}; & D^{\langle 1,1,3\rangle}\uparrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 2,2,3\rangle} \oplus D^{\langle 2,2,3\rangle};\\
D^{\langle 1\rangle}\downarrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 2,1\rangle} \oplus D^{\langle 2,1\rangle}; & D^{\langle 2,1\rangle}\uparrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 1\rangle} \oplus D^{\langle 1\rangle};\\
D^{\langle 1,2\rangle}\downarrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 1,1,2\rangle} \oplus D^{\langle 1,1,2\rangle}; & D^{\langle 1,1,2\rangle}\uparrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 1,2\rangle} \oplus D^{\langle 1,2\rangle};\\
D^{\langle 3,1\rangle}\downarrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 3,2\rangle} \oplus D^{\langle 3,2\rangle}; & D^{\langle 3,2\rangle}\uparrow^{B_1}_{\tilde{B}_1} &\cong D^{\langle 3,1\rangle} \oplus D^{\langle 3,1\rangle}.
\end{aligned}$$

**Proof.** The composition factors of the restricted and induced modules follow from the decomposition matrices of the blocks and the Branching Rule. The structures of the restricted modules then follow, since we know that the simple modules of $\tilde{B}_1$ do not self-extend. To show that the induced modules are also semi-simple, we use Frobenius reciprocity: if

$$D^{\lambda}\downarrow^{B_1}_{\tilde{B}_1} \cong D^{\tilde{\lambda}} \oplus D^{\tilde{\lambda}}$$

and

$$D^{\tilde{\lambda}}\uparrow^{B_1}_{\tilde{B}_1} \sim D^{\lambda} + D^{\lambda},$$

then

$$\begin{aligned} k \oplus k &\cong \mathrm{Hom}(D^{\lambda}{\downarrow}^{B_1}_{\tilde{B}_1}, D^{\tilde{\lambda}}) \\ &\cong \mathrm{Hom}(D^{\lambda}, D^{\tilde{\lambda}}{\uparrow}^{B_1}_{\tilde{B}_1}) \end{aligned}$$

and so $D^{\tilde{\lambda}}{\uparrow}^{B_1}_{\tilde{B}_1} \cong D^{\lambda} \oplus D^{\lambda}$. □

By the Eckmann-Shapiro relations and using the general fact that

$$\mathrm{Ext}^1(M \oplus M, N) \cong \mathrm{Ext}^1(M, N) \oplus \mathrm{Ext}^1(M, N),$$

we can determine all the Ext-spaces $\mathrm{Ext}^1_{B_1}(D^{\lambda}, D^{\mu})$ for $D^{\lambda} \ncong D^{\alpha} \ncong D^{\mu}$. This leaves us with the task of determining the Ext-spaces $\mathrm{Ext}^1_{B_1}(D^{\alpha}, D^{\lambda})$; to do this, we use Lemma 2.6 extensively.

## 3.2 Restriction to $\check{B}$

Again we consider the block $\check{B}$ of $k\mathfrak{S}_{11}$ with 3-core $(3, 1^2)$; we find that some of the simple modules of $B_1$ restrict simply to $\check{B}$.

**Lemma 3.3.**

$$\begin{aligned} D^{\langle 2,3\rangle}{\downarrow}^{B_1}_{\check{B}} &\cong D^{(9,1^2)}; \\ D^{\langle 3,2\rangle}{\downarrow}^{B_1}_{\check{B}} &\cong D^{(5,4,2)}; \\ D^{\langle 2,2,2\rangle}{\downarrow}^{B_1}_{\check{B}} &\cong D^{(6,4,1)}; \\ D^{\langle 3,1\rangle}{\downarrow}^{B_1}_{\check{B}} &\cong D^{(3^2,2^2,1)}; \\ D^{\langle 2,2,3\rangle}{\downarrow}^{B_1}_{\check{B}} &\cong D^{(6,3,2)}; \\ D^{\langle 2\rangle}{\downarrow}^{B_1}_{\check{B}} &= 0; \\ D^{\langle 1\rangle}{\downarrow}^{B_1}_{\check{B}} &= 0; \\ D^{\langle 2,2\rangle}{\downarrow}^{B_1}_{\check{B}} &= 0; \\ D^{\langle 1,2\rangle}{\downarrow}^{B_1}_{\check{B}} &= 0; \end{aligned}$$

$$D^{(9,1^2)}{\uparrow}^{B_1}_{\check{B}} \sim \begin{matrix} D^{\langle 2,3\rangle} \\ D^{\langle 2\rangle} \\ D^{\langle 2,2\rangle} \\ D^{\langle 2\rangle} \\ D^{\langle 2,3\rangle} \end{matrix} ;$$

$$D^{(5,4,2)}{\uparrow}^{B_1}_{\check{B}} \sim \begin{matrix} D^{\langle 3,2\rangle} \\ D^{\langle 1\rangle} \\ D^{\langle 3,2\rangle} \end{matrix} ;$$

$$D^{(3^2,2^2,1)}\uparrow_{\check{B}}^{B_1} \sim \begin{array}{c} D^{\langle 3,1\rangle} \\ D^{\langle 2\rangle} D^{\langle 1\rangle} \\ D^{\langle 3,1\rangle} \end{array};$$

$$D^{(6,3,2)}\uparrow_{\check{B}}^{B_1} \sim \begin{array}{c} D^{\langle 2,2,3\rangle} \\ D^{\langle 1,2\rangle} \\ D^{\langle 2,2,3\rangle} \end{array}.$$

**Proof.** The composition factors of restricted and induced modules are obtained using the Branching Rule and the decomposition matrices of $B_1$ and $\check{B}$. The socles of the induced modules are found by Frobenius reciprocity; their structures then follow from the fact that they are self-dual and that $D^{\langle 2,3\rangle}$ does not extend $D^{\langle 2,2\rangle}$. □

**Remark.** Note that we do not attempt to find the structure of $D^{(6,4,1)}\uparrow_{\check{B}_1}^{B_1}$; we do not need it, and there are very many composition factors.

This immediately gives us some of the Ext-spaces.

**Corollary 3.4.**

$$\begin{aligned} \mathrm{Ext}^1_{B_1}(D^{\langle 2,2,2\rangle}, D^{\langle 2,3\rangle}) &= 0; \\ \mathrm{Ext}^1_{B_1}(D^{\langle 2,2,2\rangle}, D^{\langle 3,2\rangle}) &= 0; \\ \mathrm{Ext}^1_{B_1}(D^{\langle 2,2,2\rangle}, D^{\langle 3,1\rangle}) &= 0. \end{aligned}$$

**Proof.** We would like to use Lemma 2.6, but here we are restricting between $\mathfrak{S}_n$ and $\mathfrak{S}_{n-2}$. But we may use the following modification: if $D^\lambda$ and $D^\mu$ are simple modules of $B_1$ with $\mathrm{Ext}^1_{B_1}(D^\lambda, D^\mu) \neq 0$, and if $\mathrm{soc}(D^\lambda\downarrow_{\check{B}})$ is a simple module $D^{\check{\lambda}}$ with $\mathrm{soc}(D^{\check{\lambda}}\uparrow^{B_1}) \cong D^\lambda$, then either

1. $\mathrm{Ext}^1_{B_1}(D^{\check{\lambda}}\uparrow^{B_1}, D^\mu) = 0$, in which case $\dim\mathrm{Ext}^1_{B_1}(D^\lambda, D^\mu)$ is the composition multiplicity of $D^\mu$ in the second Loewy layer of $D^{\check{\lambda}}\uparrow^{B_1}$, or
2. $\mathrm{Ext}^1_{\check{B}}(D^{\check{\lambda}}, D^\mu\downarrow_{\check{B}}) \neq 0$.

The proof of this is also obvious. Using this together with Lemma 3.3 and the Ext-quiver of $\check{B}$, the corollary follows. □

## 3.3 Induction to the block of $k\mathfrak{S}_{14}$ with 3-core $(3, 1^2)$

We can find more of the unknown Ext-spaces by inducing modules from $B_1$ to the block $\hat{B}_1$ of $k\mathfrak{S}_{14}$. Although we do not know the Ext-quiver of $\hat{B}_1$, it has the advantage of being self-conjugate; thus in order to find Ext-spaces in $B_1$, we may use the following general method:

- induce modules from $B_1$ to $\hat{B}_1$;
- tensor with the alternating representation by Mullineux's algorithm;

- restrict back to $B_1$ and use the known Ext-spaces.

We deal with $\hat{B}_1$ in full detail, since its Ext-quiver will then follow very easily from that of $B_1$. We denote the $p$-regular partitions of $\hat{B}_1$ using $\langle 4,5,2\rangle$-notation as defined at the beginning of this section; this is consistent with using $\langle 3,5,3\rangle$-notation for $B_1$, and it does not matter for our purposes that not all $p$-singular partitions of $\hat{B}_1$ can be displayed with this notation.

First, we use Mullineux's algorithm on the simple modules of $\hat{B}_1$; this gives the following.

**Lemma 3.5.**

$$\begin{aligned} D^{\langle 2\rangle} \otimes \mathrm{sgn} &\cong D^{\langle 1,2\rangle};\\ D^{\langle 2,2\rangle} \otimes \mathrm{sgn} &\cong D^{\langle 1,1\rangle};\\ D^{\langle 2,1\rangle} \otimes \mathrm{sgn} &\cong D^{\langle 1,2,2\rangle};\\ D^{\langle 1\rangle} \otimes \mathrm{sgn} &\cong D^{\langle 3\rangle};\\ D^{\langle 2,2,2\rangle} \otimes \mathrm{sgn} &\cong D^{\langle 1,1,2\rangle}. \end{aligned}$$

Next we find that we can determine explicitly the induction and restriction of all simples between $B_1$ and $\hat{B}_1$.

**Proposition 3.6.**

$$\begin{aligned} D^{\langle 2\rangle}\uparrow_{B_1}^{\hat{B}_1} &\cong D^{\langle 2\rangle}; & D^{\langle 2\rangle}\downarrow_{B_1}^{\hat{B}_1} &\cong D^{\langle 2\rangle};\\ D^{\langle 3,2\rangle}\uparrow_{B_1}^{\hat{B}_1} &\cong D^{\langle 1,2\rangle}; & D^{\langle 1,2\rangle}\downarrow_{B_1}^{\hat{B}_1} &\cong D^{\langle 3,2\rangle};\\ D^{\langle 2,2\rangle}\uparrow_{B_1}^{\hat{B}_1} &\cong D^{\langle 2,2\rangle}; & D^{\langle 2,2\rangle}\downarrow_{B_1}^{\hat{B}_1} &\cong D^{\langle 2,2\rangle};\\ D^{\langle 2,3\rangle}\uparrow_{B_1}^{\hat{B}_1} &\cong D^{\langle 2,1\rangle}; & D^{\langle 2,1\rangle}\downarrow_{B_1}^{\hat{B}_1} &\cong D^{\langle 2,3\rangle};\\ D^{\langle 2,2,3\rangle}\uparrow_{B_1}^{\hat{B}_1} &\cong D^{\langle 1,2,2\rangle}; & D^{\langle 1,2,2\rangle}\downarrow_{B_1}^{\hat{B}_1} &\cong D^{\langle 2,2,3\rangle};\\ D^{\langle 3\rangle}\uparrow_{B_1}^{\hat{B}_1} &\cong D^{\langle 1\rangle}; & D^{\langle 1\rangle}\downarrow_{B_1}^{\hat{B}_1} &\cong D^{\langle 3\rangle};\\ D^{\langle 3,1\rangle}\uparrow_{B_1}^{\hat{B}_1} &\cong D^{\langle 3\rangle}; & D^{\langle 3\rangle}\downarrow_{B_1}^{\hat{B}_1} &\cong D^{\langle 3,1\rangle};\\ D^{\langle 2,2,2\rangle}\uparrow_{B_1}^{\hat{B}_1} &\cong D^{\langle 2,2,2\rangle}; & D^{\langle 2,2,2\rangle}\downarrow_{B_1}^{\hat{B}_1} &\cong D^{\langle 2,2,2\rangle}; \end{aligned}$$

$$D^{\langle 1,2\rangle}\uparrow_{B_1}^{\hat{B}_1} \cong \begin{array}{c} D^{\langle 1,1,2\rangle} \\ D^{\langle 2,1\rangle} D^{\langle 2,2,2\rangle} \\ D^{\langle 1,1,2\rangle} \end{array}; \qquad D^{\langle 1,1,2\rangle}\downarrow_{B_1}^{\hat{B}_1} \cong \begin{array}{c} D^{\langle 1,2\rangle} \\ D^{\langle 2\rangle} \\ D^{\langle 2,2\rangle} \\ D^{\langle 2\rangle} \\ D^{\langle 1,2\rangle} \end{array} \oplus D^{\langle 2,2,3\rangle};$$

$$D^{\langle 1\rangle}\uparrow_{B_1}^{\hat{B}_1} \cong \begin{array}{c} D^{\langle 1,1\rangle} \\ D^{\langle 1,2\rangle} \\ D^{\langle 1,2,2\rangle} \\ D^{\langle 1,2\rangle} \\ D^{\langle 1,1\rangle} \end{array}; \qquad D^{\langle 1,1\rangle}\downarrow_{B_1}^{\hat{B}_1} \cong \begin{array}{c} D^{\langle 1\rangle} \\ D^{\langle 3,1\rangle} \\ D^{\langle 1\rangle} \end{array}.$$

**Proof.** The composition factors of the induced and restricted modules are determined using the Branching Rule together with the decomposition matrices of $B_1$ and $\hat{B}_1$; Theorem 1.1 can be used to speed up the process. In order to determine the structures of the non-simple induced and restricted modules, we note that their cosocles and socles are given by Frobenius reciprocity, and that they must be self-dual. We already know most of the Ext-quiver of $B_1$, in particular that

$$\mathrm{Ext}^1_{B_1}(D^{\langle 1,2\rangle}, D^{\langle 2,2\rangle}) = \mathrm{Ext}^1_{B_1}(D^{\langle 2\rangle}, D^{\langle 2,2,3\rangle}) = 0,$$

which gives the structure of $D^{\langle 1,1,2\rangle}\downarrow^{\hat{B}_1}_{B_1}$. For $D^{\langle 1\rangle}\uparrow^{\hat{B}_1}_{B_1}$, we need only show that $D^{\langle 1,1\rangle}$ does not extend $D^{\langle 1,2,2\rangle}$. But

$$\mathrm{Ext}^1_{\hat{B}_1}(D^{\langle 1,1\rangle}, D^{\langle 1,2,2\rangle}) \cong \mathrm{Ext}^1_{\hat{B}_1}(D^{\langle 2,2\rangle}, D^{\langle 2,1\rangle})$$

by Lemma 3.5

$$\cong \mathrm{Ext}^1_{B_1}(D^{\langle 2,2\rangle}, D^{\langle 2,3\rangle})$$

by Eckmann-Shapiro

$$= 0$$

from the part of the Ext-quiver of $B_1$ we already have. □

This gives us three more Ext-spaces, as follows.

**Proposition 3.7.**

$$\mathrm{Ext}^1_{B_1}(D^{\langle 2,2,2\rangle}, D^{\langle 2\rangle}) = 0;$$
$$\mathrm{Ext}^1_{B_1}(D^{\langle 2,2,2\rangle}, D^{\langle 3\rangle}) = 0;$$
$$\mathrm{Ext}^1_{B_1}(D^{\langle 2,2,2\rangle}, D^{\langle 2,2,3\rangle}) \cong k.$$

**Proof.** By Eckmann-Shapiro,

$$\begin{aligned}\mathrm{Ext}^1_{B_1}(D^{\langle 2,2,2\rangle}, D^{\langle 2\rangle}) &\cong \mathrm{Ext}^1_{\hat{B}_1}(D^{\langle 2,2,2\rangle}, D^{\langle 2\rangle})\\ &\cong \mathrm{Ext}^1_{\hat{B}_1}(D^{\langle 1,1,2\rangle}, D^{\langle 1,2\rangle})\\ &= 0\end{aligned}$$

by Lemma 3.5 and Lemma 2.6 and the known part of the Ext-quiver of $B_1$. The other assertions follow similarly. □

### 3.4 $D^{\langle 2,2,2\rangle}$ does not self-extend

In order to check that $D^{\langle 2,2,2\rangle}$ does not self-extend, we consider the defect one block $D$ of $k\mathfrak{S}_{12}$ with 3-core $(3, 2^2, 1^2)$. Defect one blocks of symmetric group algebras are well understood; in particular we know that

$$P(D^{(6,2^2,1^2)}) \cong \begin{matrix} D^{(6,2^2,1^2)} \\ D^{(4^2,2,1^2)} \\ D^{(6,2^2,1^2)} \end{matrix}.$$

Theorem 1.1 gives the following.

**Lemma 3.8.**

$$D^{\langle 1,2\rangle}{\downarrow}_D^{B_1} \cong D^{(6,2^2,1^2)}, \qquad D^{\langle 1\rangle}{\downarrow}_D^{B_1} \cong D^{(4^2,2,1^2)},$$

*$D^\lambda{\downarrow}_D = 0$ for any other simple module $D^\lambda$ in $B_1$.*

**Lemma 3.9.**

$$\operatorname{cosoc}(\Omega(D^{(6,2^2,1^2)}{\uparrow}^{B_1}) \cong D^{\langle 1\rangle}.$$

**Proof.** By Frobenius reciprocity and Lemma 3.8 we know that $P(D^{(6,2^2,1^2)}){\uparrow}^{B_1} \cong P(D^{\langle 1,2\rangle})$, so by inducing the short exact sequence

$$0 \longrightarrow \Omega(D^{(6,2^2,1^2)}) \longrightarrow P(D^{(6,2^2,1^2)}) \longrightarrow D^{(6,2^2,1^2)} \longrightarrow 0$$

to $B_1$, we obtain

$$\Omega(D^{(6,2^2,1^2)}){\uparrow}^{B_1} \cong \Omega(D^{(6,2^2,1^2)}{\uparrow}^{B_1}).$$

The lemma follows by Frobenius reciprocity and from the structure of $P(D^{(6,2^2,1^2)})$ given above. □

**Proposition 3.10.** *If $M$ is a $B_1$-module with simple cosocle isomorphic to $D^{\langle 1,2\rangle}$, and if $D^{\langle 1\rangle}$ does not appear as a composition factor of $M$, then $M$ is a quotient of $D^{(6,2^2,1^2)}{\uparrow}^{B_1}$.*

*In particular, $D^{\langle 1,1,2\rangle}{\downarrow}_{B_1}^{\hat{B}_1}$ is a quotient of $D^{(6,2^2,1^2)}{\uparrow}^{B_1}$.*

**Proof.** The first statement is proved exactly as in Proposition 2.2; the second statement follows from Proposition 3.6. □

**Lemma 3.11.** $\operatorname{cosoc}(\Omega(D^{\langle 1,1,2\rangle}{\downarrow}_{B_1}^{\hat{B}_1}))$ *does not contain a copy of $D^{\langle 1,2\rangle}$, i.e.*

$$\operatorname{Ext}^1_{B_1}(D^{\langle 1,1,2\rangle}{\downarrow}_{B_1}^{\hat{B}_1}, D^{\langle 1,2\rangle}) = 0.$$

**Proof.** Let $N$ be a submodule of $D^{(6,2^2,1^2)}{\uparrow}^{B_1}$ such that $D^{(6,2^2,1^2)}{\uparrow}^{B_1}\ /N \cong D^{\langle 1,1,2\rangle}{\downarrow}_{B_1}^{\hat{B}_1}$. By Proposition 3.9 we know that any factor of $\operatorname{cosoc}(\Omega(D^{\langle 1,1,2\rangle}{\downarrow}_{B_1}^{\hat{B}_1})$ isomorphic to $D^{\langle 1,2\rangle}$ must lie in the cosocle of $N$. By using the decomposition matrices and the Branching Rule, we find that $N$ contains just one copy of $D^{\langle 1,2\rangle}$. This must constitute the socle of $N$, since $D^{(6,2^2,1^2)}{\uparrow}^{B_1}$ is self-dual; since $N$ is not simple, it does not contain a copy of $D^{\langle 1,2\rangle}$ in its cosocle. The result follows. □

**Proposition 3.12.**

$$\mathrm{Ext}^1_{B_1}(D^{\langle 2,2,2\rangle}, D^{\langle 2,2,2\rangle}) = 0.$$

**Proof.** By Eckmann-Shapiro and Lemma 3.5 we have

$$\begin{aligned}\mathrm{Ext}^1_{B_1}(D^{\langle 2,2,2\rangle}, D^{\langle 2,2,2\rangle}) &\cong \mathrm{Ext}^1_{\hat{B}_1}(D^{\langle 2,2,2\rangle}, D^{\langle 2,2,2\rangle})\\ &\cong \mathrm{Ext}^1_{\hat{B}_1}(D^{\langle 1,1,2\rangle}, D^{\langle 1,1,2\rangle});\end{aligned}$$

the latter space is zero, by Lemma 2.6, Proposition 3.6 and Lemma 3.11. □

This leaves us to find $\mathrm{Ext}^1_{B_1}(D^{\langle 2,2,2\rangle}, D^\lambda)$ for $\lambda = \langle 2,2\rangle, \langle 1\rangle, \langle 1,2\rangle$. To do this we consider the projective cover of $D^{\langle 2,2,2\rangle}$; the second Loewy layer of $P(D^{\langle 2,2,2\rangle})$ gives all the Ext-spaces $\mathrm{Ext}^1_{B_1}(D^{\langle 2,2,2\rangle}, D^\lambda)$; in addition, we can easily find a filtration of this projective module by Specht modules.

## 3.5 The projective Specht module $S^{(6,4,2,1^2)}$

Consider the Specht module $S^{(6,4,2,1^2)}$ of $k\mathfrak{S}_{14}$. $(6,4,2,1^2)$ is a 3-core, so $D^{(6,4,2,1^2)}$ is the unique simple module in its block; hence the block is simple, and $D^{(6,4,2,1^2)} = S^{(6,4,2,1^2)}$ is projective. Furthermore, Theorem 1.1 implies that

$$\mathrm{soc}(S^{(6,4,2,1^2)}\downarrow_{B_1}) \cong D^{\langle 2,2,2\rangle},$$

so that

$$P(D^{\langle 2,2,2\rangle}) \cong S^{(6,4,2,1^2)}\downarrow_{B_1} \sim \begin{matrix} S^{\langle 2,2,2\rangle} \\ S^{\langle 1,2,2\rangle} \\ S^{\langle 1,2\rangle} \\ S^{\langle 1\rangle} \end{matrix}.$$

We examine the Specht module at the top of this filtration.

**Lemma 3.13.**

$$\mathrm{soc}(S^{\langle 2,2,2\rangle}) \cong D^{\langle 3\rangle}.$$

**Proof.** By [6, Theorem 8.15],

$$S^{\langle 2,2,2\rangle} \otimes \mathrm{sgn} \cong (S^{(4,3,2^2,1^2)})^*,$$

so

$$\begin{aligned}\mathrm{soc}(S^{\langle 2,2,2\rangle}) &\cong D^{(4,3,2^2,1^2)} \otimes \mathrm{sgn}\\ &\cong D^{\langle 3\rangle}\end{aligned}$$

by Mullineux. □

Now we can find the remaining unknown Ext-spaces.

**Corollary 3.14.**

$$\mathrm{Ext}^1_{B_1}(D^{\langle 2,2,2\rangle}, D^{\langle 2,2\rangle}) \cong k.$$

**Proof.** We know that

$$S^{\langle 2,2,2\rangle} \sim D^{\langle 2,2,2\rangle} + D^{\langle 3\rangle} + D^{\langle 2\rangle} + D^{\langle 2,2\rangle} + D^{\langle 2,3\rangle} + D^{\langle 3,2\rangle} + D^{\langle 2,2,3\rangle};$$

of these factors, $D^{\langle 2,2\rangle}$ only extends $D^{\langle 2\rangle}$ and possibly $D^{\langle 2,2,2\rangle}$. $D^{\langle 2,2\rangle}$ does not lie in the cosocle or the socle of $S^{\langle 2,2,2\rangle}$, so must extend (or be extended by) at least two other factors. Hence $\mathrm{Ext}^1_{B_1}(D^{\langle 2,2,2\rangle}, D^{\langle 2,2\rangle}) \neq 0$.

Given the above Specht filtration of $P(D^{\langle 2,2,2\rangle})$, we see that the only copies of $D^{\langle 2,2\rangle}$ which can lie in the second Loewy layer must lie in $S^{\langle 2,2,2\rangle}$; we have just seen that there is only one such. □

**Proposition 3.15.**

$$\mathrm{Ext}^1_{B_1}(D^{\langle 2,2,2\rangle}, D^{\langle 1\rangle}) = 0;$$
$$\mathrm{Ext}^1_{B_1}(D^{\langle 2,2,2\rangle}, D^{\langle 1,2\rangle}) = 0.$$

**Proof.** Using the above Specht filtration and the decomposition matrix of $B_1$, we find that there is exactly one copy of $D^{\langle 1\rangle}$ lying in $P(D^{\langle 2,2,2\rangle})$, namely the cosocle of $S^{\langle 1\rangle}$. But $S^{\langle 1\rangle}$ has factors other than $D^{\langle 1\rangle}$ and $D^{\langle 2,2,2\rangle}(= \mathrm{soc}(S^{\langle 1\rangle}))$. So $D^{\langle 1\rangle}$ does not lie in the second socle layer of $P(D^{\langle 2,2,2\rangle})$, and the first result follows.

There are two copies of $D^{\langle 1,2\rangle}$ in $P(D^{\langle 2,2,2\rangle})$; one of these lies in $S^{\langle 1\rangle}$, and so lies below the only copy of $D^{\langle 1\rangle}$; hence the other copy of $D^{\langle 1,2\rangle}$ lies above $D^{\langle 1\rangle}$. We claim that the copy of $D^{\langle 1,2\rangle}$ lying in $S^{\langle 1\rangle}$ does not lie in its second socle layer, and thus does not lie in the second socle layer of $P(D^{\langle 2,2,2\rangle})$. $S^{\langle 1\rangle}$ has cosocle $D^{\langle 1\rangle}$, socle $D^{\langle 2,2,2\rangle}$ and heart with factors $D^{\langle 2\rangle}$, $D^{\langle 2,2\rangle}$, $D^{\langle 3,2\rangle}$, $D^{\langle 2,2,3\rangle}$ and $D^{\langle 1,2\rangle}$. From the known part of the Ext-quiver of $B_1$ we see that $D^{\langle 2,2\rangle}$ must lie above $D^{\langle 2,2,2\rangle}$ and below $D^{\langle 2\rangle}$, and that $D^{\langle 2\rangle}$ must lie above $D^{\langle 2,2\rangle}$ and below $D^{\langle 1,2\rangle}$; hence the latter factor lies in at least the fourth socle layer. □

## 3.6 The Ext-quiver of $B_1$

We have now determined all the required Ext-spaces, and so we have the Ext-quiver of $B_1$; for comparison, we give the Ext-quiver of $\tilde{B}_1$ as well. Again our labelling of the vertices differs from that in [17], since Tan uses $\langle 3,3,4\rangle$-notation for $\tilde{B}_1$.

Note that the Ext-quiver of $B_1$ is *not* bipartite. In [15], Tan attempts to show that every weight three block of a symmetric group with abelian defect has a bipartite Ext-quiver, provided a certain conjecture holds, namely that

$$[P(D^{\alpha}) : D^{\lambda}] \leqslant 3$$

whenever we have a $[3:2]$-pair $(C, \tilde{C})$ as above, with $D^{\alpha}$ as above and $D^{\lambda}$ a non-exceptional simple module of $C$.

In characteristic three, this conjecture fails to hold; for

$$[P(D^{\langle 2,2,2\rangle}) : D^{\langle 2\rangle}] = 4,$$

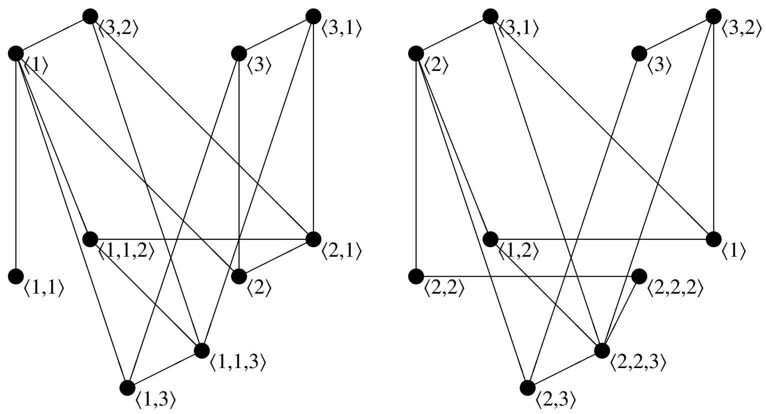



Figure 4: The Ext-quivers of $\tilde{B}_1$ and $B_1$

and this is where things begin to break down. However, the Ext-quiver is a subdivision of a bipartite graph, as defined in the previous section for the Ext-quiver of $B$: the sets

$$\{\langle 2\rangle, \langle 3\rangle, \langle 1\rangle, \langle 2,2,3\rangle\}, \{\langle 3,1\rangle, \langle 3,2\rangle, \langle 1,2\rangle, \langle 2,2,2\rangle, \langle 2,2,3\rangle\}$$

illustrate the bipartition, with the vertex labelled $\langle 2,2\rangle$ being added in the subdivision.

### 3.7 The Ext-quiver of $\hat{B}_1$

Given our work on the block $\hat{B}_1$, we are able to find its Ext-quiver very easily, and so we do this. By Eckmann-Shapiro (using Proposition 3.6) and by tensoring with the alternating representation (using Lemma 3.5), we get all the Ext-spaces between simple modules of $\hat{B}_1$ except for

$$\mathrm{Ext}^1_{\hat{B}_1}(D^{\langle 2,2\rangle}, D^{\langle 1,1\rangle}),$$
$$\mathrm{Ext}^1_{\hat{B}_1}(D^{\langle 2,2\rangle}, D^{\langle 1,1,2\rangle}),$$
$$\mathrm{Ext}^1_{\hat{B}_1}(D^{\langle 2,2,2\rangle}, D^{\langle 1,1\rangle}),$$
$$\mathrm{Ext}^1_{\hat{B}_1}(D^{\langle 2,2,2\rangle}, D^{\langle 1,1,2\rangle}).$$

But these follow immediately from Lemma 2.6 and Proposition 3.6. Hence $\hat{B}_1$ has the Ext-quiver shown. Since $\hat{B}_1$ is self-conjugate, the quiver has rotational symmetry about the central vertical axis indicated corresponding to the Mullineux involution.

Note that this Ext-quiver is also not bipartite, but is a subdivision of a bipartite graph: the sets

$$\{\langle 2\rangle, \langle 1\rangle, \langle 1,1,2\rangle, \langle 1,2,2\rangle\}, \{\langle 3\rangle, \langle 1,2\rangle, \langle 2,2,2\rangle, \langle 2,1\rangle\}$$

illustrate the bipartition, with the vertices labelled $\langle 2,2\rangle, \langle 1,1\rangle$ being added in the subdivision.

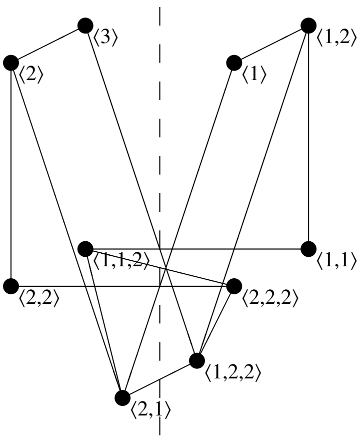



Figure 5: The Ext-quiver of $\hat{B}_1$

# A Decomposition matrices

We reproduce all the decomposition matrices used in this paper; they are taken from [7], with the exception of the matrix for the block $\hat{B}_1$ of $k\mathfrak{S}_{14}$, which was found by ad hoc means. In each matrix, the $(\lambda, \mu)$th entry is the composition multiplicity $[S^\lambda : D^\mu]$.

## A.1 The principal block of $k\mathfrak{S}_{11}$ ($\langle 5,3,3\rangle$-notation)

| | $(11)$ | $(8,3)$ | $(8,2,1)$ | $(7,3,1)$ | $(6,3,1^2)$ | $(5^2,1)$ | $(5,4,1^2)$ | $(5,3^2)$ | $(5,3,2,1)$ | $(4,3,2^2)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $(11) = \langle 1\rangle$ | 1 | | | | | | | | | |
| $(8,3) = \langle 1,1\rangle$ | 1 | 1 | | | | | | | | |
| $(8,2,1) = \langle 1,3\rangle$ | 2 | 1 | 1 | | | | | | | |
| $(7,3,1) = \langle 3\rangle$ | 1 | 1 | 1 | 1 | | | | | | |
| $(6,3,1^2) = \langle 2\rangle$ | 1 | | 1 | 1 | 1 | | | | | |
| $(5^2,1) = \langle 3,1\rangle$ | | | | 1 | | 1 | | | | |
| $(5,4,1^2) = \langle 2,1\rangle$ | | | | 1 | 1 | 1 | 1 | | | |
| $(5,3^2) = \langle 1,1,3\rangle$ | | | 1 | 1 | | 1 | | 1 | | |
| $(5,3,2,1) = \langle 1,1,2\rangle$ | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | |
| $(4,3,2^2) = \langle 3,2\rangle$ | 2 | 1 | 1 | | | | 1 | 1 | 1 | 1 |
| $(8,1^3) = \langle 1,2\rangle$ | | | 1 | | | | | | | |
| $(5,3,1^3) = \langle 1,1,1\rangle$ | 1 | | | | 1 | | 1 | | 1 | |
| $(5,2^3) = \langle 1,2,3\rangle$ | 2 | 1 | | | | | | | 1 | |
| $(5,2,1^4) = \langle 1,3,3\rangle$ | | | | | | | | 1 | 1 | |
| $(5,1^6) = \langle 1,2,2\rangle$ | | | | | | | | 1 | | |
| $(4,3,1^4) = \langle 3,3\rangle$ | 1 | | | | | 1 | 1 | 1 | 1 | 1 |
| $(3^3,2) = \langle 2,3\rangle$ | 1 | | 1 | | | | | 1 | | 1 |
| $(3^2,1^5) = \langle 2,2\rangle$ | | | | | | 1 | 1 | 1 | | 1 |
| $(2^5,1) = \langle 2,3,3\rangle$ | 1 | | | | | | | | | 1 |
| $(2^4,1^3) = \langle 2,2,3\rangle$ | | | | | | | 1 | | | 1 |
| $(2^2,1^7) = \langle 3,3,3\rangle$ | | | | | | 1 | 1 | | | |
| $(2,1^9) = \langle 2,2,2\rangle$ | | | | | | 1 | | | | |

## A.2 The block of $k\mathfrak{S}_{11}$ with 3-core $(3,1^2)$

| | $(9,1^2)$ | $(6,4,1)$ | $(6,3,2)$ | $(5,4,2)$ | $(3^2,2^2,1)$ |
|---|---|---|---|---|---|
| $(9,1^2)$ | 1 | | | | |
| $(6,4,1)$ | | 1 | | | |
| $(6,3,2)$ | 1 | 1 | 1 | | |
| $(5,4,2)$ | | 1 | 1 | 1 | |
| $(3^2,2^2,1)$ | 1 | | 1 | | 1 |
| $(6,1^5)$ | | | 1 | | |
| $(3^2,2,1^3)$ | | | 1 | 1 | 1 |
| $(3,2^3,1^2)$ | | | | | 1 |
| $(3,1^8)$ | | | | 1 | |

## A.3 The block of $k\mathfrak{S}_{11}$ with 3-core $(1^2)$

| | $(10,1)$ | $(9,2)$ | $(7,4)$ | $(7,2^2)$ | $(6,5)$ | $(6,2^2,1)$ | $(5,2^2,1^2)$ | $(4^2,3)$ | $(4^2,2,1)$ | $(4,3,2,1^2)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $(10,1)$ | 1 | | | | | | | | | |
| $(9,2)$ | 1 | 1 | | | | | | | | |
| $(7,4)$ | | 1 | 1 | | | | | | | |
| $(7,2^2)$ | 1 | 1 | 1 | 1 | | | | | | |
| $(6,5)$ | | | 1 | | 1 | | | | | |
| $(6,2^2,1)$ | 1 | 1 | 1 | 1 | 1 | 1 | | | | |
| $(5,2^2,1^2)$ | | 1 | | | 1 | 1 | 1 | | | |
| $(4^2,3)$ | | | 1 | 1 | 1 | | | 1 | | |
| $(4^2,2,1)$ | | 1 | 1 | 1 | 2 | 1 | | 1 | 1 | |
| $(4,3,2,1^2)$ | 1 | 1 | | 1 | 2 | 1 | 1 | 1 | 1 | 1 |
| $(7,1^4)$ | | | | 1 | | | | | | |
| $(6,2,1^3)$ | | | | 1 | | 1 | | | | |
| $(4^2,1^3)$ | | | | | 2 | 1 | | | 1 | |
| $(4,2^3,1)$ | 1 | 1 | | | | | 1 | | | 1 |
| $(4,2^2,1^3)$ | | | | | 1 | | 1 | 1 | | 1 |
| $(4,1^7)$ | | | | | | | | 1 | | |
| $(3^3,1^2)$ | 1 | | | 1 | | | | 1 | | 1 |
| $(3,2^4)$ | 1 | | | | | | | | | 1 |
| $(3,2^2,1^4)$ | | | | | 1 | | | 1 | 1 | 1 |
| $(3,2,1^6)$ | | | | | 2 | | | 1 | 1 | |
| $(2^3,1^5)$ | | | | | 1 | | | | 1 | |
| $(1^{11})$ | | | | | 1 | | | | | |

## A.4 The block of $k\mathfrak{S}_{12}$ with 3-core $(3,2^2,1^2)$

| | $(6,2^2,1^2)$ | $(4^2,2,1^2)$ |
|---|---|---|
| $(6,2^2,1^2)$ | 1 | |
| $(4^2,2,1^2)$ | 1 | 1 |
| $(3,2^2,1^5)$ | | 1 |

## A.5 The principal block of $k\mathfrak{S}_{12}$ ($\langle 5,4,4\rangle$-notation)

| | $(12)$ | $(11,1)$ | $(10,1^2)$ | $(9,3)$ | $(9,2,1)$ | $(8,4)$ | $(8,2^2)$ | $(7,4,1)$ | $(7,3,2)$ | $(6^2)$ | $(6,5,1)$ | $(6,4,1^2)$ | $(6,3^2)$ | $(6,3,2,1)$ | $(5^2,2)$ | $(5,4,3)$ | $(5,4,2,1)$ | $(5,3,2,1^2)$ | $(4^2,2^2)$ | $(4,3,2^2,1)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(12)=\langle 1\rangle$ | 1 | | | | | | | | | | | | | | | | | | | |
| $(11,1)=\langle 3\rangle$ | 1 | 1 | | | | | | | | | | | | | | | | | | |
| $(10,1^2)=\langle 2\rangle$ | | 1 | 1 | | | | | | | | | | | | | | | | | |
| $(9,3)=\langle \underline{1},3\rangle$ | 1 | 1 | | 1 | | | | | | | | | | | | | | | | |
| $(9,2,1)=\langle \underline{1},2\rangle$ | 2 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | |
| $(8,4)=\langle \underline{3},1\rangle$ | 1 | | | 1 | | 1 | | | | | | | | | | | | | | |
| $(8,2^2)=\langle \underline{3},2\rangle$ | 2 | 1 | | 1 | 1 | 1 | 1 | | | | | | | | | | | | | |
| $(7,4,1)=\langle \underline{2},1\rangle$ | 1 | | | 1 | 1 | 1 | | 1 | | | | | | | | | | | | |
| $(7,3,2)=\langle \underline{2},3\rangle$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | |
| $(6^2)=\langle 1,3\rangle$ | | | | | | 1 | | | | 1 | | | | | | | | | | |
| $(6,5,1)=\langle 1,2\rangle$ | | | | | | 1 | | 1 | | 1 | 1 | | | | | | | | | |
| $(6,4,1^2)=\langle 1,1\rangle$ | 1 | | | | 1 | | | 1 | | | 1 | 1 | | | | | | | | |
| $(6,3^2)=\langle 1,2,3\rangle$ | | | 1 | | 1 | | | 1 | 1 | | 1 | | 1 | | | | | | | |
| $(6,3,2,1)=\langle 1,1,3\rangle$ | 3 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | |
| $(5^2,2)=\langle 2,3\rangle$ | | | | | | | | 1 | 1 | | 1 | | | | 1 | | | | | |
| $(5,4,3)=\langle 3,1,2\rangle$ | | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 | | 1 | 1 | | | | |
| $(5,4,2,1)=\langle 3,1,1\rangle$ | 2 | | | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | | | |
| $(5,3,2,1^2)=\langle 3,3\rangle$ | 3 | 1 | | 1 | 1 | | 1 | | | 3 | 1 | 1 | 1 | 2 | | 1 | 1 | 1 | | |
| $(4^2,2^2)=\langle 2,1,1\rangle$ | 2 | | | 1 | 1 | 1 | 1 | | | 2 | | | 1 | 1 | | 1 | 1 | | 1 | |
| $(4,3,2^2,1)=\langle 2,3,3\rangle$ | 2 | 2 | 1 | 1 | 1 | | 1 | | 1 | 2 | | | 1 | 1 | | 1 | 1 | 1 | 1 | 1 |
| $(9,1^3)=\langle \underline{1},1\rangle$ | | | 1 | | 1 | | | | | | | | | | | | | | | |
| $(8,1^4)=\langle \underline{3},3\rangle$ | | | | | 1 | | 1 | | | | | | | | | | | | | |
| $(7,1^5)=\langle \underline{2},2\rangle$ | | | | | | | 1 | | 1 | | | | | | | | | | | |
| $(6,3,1^3)=\langle 1,3,3\rangle$ | 2 | | | | 1 | | 1 | | | | | 1 | | 1 | | | | | | |
| $(6,2^3)=\langle 1,1,2\rangle$ | 2 | 1 | | 1 | | 1 | 1 | | | 1 | | | | 1 | | | | | | |
| $(6,2,1^4)=\langle 1,2,2\rangle$ | | | | | | | 1 | | 1 | | | | 1 | 1 | | | | | | |
| $(6,1^6)=\langle 1,1,1\rangle$ | | | | | | | | | 1 | | | | 1 | | | | | | | |
| $(5,4,1^3)=\langle 3,1,3\rangle$ | 1 | | | | | | | | | 2 | 1 | 1 | | 1 | | | 1 | | | |
| $(5,2^3,1)=\langle 3,2,3\rangle$ | 2 | 1 | | 1 | | | | | | 1 | | | | 1 | | | | 1 | | |
| $(5,2^2,1^3)=\langle 3,2,2\rangle$ | | | | | | | | | | 2 | | | 1 | 1 | | 1 | | 1 | | |
| $(5,1^7)=\langle 3,3,3\rangle$ | | | | | | | | | | | | | 1 | | | 1 | | | | |
| $(4^3)=\langle 2,1,3\rangle$ | | | | | | 1 | 1 | | | 1 | | | | | | 1 | | | | |
| $(4^2,1^4)=\langle 2,1,2\rangle$ | 1 | | | | | | | | | 2 | 1 | | 1 | 1 | | | 1 | | 1 | |
| $(4,3,2,1^3)=\langle 2,2,3\rangle$ | 1 | 1 | | | | | 1 | | 1 | 3 | 1 | | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 1 |
| $(4,2^3,1^2)=\langle 2,2\rangle$ | | 1 | | | | | | | | 1 | | | | | | 1 | | 1 | | 1 |
| $(4,1^8)=\langle 2,2,2\rangle$ | | | | | | | | | | | | | | | 1 | 1 | | | | |
| $(3^4)=\langle 1,1,2,3\rangle$ | 1 | | | | 1 | | | | | | | | 1 | | | | | | 1 | |
| $(3^3,2,1)=\langle 1,2,3,3\rangle$ | 1 | 1 | 1 | | 1 | | 1 | | 1 | | | | 1 | | | 1 | | | 1 | 1 |
| $(3^3,1^3)=\langle 1,2,2,3\rangle$ | | 1 | | | | | 1 | | 1 | | | | | | 1 | 1 | | | | 1 |
| $(3^2,2^3)=\langle 1,1,3,3\rangle$ | | 1 | 1 | | | | | | 1 | | | | | | | | | | | 1 |
| $(3^2,2,1^4)=\langle 1,1,1,3\rangle$ | | | | | | | | | 1 | 1 | 1 | | 1 | | 1 | 1 | 1 | | 1 | 1 |
| $(3^2,1^6)=\langle 1,3,3,3\rangle$ | | | | | | | | | | 2 | 1 | | 1 | | | 1 | 1 | | 1 | |
| $(3,2^4,1)=\langle 1,1,2,2\rangle$ | 1 | 1 | | | | | | | | | | | | | | | | | 1 | 1 |
| $(3,2^3,1^3)=\langle 1,1,1,2\rangle$ | | | | | | | | | | 1 | | | | | | 1 | 1 | | 1 | 1 |
| $(3,2,1^7)=\langle 1,2,2,2\rangle$ | | | | | | | | | | 2 | 1 | | | | 1 | 1 | 1 | | | |
| $(3,1^9)=\langle 1,1,1,1\rangle$ | | | | | | | | | | | 1 | | | | 1 | | | | | |
| $(2^6)=\langle 2,2,3,3\rangle$ | 1 | | | | | | | | | | | | | | | | | | 1 | |
| $(2^4,1^4)=\langle 2,3,3,3\rangle$ | | | | | | | | | | 1 | | | | | | | 1 | | 1 | |
| $(2^3,1^6)=\langle 2,2,2,3\rangle$ | | | | | | | | | | 1 | 1 | | | | | | 1 | | | |
| $(2,1^{10})=\langle 3,3,3,3\rangle$ | | | | | | | | | | 1 | 1 | | | | | | | | | |
| $(1^{12})=\langle 2,2,2,2\rangle$ | | | | | | | | | | 1 | | | | | | | | | | |

## A.6 The principal block of $k\mathfrak{S}_{13}$ ($\langle 4,5,4\rangle$-notation)

| | $(13)$ | $(11,2)$ | $(10,3)$ | $(10,2,1)$ | $(9,2,1^2)$ | $(8,5)$ | $(8,2^2,1)$ | $(7,6)$ | $(7,5,1)$ | $(7,4,1^2)$ | $(7,3^2)$ | $(7,3,2,1)$ | $(6,5,1^2)$ | $(6,3^2,1)$ | $(5^2,3)$ | $(5^2,2,1)$ | $(5,4,3,1)$ | $(5,3^2,1^2)$ | $(4^2,3,2)$ | $(4,3^2,2,1)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(13)=\langle 2\rangle$ | 1 | | | | | | | | | | | | | | | | | | | |
| $(11,2)=\langle 3\rangle$ | 1 | 1 | | | | | | | | | | | | | | | | | | |
| $(10,3)=\langle \underline{2},3\rangle$ | 1 | 1 | 1 | | | | | | | | | | | | | | | | | |
| $(10,2,1)=\langle \underline{2},2\rangle$ | 2 | 1 | 1 | 1 | | | | | | | | | | | | | | | | |
| $(9,2,1^2)=\langle 1\rangle$ | 2 | | 1 | 1 | 1 | | | | | | | | | | | | | | | |
| $(8,5)=\langle \underline{3},2\rangle$ | 1 | | 1 | | | 1 | | | | | | | | | | | | | | |
| $(8,2^2,1)=\langle \underline{3},1\rangle$ | 2 | 1 | 1 | | 1 | 1 | 1 | | | | | | | | | | | | | |
| $(7,6)=\langle 2,3\rangle$ | | | | | | 1 | | 1 | | | | | | | | | | | | |
| $(7,5,1)=\langle 2,2\rangle$ | 1 | | 1 | | | 1 | | 1 | 1 | | | | | | | | | | | |
| $(7,4,1^2)=\langle 1,2\rangle$ | 2 | | 1 | | 1 | | | | 1 | 1 | | | | | | | | | | |
| $(7,3^2)=\langle 2,2,3\rangle$ | 1 | | 1 | 1 | | | | | 1 | | 1 | | | | | | | | | |
| $(7,3,2,1)=\langle 2,1,3\rangle$ | 4 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | |
| $(6,5,1^2)=\langle \underline{1},2\rangle$ | 1 | | | | | | | 1 | 1 | 1 | | | 1 | | | | | | | |
| $(6,3^2,1)=\langle \underline{1},3\rangle$ | 3 | | 1 | 1 | 1 | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | |
| $(5^2,3)=\langle 3,2,2\rangle$ | | | | | | | | 1 | 1 | | 1 | | | | 1 | | | | | |
| $(5^2,2,1)=\langle 3,1,2\rangle$ | 2 | | 1 | | | | | 2 | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | | | | |
| $(5,4,3,1)=\langle 1,3\rangle$ | 2 | | 1 | | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | |
| $(5,3^2,1^2)=\langle 3,3\rangle$ | 3 | 1 | 1 | | 1 | | 1 | 3 | | 1 | | 2 | 1 | 1 | | 1 | 1 | 1 | | |
| $(4^2,3,2)=\langle 1,2,2\rangle$ | 2 | | 1 | | 1 | 1 | 1 | 2 | | | | 1 | | 1 | | 1 | 1 | | 1 | |
| $(4,3^2,2,1)=\langle 2,2,3,3\rangle$ | 3 | 1 | 1 | 1 | 1 | | 1 | 2 | | | 1 | 1 | | 1 | | 1 | 1 | 1 | 1 | 1 |
| $(10,1^3)=\langle \underline{2},1\rangle$ | | | | 1 | | | | | | | | | | | | | | | | |
| $(8,2,1^3)=\langle \underline{3},3\rangle$ | | | | | 1 | | 1 | | | | | | | | | | | | | |
| $(7,3,1^3)=\langle 2,3,3\rangle$ | 2 | | | | 1 | | 1 | | | 1 | | 1 | | | | | | | | |
| $(7,2^3)=\langle 2,1,2\rangle$ | 2 | 1 | 1 | | | 1 | 1 | 1 | | | | 1 | | | | | | | | |
| $(7,2,1^4)=\langle 2,2,2\rangle$ | | | | | | | 1 | | | | 1 | 1 | | | | | | | | |
| $(7,1^6)=\langle 2,1,1\rangle$ | | | | | | | | | | | 1 | | | | | | | | | |
| $(6,2,1^5)=\langle \underline{1},1\rangle$ | | | | | | | | | | | 1 | 1 | | 1 | | | | | | |
| $(5^2,1^3)=\langle 3,2,3\rangle$ | 1 | | | | | | | 2 | | 1 | | 1 | 1 | | | 1 | | | | |
| $(5,2^4)=\langle 3,1,3\rangle$ | 2 | 1 | 1 | | | | | 1 | | | | 1 | | | | | | 1 | | |
| $(5,2^2,1^4)=\langle 3,1,1\rangle$ | | | | | | | | 2 | | | | 1 | | 1 | | | 1 | 1 | | |
| $(5,2,1^6)=\langle 3,3,3\rangle$ | | | | | | | | | | | | | | 1 | | | 1 | | | |
| $(4^3,1)=\langle 1,2,3\rangle$ | | | | | | 1 | 1 | 1 | | | | | | | | | 1 | | | |
| $(4^2,1^5)=\langle 1,1,2\rangle$ | 1 | | | | | | | 2 | | | | 1 | 1 | 1 | | 1 | | | 1 | |
| $(4,3^3)=\langle 1,2,2,3\rangle$ | 1 | | | | 1 | | | | | | | | | 1 | | | | | 1 | |
| $(4,3^2,1^3)=\langle 2,2,2,3\rangle$ | 1 | 1 | | | | | 1 | 3 | | | 1 | 1 | | | 1 | 1 | 1 | 1 | | 1 |
| $(4,3,2^3)=\langle 1,2,3,3\rangle$ | 2 | 1 | 1 | 1 | | | | 2 | | | 1 | 1 | | | | 1 | | 1 | | 1 |
| $(4,3,2,1^4)=\langle 1,1,2,3\rangle$ | 1 | | | | | | | 4 | | | 1 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 1 |
| $(4,3,1^6)=\langle 2,3,3,3\rangle$ | | | | | | | | 2 | | | | | 1 | 1 | | 1 | 1 | | 1 | |
| $(4,2^4,1)=\langle 1,2,2,2\rangle$ | 1 | 1 | | | | | | 1 | | | | | | | | | | 1 | | 1 |
| $(4,2^3,1^3)=\langle 1,1,2,2\rangle$ | | | | | | | | 2 | | | | | | | | 1 | 1 | 1 | | 1 |
| $(4,2,1^7)=\langle 2,2,2,2\rangle$ | | | | | | | | 2 | | | | | | | 1 | 1 | 1 | | | |
| $(4,1^9)=\langle 1,1,1,2\rangle$ | | | | | | | | | | | | | | | 1 | | | | | |
| $(3^3,2^2)=\langle 1,3,3\rangle$ | 1 | | | 1 | | | | | | | 1 | | | | | | | | | 1 |
| $(3^3,1^4)=\langle 1,1,3\rangle$ | | | | | | | | 1 | | | 1 | | | | 1 | 1 | | | | 1 |
| $(3,2^4,1^2)=\langle 1,1\rangle$ | 1 | | | | | | | 1 | | | | | | | | 1 | | | 1 | 1 |
| $(3,2,1^8)=\langle 1,1,1\rangle$ | | | | | | | | 2 | | | | | 1 | | 1 | 1 | | | | |
| $(2^6,1)=\langle 1,1,3,3\rangle$ | 1 | | | | | | | | | | | | | | | | | | 1 | |
| $(2^5,1^3)=\langle 1,3,3,3\rangle$ | | | | | | | | 1 | | | | | | | | 1 | | | 1 | |
| $(2^3,1^7)=\langle 1,1,1,3\rangle$ | | | | | | | | 1 | | | | | 1 | | | 1 | | | | |
| $(2^2,1^9)=\langle 3,3,3,3\rangle$ | | | | | | | | 1 | | | | | 1 | | | | | | | |
| $(1^{13})=\langle 1,1,1,1\rangle$ | | | | | | | | 1 | | | | | | | | | | | | |

## A.7 The block of $k\mathfrak{S}_{13}$ with 3-core $(3, 1)$ ($\langle 3, 5, 3\rangle$-notation)

| | $(12,1)$ | $(9,4)$ | $(9,2^2)$ | $(7,4,2)$ | $(6,5,2)$ | $(6,4,3)$ | $(6,4,2,1)$ | $(6,3,2,1^2)$ | $(5,4,2,1^2)$ | $(4^2,2^2,1)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $(12,1) = \langle 2\rangle$ | 1 | | | | | | | | | |
| $(9,4) = \langle 2,2\rangle$ | 1 | 1 | | | | | | | | |
| $(9,2^2) = \langle 2,3\rangle$ | 2 | 1 | 1 | | | | | | | |
| $(7,4,2) = \langle 3\rangle$ | 1 | 1 | 1 | 1 | | | | | | |
| $(6,5,2) = \langle 3,2\rangle$ | | | | 1 | 1 | | | | | |
| $(6,4,3) = \langle 2,2,3\rangle$ | | | 1 | 1 | 1 | 1 | | | | |
| $(6,4,2,1) = \langle 2,2,2\rangle$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | |
| $(6,3,2,1^2) = \langle 1,2\rangle$ | 2 | 1 | 1 | | | 1 | 1 | 1 | | |
| $(5,4,2,1^2) = \langle 1\rangle$ | 1 | 1 | | | 1 | 1 | 1 | 1 | 1 | |
| $(4^2,2^2,1) = \langle 3,1\rangle$ | 2 | 1 | 1 | | | 1 | | 1 | 1 | 1 |
| $(9,1^4) = \langle 2,1\rangle$ | | | 1 | | | | | | | |
| $(6,4,1^3) = \langle 1,2,2\rangle$ | | | | | | | 1 | | | |
| $(6,2^3,1) = \langle 1,2,3\rangle$ | 2 | 1 | | | | | | 1 | | |
| $(6,2^2,1^3) = \langle 2,3,3\rangle$ | | | | | | 1 | | 1 | | |
| $(6,1^7) = \langle 1,1,2\rangle$ | | | | | | 1 | | | | |
| $(4^2,2,1^3) = \langle 3,3\rangle$ | 1 | | | | 1 | 1 | | 1 | 1 | 1 |
| $(3^4,1) = \langle 1,3\rangle$ | 1 | | 1 | | | 1 | | | | 1 |
| $(3^2,2,1^5) = \langle 1,1\rangle$ | | | | | 1 | 1 | | | 1 | 1 |
| $(3,2^5) = \langle 1,3,3\rangle$ | 1 | | | | | | | | | 1 |
| $(3,2^3,1^4) = \langle 1,1,3\rangle$ | | | | | | | | | 1 | 1 |
| $(3,2^2,1^6) = \langle 3,3,3\rangle$ | | | | | 1 | | | | 1 | |
| $(3,1^{10}) = \langle 1,1,1\rangle$ | | | | | 1 | | | | | |

## A.8 The block of $k\mathfrak{S}_{14}$ with 3-core $(3, 1^2)$ ($\langle 4, 5, 2\rangle$-notation)

| | $(12,1^2)$ | $(9,4,1)$ | $(9,3,2)$ | $(8,4,2)$ | $(6^2,2)$ | $(6,4^4)$ | $(6,4,2^2)$ | $(6,3,2^2,1)$ | $(5,4,2^2,1)$ | $(4^2,2^2,1^2)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $(12,1^2) = \langle 2\rangle$ | 1 | | | | | | | | | |
| $(9,4,1) = \langle 2,2\rangle$ | 1 | 1 | | | | | | | | |
| $(9,3,2) = \langle 2,1\rangle$ | 2 | 1 | 1 | | | | | | | |
| $(8,4,2) = \langle 1\rangle$ | 1 | 1 | 1 | 1 | | | | | | |
| $(6^2,2) = \langle 1,2\rangle$ | | | | 1 | 1 | | | | | |
| $(6,4^4) = \langle 1,2,2\rangle$ | | | 1 | 1 | 1 | 1 | | | | |
| $(6,4,2^2) = \langle 2,2,2\rangle$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | |
| $(6,3,2^2,1) = \langle 1,1,2\rangle$ | 2 | 1 | 1 | | | | 1 | 1 | | |
| $(5,4,2^2,1) = \langle 1,1\rangle$ | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | |
| $(4^2,2^2,1^2) = \langle 3\rangle$ | 1 | | 1 | | 1 | 1 | | 1 | 1 | 1 |
| $(9,1^5) = \langle 2,3\rangle$ | | | 1 | | | | | | | |
| $(6,4,1^4) = \langle 2,2,3\rangle$ | | | | | | | 1 | | | |
| $(6,3,2,1^3) = \langle 1,2,3\rangle$ | | | 1 | | | 1 | 1 | 1 | | |
| $(6,2^3,1^2) = \langle 3,2\rangle$ | | | | | | | | 1 | | |
| $(6,1^8) = \langle 2,3,3\rangle$ | | | | | | 1 | | | | |
| $(5,4,2,1^3) = \langle 1,3\rangle$ | | | | | 2 | 1 | 1 | 1 | 1 | |
| $(3^4,1^2) = \langle 3,1\rangle$ | 1 | | 1 | | | 1 | | | | 1 |
| $(3^2,2^4) = \langle 1,1,3\rangle$ | 1 | | | | | | | | | 1 |
| $(3^2,2^2,1^4) = \langle 1,1,1\rangle$ | | | | | 1 | 1 | | | 1 | 1 |
| $(3^2,2,1^6) = \langle 1,3,3\rangle$ | | | | | 2 | 1 | | | 1 | |
| $(3,2^3,1^5) = \langle 3,3\rangle$ | | | | | 1 | | | | 1 | |
| $(3,1^{11}) = \langle 3,3,3\rangle$ | | | | | 1 | | | | | |

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