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Ribbon blocks for centraliser algebras of symmetric groups

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Abstract

Suppose l, m are natural numbers with $l \leqslant m$, and $\mathbb{F}$ a field of characteristic p, and let $\mathcal{C}^{\mathbb{F}}_{l,m}$ denote the centraliser of the group algebra $\mathbb{F}S_l$ inside $\mathbb{F}S_m$. Ellers and Murray give a conjectured classification of the blocks of $\mathcal{C}^{\mathbb{F}}_{l,m}$, in terms of the p-blocks of S_l and S_m. We prove this conjecture for a family of blocks that we call *ribbon blocks* and *belt blocks*. These are the blocks containing Specht modules labelled by skew-partitions having no repeated entries in their p-content.

Contents

1 Introduction

Let G be a finite group, H a subgroup of G and $\mathbb{F}$ an algebraically closed field. The *centraliser algebra* $\mathbb{F}G^H$ is defined by

$$\mathbb{F}G^H = \{a \in \mathbb{F}G \mid ah = ha \ \forall h \in H\}.$$

This algebra appears in a series of papers by Ellers [E1, E2, E3, E4] mainly motivated by the attempt to extend some *local-global* conjectures and theorems whenever $\mathbb{F}$ has positive characteristic.

Such statements relate representation-theoretic information on $\mathbb{F}G^H$ to local information. In particular, Ellers' main motivation lies in constructing a theory in which Alperin's Weight Conjecture and Brauer's First Main Thoerem are just special cases of a more general setting.

With this in mind, the centraliser algebra has become an important object of study. If $\mathbb{F}$ has characteristic zero, then a more general argument by Curtis and Reiner [CR, Section 11D] shows that $\mathbb{F}G^H$ is a semisimple algebra, with a straightforward construction of its simple modules. So the main focus is on the case where $\mathbb{F}$ has positive characteristic.

In this paper we focus on the case where both G and H are symmetric groups. Let l, m be non-negative integers with $l \leqslant m$. The symmetric group S_l is naturally a subgroup of S_m and hence it makes sense to consider the centraliser algebra $\mathbb{F}S_m^{S_l} := \mathcal{C}_{l,m}^{\mathbb{F}}$. The main goal is to find a suitable labelling for the simple $\mathcal{C}_{l,m}^{\mathbb{F}}$-modules and a complete description of its blocks, i.e. minimal two-sided ideals whose direct sum gives the whole algebra. A powerful result would be to give answers to these issues in terms of the combinatorics of integer partitions, somehow extending the symmetric group case pioneered by James [J] which we recall in Section 2. Following [K], a complete set of pairwise non-isomorphic simple $\mathcal{C}_{l,m}^{\mathbb{F}}$-modules when $\mathrm{char}(\mathbb{F}) = 0$ is given by modules of the form

$$\mathrm{S}^{\mu\setminus\lambda} := \mathrm{Hom}_{\mathbb{F}S_l}(\mathrm{S}^{\mu}\downarrow_{S_l}, \mathrm{S}^{\lambda})$$

where μ and λ denote partitions of m and l, respectively, and $\mu \setminus \lambda$ is a *skew-partition* in the sense of Section 2.3. In view of the similarity with the symmetric group case, we name these modules *Specht modules*. The Specht modules are also defined in positive characteristic, where they are generally reducible, and (unlike in the symmetric group case) there does not seem to be a straightforward way to obtian the simple modules from the Specht modules. In positive characteristic, a full description of the simple modules of $\mathcal{C}_{l,m}^{\mathbb{F}}$ is far out of reach.

The goal of the present paper is to characterise certain blocks of $\mathcal{C}_{l,m}^{\mathbb{F}}$. As in the symmetric group case, describing the blocks amounts to finding the appropriate subdivision of the set of Specht modules (or equivalently, of the set of skew-partitions). In [EM2], Ellers and Murray suggested a classification of the blocks in positive characteristic and checked it for $m - l \leqslant 3$. Their conjecture (Conjecture 2.5) is inspired by the Nakayama Conjecture (Theorem 2.1) and hence is in terms of cores of partitions. We define a *combinatorial block* to be a set of skew-partitions which is predicted to comprise a single block by Conjecture 2.5. The work of Ellers and Murray shows that every combinatorial block is a union of blocks, so the remaining task to prove their conjecture is to show that the Specht modules labelled by two skew-partitions in the same combinatorial block really do lie in the same block.

In this paper we prove Ellers and Murray's conjecture for a class of combinatorial blocks of $\mathcal{C}_{l,m}^{\mathbb{F}}$ we call *ribbon blocks* and *belt blocks*. These are precisely the combinatorial blocks containin skew-partitions having no repetitions in their content (see Section 2.3). In Section 3 we show that (a proper approximation of) the decomposition matrix of a ribbon or belt block is a connected matrix; this implies that combinatorial blocks of this kind are blocks of the centraliser algebra.

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2 Background and notation

In this chapter we introduce all the background we need in Section 3. We begin with some basic notations. Let R be a commutative ring with field of fractions $\mathbb{F}$. Let $\mathfrak{p}$ be a maximal ideal of R and denote by k the residue field $R/\mathfrak{p}$ of positive characteristic p. We suppose that both $\mathbb{F}$ and k are algebraically closed.

If $e \geqslant 2$ is an integer, then we write $\bar{a}$ to mean $a + e\mathbb{Z}$, for $a \in \mathbb{Z}$, so that $\mathbb{Z}/e\mathbb{Z} = \{\bar{a} \mid a \in \mathbb{Z}\} = \{\bar{0}, \bar{1}, \dots, \overline{e-1}\}$.

In this paper, $\mathbb{N}$ denotes the set of positive integers.

2.1 Partitions and Specht modules

In this section we introduce the main combinatorial objects in this paper. A *partition* is a non-increasing sequence $\nu = (\nu_1, \nu_2, \dots)$ of non-negative integers such that $\nu_N = 0$ for $N \gg 0$. The integer ν_k is called the k-th *part* of ν and the number of non-zero parts is referred as the *length* of ν and denoted by $l(\nu)$. The (finite) sum $\nu_1 + \cdots + \nu_{l(\nu)}$ is referred as the *size* of ν and denoted by $|\nu|$. If $n \geqslant 0$ and ν is a partition of size n, we say that ν *is a partition of* n and we write $\nu \vdash n$. We denote by $\varnothing$ the unique partition of 0. When writing partitions we omit the trailing zeroes and group together equal parts with a superscript, e.g. $(3, 2^3, 1)$ stands for the length 5 partition $(3, 2, 2, 2, 1, 0, \dots) \vdash 10$. We denote by $\mathcal{P}$ the set of all partitions and by $\mathcal{P}_n$ the set of partitions of a fixed non-negative integer n.

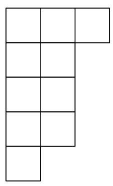
The set of partitions $\mathcal{P}_n$ plays the role of a powerful labelling set for a relevant class of RS_n, $\mathbb{F}S_n$ and kS_n-modules known in the literature as *Specht modules*. The set of Specht modules $\{\mathrm{S}^\nu \mid \nu \in \mathcal{P}_n\}$ is a crucial object in the study of the representation theory of the symmetric group S_n because it provides a complete set of pairwise non-isomorphic simple $\mathbb{F}Sn$-modules (in fact, more generally, over any characteristic zero field). If $\mathbb{F}$ is replaced by the residue field k, an analogous statement holds if $n < p$. When $n \geqslant p$, a complete family of pairwise non-isomorphic simple kS_n-modules can be constructed from the Specht modules: if μ is a *p-regular* partition (meaning that it does not have p equal positive parts), then the Specht module S^μ over k has a unique simple quotient D^μ, and the simple modules obtained in this way give a complete set of non-isomorphic simple kS_n-modules. The main outstanding problem in the modular representation theory of the symmetric group is the determination of the *decomposition numbers* $[\mathrm{S}^\lambda : \mathrm{D}^\mu]$ for all partitions λ, μ of n with μ p-regular. This problem goes back to the work of Robinson in the 1960s [Ro2], but remains unsolved in general.

2.2 Young diagrams and standard tableaux

We now introduce a good method for visualising partitions. The *Young diagram* of a partition ν is the set

$$[\nu] = \{(r, c) \in \mathbb{N}^2 \mid c \leqslant \nu_r\}.$$

The elements of $[\nu]$, and more generally of $\mathbb{N}^2$, are called *nodes*. We draw a Young diagram as an array of boxes in the plane such that the horizontal axis is oriented left-to-right and the vertical axis top-to-bottom (commonly referred as the *English* notation). For example, $[(3, 2^3, 1)]$ appears as follows.



Motivated by the drawing, we say that the node (r, c) is *above* the node (s, d), or that (s, d) is *below* (r, c), if $r < s$. Analogously, we say that (r, c) lies *to the left* of (s, d), or that (s, d) lies *to the right* of (r, c), whenever $c < d$.

Fix e is a positive integer. The *content* of a node (r, c) is the integer $c - r$, and the *e-residue* of (r, c) is $\overline{c - r}$. Note that nodes lying in the same diagonal of $\mathbb{N}^2$ have the same content and therefore the same e-residue for all e. If $\nu \in \mathcal{P}$, we define the *e-content* and we write $\mathrm{cont}_e(\nu)$, for the multiset of e-residues of the nodes in $[\nu]$. For example,

$$\mathrm{cont}_3(3,2^3,1) = \{\overline{-4},\overline{-3},\overline{-2},\overline{-2},\overline{-1},\overline{-1},\bar{0},\bar{0},\bar{1},\bar{2}\} = \{\bar{0},\bar{0},\bar{0},\bar{1},\bar{1},\bar{1},\bar{2},\bar{2},\bar{2},\bar{2}\}.$$

The *rim* of a partition ν is the set of nodes $(r,c) \in [\nu]$ such that $(r+1,c+1) \notin [\nu]$. We regard this set as ordered from from SW to NE saying that a node (r,c) comes after (s,d) if (r,c) lies above or to the right of (s,d). For $h \geqslant 1$, we define a *removable h-hook* of ν to be an interval $(r_1,c_1),\dots,(r_h,c_h)$ of the rim of ν such that $(r_1+1,c_1) \notin [\nu]$ and $(r_h,c_h+1) \notin [\nu]$ and . We call (r_1,c_1) and (r_h,c_h) the *foot node* and the *hand node* of the removable h-hook, respectively. By definition the nodes of a removable h-hook can be removed from $[\nu]$ to give the Young diagram of a partition of $|\nu| - h$. If $h = 1$, a removable 1-hook is simply called a *removable node*. Dually, we define the *neighbours* of $[\nu]$ to be the nodes $(s,d) \notin [\nu]$ such that either $s = 1$ or $d = 1$, or $(s-1,d-1) \in [\nu]$. The neighbours of ν are naturally ordered in increasing order of content. For $h \geqslant 1$, an interval of neighbours $(s_1,d_1),\dots,(s_h,d_h)$ of $[\nu]$ is called an *addable h-hook* if $d_1 = 1$ or $(s_1,d_1-1) \in [\nu]$, and $s_h = 1$ or $(s_h-1,d_h) \in [\nu]$. As in the dual case, we call (s_1,d_1) the *foot node* and (s_h,d_h) the *hand node* of the addable h-hook. Again we can see that, pictorially, an addable h-hook is a series of adjacent nodes which can be added to $[\nu]$ to obtain the Young diagram of a partition of $|\nu| + h$. An addable 1-hook is called an *addable node*.

Fix $n \geqslant 0$ and $\nu \in \mathcal{P}_n$. We define a *ν-tableau* as a bijection $T : [\nu] \to \{1,\dots,n\}$ represented by filling the nodes of the Young diagram $[\nu]$ with their images under T. A ν-tableau is called *standard* if its entries increase along rows and down columns.

Example. Let $\nu = (5,3,2,1) \vdash 11$. The following are two ν-tableaux, the one on the left being standard.

1	3	7	8	10
2	5	9		
4	6			
11				

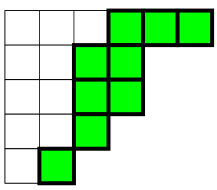
4	9	2	8	10
11	7	3		
1	6			
5				

2.3 Skew-partitions and skew Specht modules

This section gets the reader in touch with the combinatorial protagonist of the paper. Since the algebra of interest – as we shall see in Section 2.7 – relies on two natural numbers, it seems natural to consider a relation of *containment* between pairs of partitions.

Let $\lambda, \mu \in \mathcal{P}$. We say that λ *lies inside* μ if $[\lambda] \subseteq [\mu]$. In this case we say that $\mu \setminus \lambda$ is a *skew-partition*. We immediately see that, if $\mu \setminus \lambda$ is a skew-partition, then $|\mu| \geqslant |\lambda|$ with equality holding if and only if $\mu = \lambda$. We call the non-negative integer $|\mu| - |\lambda|$ the *size* of $\mu \setminus \lambda$. Given two non-negative integers $l \leqslant m$, we set $\mathcal{P}_{l,m} := \{\mu \setminus \lambda \mid \lambda \in \mathcal{P}_l,\ \mu \in \mathcal{P}_m\}$.

It is useful to visualise skew-partitions as Young diagrams. For a skew-partition $\mu \setminus \lambda$, we define the *skew-diagram* $[\mu \setminus \lambda]$ as the set subset of $\mathbb{N}^2$ comprising nodes $[\mu] \setminus [\lambda]$. For example, if $\mu = (6,4^2,3,2) \vdash 19$ and $\lambda = (3,2^3,1) \vdash 10$, the skew-diagram $[\mu \setminus \lambda]$ comprises the coloured nodes in the picture below.



Note that two different skew-partitions can have the same Young diagram; for example, $[(3,2) \setminus (3,1)] = [(2^2,1) \setminus (2,1^2)]$.

We remark that if λ lies inside μ, the Young diagram $[\lambda]$ can be recovered by repeatedly deleting removable hooks from $[\mu]$. The union of these hooks comprises the skew-diagram $[\mu \setminus \lambda]$. Note that this series of removals is not unique.

Similarly to the case of partitions, for e a positive integer, we define the *e-content* of a skew-partition $\mu \setminus \lambda$, and we write $\mathrm{cont}_e(\mu \setminus \lambda)$, as the multiset of e-residues of the nodes in $[\mu \setminus \lambda]$, e.g.

$$\mathrm{cont}_4((6,4^2,3,2) \setminus (3,2^3,1)) = \{\overline{-3},\overline{-1},\bar{0},\bar{1},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\} = \{\bar{0},\bar{0},\bar{1},\bar{1},\bar{1},\bar{1},\bar{2},\bar{3},\bar{3}\}.$$

In conclusion to this section, we generalise the ideas of tableau and standard tableau to the case of skew-partitions. If $\mu \setminus \lambda$ is a skew-partition of size n, a $\mu \setminus \lambda$*-tableau* is a bijection $T : [\mu \setminus \lambda] \to \{1, \ldots, n\}$ represented by filling the nodes of $[\mu \setminus \lambda]$ with their images under T. A tableau is *standard* if its entries increase along rows and down columns. Standard *skew-tableaux* play a role in the representation theory of symmetric groups: if $\mu \setminus \lambda \in \mathcal{P}_{l,m}$, the set of standard $\mu \setminus \lambda$-tableaux is in bijection with the paths between the Specht modules S^{λ} and S^{μ} in the branching graph of symmetric groups (see for example [S, Theorem 2.8.3]). The branching rule itself also ensures that if A is a commutative ring A, then the A-module

$$\mathrm{S}^{\mu\setminus\lambda} := \mathrm{Hom}_{AS_l}(\mathrm{S}^{\lambda}, \mathrm{S}^{\mu}\downarrow_{S_l})$$

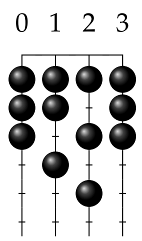
is non-trivial if and only if $\mu \setminus \lambda$ is a skew-partition, in which the dimension of $\mathrm{S}^{\mu\setminus\lambda}$ equals the number of standard $\mu \setminus \lambda$-tableaux. Modules of this kind will come back in Section 2.7 and will be the heart of our treatment. We remark that analogous modules for the affine Hecke algebra were studied by Ram [Ra].

From now on throughout this paper we will abuse notation by omitting square brackets and not distinguishing between partitions and Young diagrams and between skew-partitions and skew-diagrams.

2.4 The abacus

In this section we introduce an idea of James and Kerber [JK] that gives another combinatorial perspective to the objects mentioned in Sections 2.1 and 2.3. Fix e a positive integer. We consider an abacus display with e vertical runners labelled from left to right by the symbols $0, \ldots, e-1$. We index the *positions* in the i-th runner by the integers $i, i+e, i+2e, \ldots$ from the top down. Then we align runners so that position x is immediately to the right of position $x-1$ whenever $e \nmid x$. Each position in the abacus is either a *bead* ● or a *space* ┼.

Now consider a partition ν and an integer $C \geqslant l(\nu)$. We put a bead in the abacus display just constructed at position $\nu_j - j + C$ for $1 \leqslant j \leqslant C$. We refer to this drawing as the *abacus configuration* of ν (of charge C) and we denote it by $\mathrm{Ab}_e(\nu)$, or simply $\mathrm{Ab}(\nu)$ if e is clear from the context. The abacus configuration of a skew-partition $\mu \setminus \lambda$ is the pair of abacus configurations $\mathrm{Ab}(\mu)$ and $\mathrm{Ab}(\lambda)$ with the same charge. For convenience we always choose $C \equiv 0 \pmod{e}$ (this means that the total number of beads in an abacus display is a multiple of e). Observe that increasing C by e entails adding a row of beads at the top of the abacus configuration. Note that, by construction, every runner of the abacus has infinitely many consecutive spaces downwards. We will use the convention that all positions below those shown are spaces. The picture below exhibits the abacus configuration of $(7,3,2^2,1^2) \vdash 16$ in an abacus display with 4 runners and 12 beads.



We remark that given the abacus configuration $\mathrm{Ab}(\nu)$ with charge C, we can read off the j-th part of ν counting the number of spaces which precede the j-th largest bead position in $\mathrm{Ab}(\nu)$. Note that, by definition, the abacus encodes the rim of ν: a node (r,c) corresponds to the position marked by $c-r+C$; this position is a bead if and only if (r,c) lies below the next node in the rim of ν, or $(r,c)=(1,\lambda_1)$, and otherwise this position is a space.

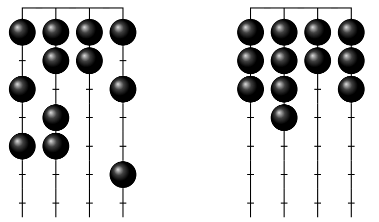
We remark that for $h \geqslant 1$, a removable h-hook arises from a position x such that x is a bead and $x-h$ is a space in $\mathrm{Ab}(\nu)$. The removal of this h-hook from ν corresponds to moving the bead at x into the space at $x-h$. Similarly, adding an addable h-hook corresponds to moving a bead at position x to position $x+h$, for some x. We will say that a removable or addable hook is *at runner i* if it corresponds to a position $x \equiv i \pmod e$. In the abacus configuration above, for example, $(7,3,2^2,1^2)$ has a removable 7-hook at runner 1 since position 13 is a bead and position 6 is a space. Switching these two positions gives the abacus configuration of the partition $(7,1^2) \vdash 9$ obtained by removing nodes $(6,1)$,$(5,1)$,$(4,1)$,$(4,2)$,$(3,2)$,$(2,2)$ and $(2,3)$ from $(7,3,2^2,1^2)$.

2.5 p-cores and p-blocks

Our main interest in this paper is understanding blocks, and to prepare the ground we recall the combinatorics underlying the p-block theory of the symmetric groups. The theory of p-blocks of finite groups goes back to the early work of Brauer, and the p-blocks for the symmetric group were first studied by Nakayama [N], who conjectured a classification proved simultaneously by Brauer and Robinson [B, Ro1].

Fix an integer $e \geqslant 2$, we say that a partition is an *e-core* if it has no removable e-hooks. Given $\nu \in \mathcal{P}$, we define the *e-core of* ν as the partition obtained by successively deleting removable e-hooks from ν until an e-core is reached. We denote this partition by $\mathrm{core}_e(\nu)$. In terms of the abacus, evaluating $\mathrm{core}_e(\nu)$ coincides with sliding all beads up as far as possible in the runners of $\mathrm{Ab}(\nu)$. Since this process clearly produces a well-defined abacus configuration, we deduce that the e-core of a partition is well-defined. We define the *e-weight* of ν, written $\mathrm{weight}_e(\nu)$, to be the number of e-hooks removed in evaluating $\mathrm{core}_e(\nu)$. Note that $|\nu| = |\mathrm{core}_e(\nu)| + e\,\mathrm{weight}_e(\nu)$. For $i=0,\dots,e-1$, we write $\nu^{(i)}$ for the partition which corresponds to the i-th runner of $\mathrm{Ab}(\nu)$ considered as an abacus configuration with one single runner (with positions properly numbered). The e-tuple of partitions $(\nu^{(0)},\dots,\nu^{(e-1)})$ is called the *e-quotient* of ν. From the abacus interpretation of the weight, we immediately deduce that $\mathrm{weight}_e(\nu) = |\nu^{(0)}| + \cdots + |\nu^{(e-1)}|$.

Example. Let $\nu = (12,7^2,5,4,2,1^2) \vdash 39$ and $e=4$. Then $\mathrm{core}_4(\nu) = (2,1) \vdash 3$, as we see from the following abacus configurations.



$\mathrm{Ab}(\nu)$ $\mathrm{Ab}(\mathrm{core}_4(\nu))$

We find that $\mathrm{weight}_4(\nu) = 9$, and the 4-quotient of ν is $((2,1),(1^2),\varnothing,(3,1))$.

Now let n be a positive integer, and consider the p-blocks of S_n; that is, the indecomposable summands of the algebra kS_n. It is a standard fact that each Specht module is a module for a single p-block. Conversely, every block contains at least one Specht module (because every simple module is a composition factor of a Specht module), so in order to describe the p-block structure of S_n it suffices to say when two Specht modules lie the same block. The following famous result, known as the *Nakayama conjecture*, gives a combinatorial condition to understand when two Specht modules belong to the same p-block of S_n.

Theorem 2.1. *Let $\alpha, \beta \in \mathcal{P}_n$. The Specht modules S^α, S^β belong to the same p-block of S_n if and only if* $\mathrm{core}_p(\alpha) = \mathrm{core}_p(\beta)$.

Theorem 2.1 was stated by Nakayama in 1940 and proved independently by Brauer and Robinson. Thorough the paper we will abuse notation regarding a p-block B of S_n as the subset of partitions of n which labels the Specht kS_n-modules belonging to B.

Clearly if two partitions of n have the same p-core then they have the same p-weight. Given a p-block B of S_n, we denote by γ_B and w_B, respectively, the p-core and the p-weight of *any* partition in B. It is clear that (the abacus configuration of) all partitions in B can be recovered starting from γ_B and then sliding w_B times beads down in $\mathrm{Ab}(\gamma_B)$ in all possible ways.

We now give some more conditions equivalent to those in Theorem 2.1. Note that nothing of what follows requires p to be a prime number. We need all the abacus configurations to contain the same number of beads. Since the length of a partition of n is at least n, to ensure this is enough to fix the (common) charge C of the abaci to be at least n, maintaining the convention that $C \equiv 0 \pmod p$. For every $\nu \in \mathcal{P}_n$ and every $i = 0, \dots, p-1$, let $b_i(\nu)$ denote the number of beads in the i-th runner of the abacus configuration $\mathrm{Ab}_p(\nu)$. It turns out that the p-tuple $(b_0(\nu), \dots, b_{p-1}(\nu))$ is another invariant of the p-block to which ν belongs. Actually, by definition, the p-core of a partition is obtained by only sliding beads up in its abacus configuration. It follows that the number of beads in every runner remains unchanged at the end of the process. In particular, if α and β are in the same p-block B, then $b_i(\alpha) = b_i(\gamma_B) = b_i(\beta)$ for all $i = 0, \dots, p-1$. (Note that the numbers b_i depend on the choice of the charge; in fact C can be recovered as $b_0 + \cdots + b_{p-1}$. If C is increased by p, then each b_i is increased by 1.)

Analysing the integers just defined, Ellers and Murray [EM2, Lemma 7] derived the following result which we will need later.

Proposition 2.2. *Let B be a p-block of S_n and let $\alpha, \beta \in B$. Suppose that for some $h \geqslant 1$ and some $i \in \{0, \dots, p-1\}$, α has a removable h-hook at runner i. Then at least one of the following occurs:*

- ◇ *α has an addable h-hook at runner $j \equiv i - h \pmod p$;*
- ◇ *β has a removable h-hook at runner i.*

Another reinterpretation of the combinatorial condition for two Specht modules to be in the same block concerns p-contents. Recall that the p-content of a partition λ is the multiset of p-residues of the nodes of λ. The following result goes back to Littlewood [L].

Theorem 2.3. *Suppose $\lambda, \mu \in \mathcal{P}_n$. Then λ and μ have the same p-core if and only if they have the same p-content.*

So we can label a p-block B of S_n by a multiset of cardinality n of elements of $\mathbb{Z}/p\mathbb{Z}$, corresponding to the p-content of any partition in B. Using this interpretation, we immediately gain a useful fact which will come in handy in Section 3.2.

Proposition 2.4. *Let m be an integer with $m \geqslant p$. If $\mu \setminus \lambda \in \mathcal{P}_{m-p,m}$ and $\mathrm{cont}_p(\mu \setminus \lambda) = \{\bar{0}, \dots, \overline{p-1}\}$, then $\mathrm{core}_p(\lambda) = \mathrm{core}_p(\mu)$ and $\mathrm{weight}_p(\lambda) = \mathrm{weight}_p(\mu) - 1$.*

Proof. Let $\nu = \mathrm{core}_p(\lambda)$, let B be the block of S_{m-p} containing λ, and let C be the block of S_m with p-core ν. Observe that the nodes of a rim p-hook have contents $a, a+1, \dots, a+p-1$ for some integer a, and therefore have p-residues $\bar{0}, \bar{1}, \dots, \overline{p-1}$ in some order. Therefore the p-content of C is obtained from the p-content of B by adding one copy of each element $\bar{0}, \bar{1}, \dots, \overline{p-1}$. From Theorem 2.3, this means that μ lies in C. □

2.6 The degenerate affine Hecke algebra

The next two sections are devoted to introducing the main algebraic objects in the paper. Before meeting the algebra which is at the heart of the treatment, we introduce an auxiliary structure and some interesting related concepts. Let A be a field and n be a positive integer.

Definition. The *degenerate affine Hecke algebra of degree n over A*, denoted by $\mathcal{H}_n^A$, is the unital, associative A-algebra with generators $z_1, \dots, z_n, s_1, \dots, s_{n-1}$ subject to the following relations:

1. $z_i z_j = z_j z_i$ for all i, j;
2. $(s_i s_j)^{m_{ij}} = 1$, where $m_{ii} = 1$, $m_{i(i+1)} = 3$ and $m_{ij} = 2$ for $|i-j| > 1$;
3. $s_i z_j = z_j s_i$, for $j \neq i, i+1$;
4. $s_i z_i = z_{i+1} s_i - 1$.

Looking at the relation (1), we see that the *polynomial* generators $z_1, \dots, z_n$ generate a subalgebra $A[z_1, \dots, z_n]$ of $\mathcal{H}_n^A$ isomorphic to the polynomial algebra in n indeterminates over A. Moreover, this is a maximal commutative subalgebra of $\mathcal{H}_n^A$. Considering relations in (2), we also observe that the *Coxeter* generators $s_1, \dots, s_{n-1}$ generate a subalgebra of $\mathcal{H}_n^A$ isomorphic to AS_n. In fact, as A-modules, $\mathcal{H}_n^A \cong A[z_1, \dots, z_n] \otimes_A AS_n$. We remark that the centre $Z(\mathcal{H}_n^A)$ of $\mathcal{H}_n^A$ is given by the ring of symmetric polynomials in the indeterminates $z_1, \dots, z_n$ [K, Theorem 3.3.1].

We now focus on the polynomial subalgebra in the case where A is an algebraically closed field. Since $A[z_1, \dots, z_n]$ is commutative, every simple $A[z_1, \dots, z_n]$-module M is one-dimensional and corresponds (via a 1:1 correspondence) to the n-tuple $(a_1, \dots, a_n) \in A^n$ such that z_i acts as the scalar a_i on M. We define the *formal character* of a $A[z_1, \dots, z_n]$-module as the formal $\mathbb{Z}$-linear combination of the elements of A^n corresponding to its composition factors. The *formal character* of a finite dimensional $\mathcal{H}_n^A$-module M is then defined as the formal character of its restriction $M \downarrow_{A[z_1,\dots,z_n]}$. More formally, following [K], one identifies $A[z_1, \dots, z_n]$ with the *parabolic subalgebra*

$$\mathcal{H}^A_{(1,\dots,1)} \cong \mathcal{H}_1^A \otimes_A \cdots \otimes_A \mathcal{H}_1^A$$

of $\mathcal{H}_n^A$, and defines the formal character $\mathrm{ch}(M)$ of M as the image of the class $[M]$ in the Groethendieck group $K(\mathcal{H}_n^A\text{-mod})$, under the homomorphism

$$\mathrm{ch} : K(\mathcal{H}_n^A\text{-mod}) \longrightarrow K(\mathcal{H}^A_{(1,\dots,1)}\text{-mod})$$

induced by the restriction from $\mathcal{H}_n^A$ to $\mathcal{H}^A_{(1,\dots,1)}$. By [K, Theorem 5.3.1], ch is injective, therefore the composition factors of an $\mathcal{H}_n^A$-module are determined by its formal character. This fact will be particularly useful in what follows due to the strong relationship between the degenerate affine Hecke algebra and the algebra we are going to meet in the next section.

Before going on we associate another important object to any indecomposable $\mathcal{H}_n^A$-module M. Let $(a_1, \dots, a_n)$ be a summand in the formal character of M. The *central character* of M is defined as the map $Z(\mathcal{H}_n^A) \to A$ such that $f \mapsto f(a_1, \dots, a_n)$ for all $f \in Z(\mathcal{H}_n^A)$. Since M is indecomposable, all summands of $\mathrm{ch}(M)$ lie in the same S_n-orbit [K, Lemma 4.2.2], so the previous definition does not depend on the choice of the summand $(a_1, \dots, a_n)$. So we say that $\{a_1, \dots, a_n\}$ is the central character of M and we write $\mathcal{C}(M)$ for it. Then two indecomposable $\mathcal{H}_n^A$-modules lie in the same block if and only if they have the same central character.

2.7 The centraliser algebra for symmetric groups

We are ready to meet the algebraic protagonist of the present work. Most of the material in this section is taken from [EM2]. Let A be a commutative ring and take m, l non-negative integers with $l \leqslant m$ and set $n := m - l$. Then AS_l is an A-subalgebra of the group ring AS_m. We define the *centraliser algebra* $\mathcal{C}^A_{l,m}$ to be the centraliser of AS_l in AS_m; that is, the algebra

$$\mathcal{C}^A_{l,m} := \{a \in AS_m \mid ab = ba \text{ for every } b \in AS_l\}.$$

This algebra has been studied extensively by Ellers and Murray [EM1, EM2], following earlier more general work on centraliser algebras for subgroups of finite groups by Ellers. However, $\mathcal{C}^A_{l,m}$ remains poorly understood, and in particular its blocks are not yet known. This is the focus of the present paper.

By the very effective description in [K, Section 2.1] we know a straightforward generating set for $\mathcal{C}^A_{l,m}$. This is the union of three subsets:

- ⋄ the centre $Z(AS_m)$;
- ⋄ the symmetric group (isomorphic to S_n) on the symbols $\{l+1, \dots, m\}$;
- ⋄ the set of *Jucys–Murphy elements* $L_j = (1\,j) + (2\,j) + \cdots + (j-1\,j)$ for $j = l+1, \dots, m$.

We introduce a family of $\mathcal{C}^A_{l,m}$-modules which will be at the heart of our treatment. Suppose that $\lambda \vdash l$ and $\mu \vdash m$. Then the A-module $\mathrm{S}^{\mu\setminus\lambda} = \mathrm{Hom}_{AS_l}(\mathrm{S}^\lambda, \mathrm{S}^\mu \downarrow_{S_l})$ introduced at the end of Section 2.3 is naturally a module for $\mathcal{C}^A_{l,m}$, via the action

$$(c\phi)(v) = c\phi(v) \qquad \text{for } c \in \mathcal{C}^A_{l,m},\ \phi \in \mathrm{S}^{\mu\setminus\lambda},\ v \in \mathrm{S}^\lambda.$$

As already remarked, $\mathrm{S}^{\mu\setminus\lambda}$ is non-trivial if and only if $\mu \setminus \lambda \in \mathcal{P}_{l,m}$. To emphasise the similarity with the symmetric group case, we refer to these $\mathcal{C}^A_{l,m}$-modules as *Specht modules*.

We now recall our p-modular system $(R, \mathbb{F}, k)$, where p is the prime characteristic of the residue field k. Since $\mathrm{char}(\mathbb{F}) = 0$ the algebras $\mathbb{F}S_m$ and $\mathbb{F}S_l$ are both semisimple by Maschke's Theorem. The general theory of centraliser algebras then implies that $\mathcal{C}^{\mathbb{F}}_{l,m}$ is semisimple as well, with a complete irredundant set of simple modules given by the set

$$\{\mathrm{S}^{\mu\setminus\lambda} \mid \mu \setminus \lambda \in \mathcal{P}_{l,m}\}$$

of Specht modules.

Things change if the fraction field $\mathbb{F}$ is replaced by the residue field k: the centraliser algebra $\mathcal{C}^k_{l,m}$ is in general not semisimple and its Specht modules generally fail to be simple. The decomposition number problem for $\mathcal{C}^k_{l,m}$ asks for the composition factors of the Specht modules; unfortunately, at present we do not have a good labelling of simple module for $\mathcal{C}^k_{l,m}$. This paper is concerned with determining the *p-blocks* of $\mathcal{C}^k_{l,m}$. The Specht module $\mathrm{S}^{\mu\setminus\lambda}$ over k is a p-modular reduction of the corresponding (simple) Specht module over $\mathbb{F}$. So (analogously to the case of symmetric groups) describing the p-block structure of $\mathcal{C}^k_{l,m}$ reduces to finding the appropriate partition of the set of Specht $\mathcal{C}^k_{l,m}$-modules. For this, we use an approach pioneered by Ellers and Murray, which we describe in the next section.

2.8 Combinatorial blocks

Let l, m, n be as in Section 2.7. We have pointed out that in order to find the decomposition of $\mathcal{C}^k_{l,m}$ into p-blocks, we need to look for a way to partition the set of Specht $\mathcal{C}^k_{l,m}$-modules. We know by Theorem 2.1 how the Specht modules for the symmetric group split into p-blocks, this is a characterisation based on the concept of p-core. Our main conjecture (which is implicit in the work of Ellers and Murray [EM2]) predicts that the case of the centraliser algebra works in the same way.

Conjecture 2.5. *Let $\mu_1 \setminus \lambda_1, \mu_2 \setminus \lambda_2 \in \mathcal{P}_{l,m}$. The Specht $\mathcal{C}^k_{l,m}$-modules $\mathrm{S}^{\mu_1\setminus\lambda_1}, \mathrm{S}^{\mu_2\setminus\lambda_2}$ belong to the same p-block of $\mathcal{C}^k_{l,m}$ if and only if $\mathrm{core}_p(\mu_1) = \mathrm{core}_p(\mu_2)$ and $\mathrm{core}_p(\lambda_1) = \mathrm{core}_p(\lambda_2)$.*

We can consider Conjecture 2.5 in terms of idempotents. We say that a non-zero central idempotent of $\mathcal{C}^k_{l,m}$ is a *block idempotent* if it is primitive, i.e. it cannot be written as the sum of two non-zero central idempotents. The blocks of $\mathcal{C}^k_{l,m}$ are then precisely the algebras $e\mathcal{C}^k_{l,m}$ for e a primitive central idempotent.

In the case of the symmetric group S_n, Theorem 2.1 tells that there is a bijection between the set of p-block idempotents of kS_n and the set of p-core partitions which are the p-core of at least one partition in $\mathcal{P}_n$. Given p-blocks B and B' of kS_m and kS_l respectively, let e_B and $e_{B'}$ the corresponding p-block idempotents. Then it is straightforward to see that $e_B\, e_{B'}$ is a central idempotent of $\mathcal{C}^k_{l,m}$, so is a sum of block idempotents, and $e_B\, e_{B'}\mathcal{C}^k_{l,m}$ is a (possibly zero) sum of p-blocks of $\mathcal{C}^k_{l,m}$. Moreover, $e_B\, e_{B'}$ acts as the identity on $\mathrm{S}^{\mu\setminus\lambda}$ if S^{μ} lies in B and S^{λ} lies in B' (that is, if $\mathrm{core}_p(\mu) = \gamma_B$ and $\mathrm{core}_p(\lambda) = \gamma_{B'}$), and otherwise $e_B\, e_{B'}\, \mathrm{S}^{\mu\setminus\lambda} = 0$. If $e_B\, e_{B'} \neq 0$, then we refer to $e_B\, e_{B'}\mathcal{C}^k_{l,m}$ as a *combinatorial block* of $\mathcal{C}^k_{l,m}$. Conjecture 2.5 then predicts that every combinatorial block is a p-block. Since the sum of the elements $e_B\, e_{B'}$ over all choices of B and B' is 1, this would then mean that every p-block of $\mathcal{C}^k_{l,m}$ is a combinatorial block. We will abuse notation by identifying a combinatorial block with the set of Specht modules in that combinatorial block, or with the set of labelling skew-partitions.

In [EM2], Ellers and Murray prove Conjecture 2.5 in the case $n \leqslant 3$. The aim of the present paper is to prove Conjecture 2.5 for a large family of combinatorial blocks of $\mathcal{C}^k_{l,m}$. We follow the idea of Ellers and Murray [EM2, Section 1.2] to exploit the relationship between the centraliser algebra $\mathcal{C}^k_{l,m}$ and the degenerate affine Hecke algebra $\mathcal{H}^k_n$. Their work shows that every $\mathcal{C}^k_{l,m}$-module naturally admits the structure of an $\mathcal{H}^k_n$-module: the symmetric group on $\{l+1, \ldots, m\}$ and the Jucys–Murphy elements $L_{l+1}, \ldots, L_m$ generate a subalgebra of $\mathcal{C}^k_{l,m}$ which is a quotient of $\mathcal{H}^k_n$; restricting to this subalgebra and then inflating yields an $\mathcal{H}^k_n$-module. With respect to this structure, every Specht module is equipped with a formal character and a central character and it is an easy task to combinatorially evaluate these. Crucially, two $\mathcal{C}^k_{l,m}$-modules in the same combinatorial block have the same composition factors if and only if the corresponding $\mathcal{H}^k_n$-modules do. This enables us to work with formal characters of Specht modules to prove the p-block structure.

Let $\mu \setminus \lambda \in \mathcal{P}_{l,m}$ and consider a standard $\mu \setminus \lambda$-tableau T. We associate to T the permutation $a_T = (a_1, \ldots, a_n)$ of $\mathrm{cont}_p(\mu \setminus \lambda)$ such that a_j is the p-residue of the node containing with number j in T, for $j = 1, \ldots, n$. The formal character of the Specht module $\mathrm{S}^{\mu\setminus\lambda}$, we denote by $\mathrm{ch}(\mu \setminus \lambda)$, is the formal sum of the n-tuples a_T with T varying in the set of standard $\mu \setminus \lambda$-tableaux.

Example. Let $p = 3$ and $\mu \setminus \lambda = (4^2, 2) \setminus (2^2, 1) \in \mathcal{P}_{5,10}$. The formal character of the Specht $\mathcal{C}^k_{5,10}$-module $\mathrm{S}^{\mu\setminus\lambda}$ is as follows (we draw the skew-diagram $\mu \setminus \lambda$ filling every node with its 3-residue, omitting bars):

		2	0
		1	2
	2		

$$2 \cdot (2,2,0,1,2) + 2 \cdot (2,2,1,0,2) + 2 \cdot (2,0,1,2,2) + 2 \cdot (2,1,0,2,2) + (2,0,2,1,2) + (2,1,2,0,2).$$

By construction, every term of the formal character $\mathrm{ch}(\mu \setminus \lambda)$ is a permutation of the p-content of $\mu \setminus \lambda$. The central character of the Specht module $S^{\mu\setminus\lambda}$, as an $\mathcal{H}_n^k$-module, then exactly coincides with the multiset $\mathrm{cont}_p(\mu \setminus \lambda)$. We observe the following fact.

Proposition 2.6. *Let* $\mu_1 \setminus \lambda_1$ *and* $\mu_2 \setminus \lambda_2$ *be skew-partitions in the same combinatorial block of* $\mathcal{C}_{l,m}^k$*. Then* $\mathrm{cont}_p(\mu_1 \setminus \lambda_1) = \mathrm{cont}_p(\mu_2 \setminus \lambda_2)$.

Proof. This follows from Theorem 2.3. □

Motivated by the above result, we say that a combinatorial block B has central character $\mathcal{C}(B)$ equal to the p-content of *any* skew-partition in B.

One of the main tools in the paper is the *decomposition matrix*. Let B be a combinatorial block and let D_B be the matrix whose rows are indexed by B and columns by those simple $\mathcal{H}_n^k$-modules appearing as composition factors of some Specht module in B. To detect the entries of the row of D_B labelled by the skew-partition $\mu \setminus \lambda$, write $\mathrm{ch}(\mu \setminus \lambda)$ as a $\mathbb{Z}$-linear combination of formal characters of simple $\mathcal{H}_n^k$-modules. The coefficients in this expression give the desired decomposition numbers. B is then a p-block of $\mathcal{C}_{l,m}^k$ if and only if the decomposition matrix D_B is a connected matrix (i.e. if it cannot be put in block-diagonal form by permuting rows and columns).

We end this section and this chapter with a crucial remark.

Remark. Given a skew-partition $\mu \setminus \lambda$, fill the boxes of the skew-diagram $\mu \setminus \lambda$ with their p-residues, and define the *p-shape* of $\mu \setminus \lambda$ to be the multiset of connected components of the resulting diagram. It is easy to see that if two skew-partitions in B have the same p-shape, then the corresponding Specht modules have the same formal character, and therefore the same composition factors (with multiplicity) as $\mathcal{H}_n^k$-modules. This means that for our purposes, we can largely shorten the number of rows of D_B relabelling them with the set $\mathcal{B}$ that comes out quotienting B by the equivalence relation that identifies skew-partitions with the same p-shape. We say that $\mathcal{B}$ is the *set of p-shapes* of B. We regard the elements of $\mathcal{B}$ as $\mathcal{H}_n^k$-modules and when talking about the formal character and the composition factors of a p-shape X we mean those of *any* Specht module in B whose p-shape is X. Also, saying that a p-shape belongs to a combinatorial block we mean that the combinatorial block contains a skew-partition with that p-shape.

When drawing p-shapes, we follow [EM2] by drawing the connected components in a diagonal line in arbitrary order.

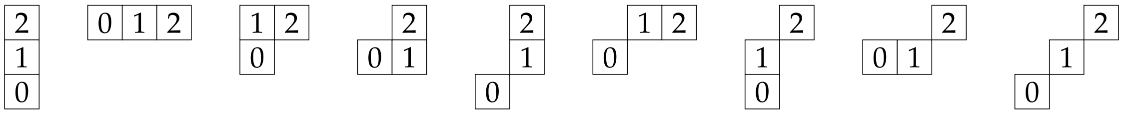
For example, take $p = 3$, and consider the skew-partitions $(4,1) \setminus (2)$ and $(3,2) \setminus (2)$:

Both skew-partitions have the same 3-shape and the corresponding Specht modules both have formal character $(2,0,2) + 2 \cdot (2,2,0)$.

3 Ribbon blocks and belt blocks

This chapter is devoted to establishing Conjecture 2.5 for a special family of combinatorial blocks of the centraliser algebra. Again let m, l be non-negative integers with $m \geqslant l$ and let $n = m - l$. Recall that $(R, \mathbb{F}, k)$ is a p-modular system where p denotes the (prime) characteristic of the residue field k. Let B be a combinatorial block of $\mathcal{C}_{l,m}^k$. The central character of B is a multiset of elements of $\mathbb{Z}/p\mathbb{Z}$. We will be concerned with the case where there are no repeated entries in the central character, so that the central character is a subset of $\mathbb{Z}/p\mathbb{Z}$. In this case clearly $n \leqslant p$ and we say that B is a *generalised ribbon block* if $n < p$ (so that the central character is a proper subset of $\mathbb{Z}/p\mathbb{Z}$), or a *belt block* if $n = p$. We will assume for the rest of Section 3 that B is either a generalised ribbon block or a belt block.

In the last section we explained that when working modulo p we can identify skew-partitions with the same formal character (since the rows of the decomposition matrix D_B labelled by them are identical), so from now on we will be concerned with just the set $\mathcal{B}$ of p-shapes of B. Firstly it is a good idea to give a visualisation of the elements of $\mathcal{B}$. If $\mu \setminus \lambda$ is a skew-partition in B, then its skew-diagram does not contain a 2×2-subdiagram (because otherwise it would have boxes with equal p-residue, contrarily to our assumption on B) and hence it is a disjoint union of rim-hooks of μ. If X is the p-shape of $\mu \setminus \lambda$, we refer to a *connected component* of X for any of these rim-hooks and we say that X is a *ribbon* if $\mu \setminus \lambda$ is connected, i.e. it is a single rim-hook of μ. For example, if $p > n = 3$, and B has central character $\{\bar{0}, \bar{1}, \bar{2}\}$, then the p-shapes in $\mathcal{B}$ are among the nine drawings below (the first four on the left being the possible ribbons).



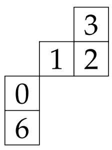
We close this long preface noting that, since there are no repetitions in $\mathcal{C}(B)$, if X is a p-shape in $\mathcal{B}$, there is no risk of confusion writing $\boxed{i}$ for the unique node labelled by i in X, for every $i \in \mathcal{C}(B)$.

3.1 Generalised ribbon blocks

In this section we fix a generalised ribbon block B, with $\mathcal{C} = \mathcal{C}(B) \subset \mathbb{Z}/p\mathbb{Z}$ its central character. Our aim is to prove Conjecture 2.5 for B.

Before starting we give another combinatorial perspective to the p-shapes in a generalised ribbon block. The reason why we do this will be clarified in the next section. We define an *arrow graph* to be a directed graph with vertex set $\mathcal{C}$ such that every arrow has the form $i \to i+1$ or $i+1 \to i$. Given a p-shape X, we define an associated arrow graph Γ_X by drawing an arrow $i \to i+1$ [resp. $i+1 \to i$] whenever $\boxed{i}$ lies on the left of [resp. below] $\boxed{i{+}1}$ in X, or no arrow between i and $i+1$ if these nodes lie in different connected components of X. It is clear that the set of all possible p-shapes is in bijection with the set of all possible arrow graphs. We define a p-shape to be *maximal* if its arrow graph is maximal, i.e. contains an arrow between i and $i+1$ whenever $i, i+1 \in \mathcal{C}$. Note that starting from an arrow graph we can recover the corresponding p-shape by mirroring the above procedure. Moreover, we can evaluate its formal character by taking the formal sum of all permutations $(a_1, \ldots, a_n)$ of $\mathcal{C}$ with the property that i appears before j for every arrow $i \to j$.

Example. Suppose $p = 7$ and $\mathcal{C} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{6}\}$. We give an example of a p-shape and the corresponding arrow graph (omitting bars).



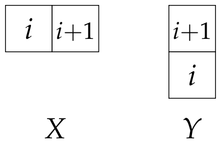
$$6 \longleftarrow 0 \qquad 1 \longrightarrow 2 \longleftarrow 3$$

The corresponding formal character is then

$$\begin{aligned} &(0,1,3,2,6) + (0,1,3,6,2) + (0,1,6,3,2) + (0,3,1,2,6) + (0,3,1,6,2) \\ &+ (0,3,6,1,2) + (0,6,1,3,2) + (0,6,3,1,2) + (1,0,3,2,6) + (1,0,3,6,2) \\ &+ (1,0,6,3,2) + (1,3,0,6,2) + (1,3,0,2,6) + (1,3,2,0,6) + (3,0,1,2,6) \\ &+ (3,0,1,6,2) + (3,0,6,1,2) + (3,1,0,6,2) + (3,1,0,2,6) + (3,1,2,0,6). \end{aligned}$$

Remark. Let $\mu \setminus \lambda$ be a skew-partition with p-shape X. Consider the abacus displays for μ and λ with the same number of beads. For each $i \in \mathcal{C}$, let x_i be the position of $\mathrm{Ab}(\mu)$ corresponding to the node $\boxed{i}$ (that is a node of the rim of μ by assumption). We look at the arrow graph of X to detect the content of the position x_i in $\mathrm{Ab}(\mu)$ and $\mathrm{Ab}(\lambda)$: x_i is a space in both $\mathrm{Ab}(\mu)$ and $\mathrm{Ab}(\lambda)$ if there is an arrow $i \to i+1$, is a bead in both $\mathrm{Ab}(\mu)$ and $\mathrm{Ab}(\lambda)$ if there is an arrow $i+1 \to i$, is a bead in $\mathrm{Ab}(\mu)$ and a space in $\mathrm{Ab}(\lambda)$ if there is no arrow between i and $i+1$.

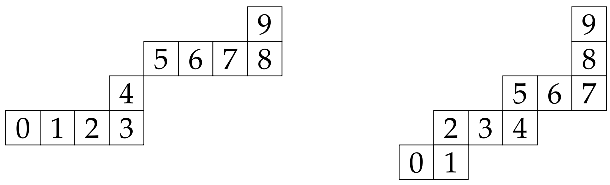
Now we introduce a fundamental tool in our treatment. Let X, Y be two p-shapes and let $\mathrm{D}(X,Y)$ the subset of $\mathcal{C}$ containing those integers i such that $i+1 \in \mathcal{C}$ and $\boxed{i}$ lies on the left of $\boxed{i+1}$ in X and below $\boxed{i+1}$ in Y (see the picture below), or vice versa.



Define the *distance* $\mathrm{d}(X,Y)$ *between* X *and* Y as the cardinality of $\mathrm{D}(X,Y)$. This concept can be rephrased also in terms of arrow graphs: take Γ_X and Γ_Y and denote by $\Gamma_X \cup \Gamma_Y$ the picture obtained by overlapping these two arrow graphs. Then $\mathrm{d}(X,Y)$ coincides with the number of edges where two differently oriented arrows overlap in $\Gamma_X \cup \Gamma_Y$.

Note that d is *not* a metric: $\mathrm{d}(X,Y)$ can be zero even when $X \neq Y$.

Example. Suppose $p \geqslant 11$ and $\mathcal{C} = \{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6},\bar{7},\bar{8},\bar{9}\}$ and consider the two following p-shapes:



Then $\mathrm{D}(X,Y) = \{\bar{1},\bar{3},\bar{7}\}$ and $\mathrm{d}(X,Y) = 3$. The union of the two arrow graphs $\Gamma_X \cup \Gamma_Y$ is given as follows, with Γ_X in red and Γ_Y in blue.

We start our analysis of generalised ribbon blocks by observing that the formal characters of two different maximal p-shapes have no summands in common.

Lemma 3.1. *Consider* $\mathbf{a} = (a_1, \ldots, a_n)$ *where* $a_1, \ldots, a_n$ *are the elements of* $\mathcal{C}$ *in some order. Then* $\mathbf{a}$ *appears in the formal character of exactly one maximal* p*-shape.*

Proof. We construct the unique maximal p-shape X such that $\mathbf{a}$ is a term of the formal character $\mathrm{ch}(X)$. Start drawing $\boxed{i}$, for some $i \in \mathcal{C}$ such that $i-1 \notin \mathcal{C}$. Then consider the order of i and $i+1$ in $\mathbf{a}$: if $i+1$ occurs before [resp. after] i, then draw $\boxed{i+1}$ just above [resp. to the right of] the node $\boxed{i}$. Now do the same for $i+1$ and $i+2$, and continue until an element $k \in \mathcal{C}$ is reached for which $k+1 \notin \mathcal{C}$. This determines one component of X. Now repeat with the other elements $i \in \mathcal{C}$ for which $i - i \notin \mathcal{C}$ to obtain the other components.

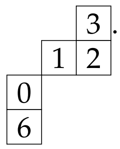
By construction, $\mathbf{a}$ is a summand of $\mathrm{ch}(X)$. We now see that X is unique. If Y is different maximal p-shape, then the set $\mathrm{D}(X,Y)$ is non-empty, because Γ_X and Γ_Y both have arrows joining i and $i+1$ whenever $i, i+1 \in \mathcal{C}$. Take $i \in \mathrm{D}(X,Y)$, and suppose that i appears before $i+1$ in $\mathbf{a}$ (the other case being similar). This means that we have an arrow $i \to i+1$ in Γ_X, and therefore an arrow $i \leftarrow i+1$ in Γ_Y, so that $\mathbf{a}$ does not appear in $\mathrm{ch}(Y)$. $\square$

Corollary 3.2. *Different maximal p-shapes have no composition factors in common.*

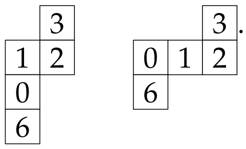
The next three results will finally give us a labelling set for the columns of the decomposition matrix of a ribbon block. In Lemma 3.3, if X is a p-shape, a standard X-tableau indicates a standard tableau of *any* skew-partition with p-shape X. We begin with a definition.

Definition. Let X be a p-shape. A *refinement* of X is a maximal p-shape Y that can be obtained joining the connected components of X together.

Example. Suppose $p = 7$, and let X be the p-shape from an earlier example:



Then X has exactly two refinements, as follows.



Lemma 3.3. *Let X be a p-shape. The formal character of X is the sum of the formal characters of the refinements of X.*

Proof. This follows from a correspondence between standard tableaux: given a refinement Y of X, each standard Y-tableau gives rise to a standard X-tableau (just by breaking into individual connected components). Conversely, each standard X-tableau arises in this way for a unique refinement Y: for each i such that the nodes $\boxed{i}$ and $\boxed{i+1}$ lie in different connected components of X, the relative order of the entries $i, i+1$ tells us how they need to be joined together if the resulting tableau is to be standard. □

Corollary 3.4. *If X is a p-shape, then the composition factors of X are the composition factors of the refinements of X.*

Proof. This follows from the fact that the formal characters of simple modules of the degenerate affine Hecke algebra are linearly independent [K, Theorem 5.3.1]. □

Theorem 3.5. *Let X and Y be two p-shapes. The following are equivalent:*

1. *X and Y have a composition factor in common;*
2. *The formal characters of X and Y have a term in common;*
3. $\mathrm{d}(X, Y) = 0$;
4. *There is a maximal p-shape which is a refinement of both X and Y.*

Proof.

(1⇒ 2) This follows from the fact that the formal character of a $\mathcal{H}_n^k$-module is the sum of the formal characters of its composition factors.

(2 ⇒ 3) If there exists an integer $i \in D(X,Y)$, then $i+1$ occurs before i in every summand of $\mathrm{ch}(X)$ and after i in every summand of $\mathrm{ch}(Y)$, or vice versa. So $\mathrm{ch}(X)$ and $\mathrm{ch}(Y)$ have no terms in common.

(3 ⇒ 4) Consider the graph $\Gamma_X \cup \Gamma_Y$. Since $\mathrm{d}(X,Y) = 0$, this is an admissible arrow graph. Then every arrow graph obtained by filling the vacant edges in $\Gamma_X \cup \Gamma_Y$ corresponds to a refinement of both X and Y.

(4 ⇒ 1) This follows from Corollary 3.4. □

Theorem 3.5 tells us that the set of maximal p-shapes serves as a labelling set for an approximation to the decomposition matrix of B: by Corollary 3.4 any simple module S occurs as a composition factor of a unique maximal p-shape Y (say with multiplicity c). So by Corollary 3.4, if X is any p-shape then S occurs as a composition factor of X with multiplicity c if Y is a refinement of X, and 0 otherwise.

So we can define a matrix D_B with rows indexed by the set of p-shapes, and columns indexed by the set of maximal p-shapes, with entry $d_{XY} = 1$ if Y is a refinement of X and 0 otherwise. Then the actual decomposition matrix of B is obtained from D_B by possibly duplicating rows (because a given p-shape may correspond to more than one Specht module) and possibly duplicating and rescaling columns (because a Specht module labelled by a maximal p-shape might be reducible). As a consequence, to show that B is a single p-block of $\mathcal{C}^k_{l,m}$, we just need to show that D_B is a connected matrix, and our problem becomes purely combinatorial.

(In fact we suspect that every Specht module labelled by a maximal p-shape is simple, but we could not find a proof of this statement, and we do not need it.)

Example. Suppose $p = 5$, $l = 5$ and $m = 8$, and let B be the block consisting of Specht modules $\mathrm{S}^{\mu\setminus\lambda}$ where $\mathrm{core}_5(\mu) = (2,1)$ and $\mathrm{core}_5(\lambda) = \varnothing$. Then $\mathcal{C} = \{\bar{0}, \bar{1}, \bar{4}\}$, and the p-shapes in B are as follows:

$$X_1 = \begin{array}{|c|c|c|}\hline 4 & 0 & 1 \\ \hline\end{array}, \quad X_2 = \begin{array}{cccc} \cline{3-4} & & \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} \\ \cline{1-1}\cline{3-4} \multicolumn{1}{|c|}{4} & & & \\ \cline{1-1} \end{array}, \quad X_3 = \begin{array}{cc} \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} \\ \hline \multicolumn{1}{|c|}{4} & \\ \cline{1-1} \end{array}, \quad X_4 = \begin{array}{ccc} \cline{3-3} & & \multicolumn{1}{|c|}{1} \\ \cline{1-1}\cline{3-3} \multicolumn{1}{|c|}{0} & & \\ \cline{1-1} \multicolumn{1}{|c|}{4} & & \\ \cline{1-1} \end{array}, \quad X_5 = \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline 4 \\ \hline \end{array}.$$

(For example, the p-shape X_2 arises from the skew-partitions $(7,1) \setminus (5)$ and $(5,3) \setminus (4,1)$.)

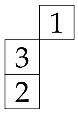
The matrix D_B is then given as follows.

	X_1	X_3	X_5
X_1	1	0	0
X_2	1	1	0
X_3	0	1	0
X_4	0	1	1
X_5	0	0	1

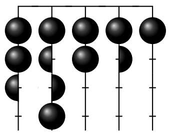
Definition. Let X, Y be two p-shapes in $\mathcal{B}$. We say that X and Y are *p-linked* if there exists a series of p-shapes $X = X_1, X_2, \ldots, X_{t-1}, X_t = Y$ in $\mathcal{B}$ such that $\mathrm{d}(X_j, X_{j+1}) = 0$ for every $1 \leqslant j \leqslant t-1$.

In what follows we will mainly work with abacus configurations. We remark that if $\mu \setminus \lambda$ is a skew-partition in B, a connected component of its p-shape corresponds to a rim-hook of μ. If $h, f \in \mathbb{Z}$ are the positions corresponding to the hand node and foot node of the rim-hook, respectively, then h is a bead and $f-1$ is a space in $\mathrm{Ab}(\mu)$, and vice versa in $\mathrm{Ab}(\lambda)$. These pairs of positions are the only in which the contents in $\mathrm{Ab}(\mu)$ and $\mathrm{Ab}(\lambda)$ differ. We represent $\mathrm{Ab}(\mu)$ and $\mathrm{Ab}(\lambda)$ simultaneously in an abacus configuration $\mathrm{Ab}(\mu \setminus \lambda)$, putting a *right half-bead* ◗ at position h and a *left half-bead* ◖ at position $f-1$. (So the right half-beads are precisely the beads occurring in the abacus configuration of μ but not of λ, and vice versa for the left half-beads.)

Example. Take $p = 5$ and $\mu \setminus \lambda = (7,3,1^2) \setminus (7,2)$, giving the following p-shape.



This p-shape has two connected components, so that $\mu \setminus \lambda$ comprises two rim-hooks. We draw the abacus configuration $\mathrm{Ab}(\mu \setminus \lambda)$ with half-beads.



Our aim is to show that any two p-shapes in B are p-linked. Consider a pair of p-shapes $X, Y \in \mathcal{B}$ such that $\mathrm{d}(X,Y) > 0$. Let $\mu_X \setminus \lambda_X$ and $\mu_Y \setminus \lambda_Y$ be skew-partitions in B having p-shape X and Y, respectively. We write $\mathrm{Ab}(X)$ and $\mathrm{Ab}(Y)$ for the abacus displays $\mathrm{Ab}(\mu_X \setminus \lambda_X)$ and $\mathrm{Ab}(\mu_Y \setminus \lambda_Y)$, respectively. Consider $\mathcal{C}$ the central character of the generalised ribbon block $\mathcal{B}$; recall that this is a (disjoint) union of intervals of $\mathbb{Z}/p\mathbb{Z}$. For every $j \in \mathcal{C}$, consider the (unique) node $\boxed{j}$ in $\mu_X \setminus \lambda_X$ and $\mu_Y \setminus \lambda_Y$ and denote by x_j and y_j the positions corresponding to these nodes in the abacus displays $\mathrm{Ab}(X)$ and $\mathrm{Ab}(Y)$, respectively. The choice of the charge ensures that positions x_j and y_j are both in the j-th runner.

Note that in $\mathrm{Ab}(X)$, position x_j is

- ◇ a bead, if node $\boxed{j{+}1}$ lies immediately above $\boxed{j}$ in $\mu_X \setminus \lambda_X$;
- ◇ a space, if $\boxed{j{+}1}$ lies immediately to the right of $\boxed{j}$ in $\mu_X \setminus \lambda_X$;
- ◇ a right half-bead, if $\boxed{j}$ and $\boxed{j{+}1}$ are in different components of $\mu_X \setminus \lambda_X$.

In particular, x_j cannot be a left half-bead in $\mathrm{Ab}(X)$. A similar statement applies to y_j, so that $j \in \mathrm{D}(X,Y)$ if and only if x_j is a bead in $\mathrm{Ab}(X)$ and y_j is a space in $\mathrm{Ab}(Y)$, or vice versa.

Fix $i \in \mathrm{D}(X,Y)$, and assume without loss of generality that the node $\boxed{i}$ lies on the left of $\boxed{i{+}1}$ in X and below it in Y. Then x_i is a space in $\mathrm{Ab}(X)$ and y_i is a bead in $\mathrm{Ab}(Y)$. We let r be maximal such that $i, i+1, \dots, i+r \in \mathcal{C}$ (where we add modulo p); note that there must be such an r because by assumption $\mathcal{C} \neq \mathbb{Z}/p\mathbb{Z}$. Observe that if $j \in \{i, \dots, i+r\}$ and node $\boxed{j}$ lies in the same connected component of $\mu_X \setminus \lambda_X$ as $\boxed{i}$, then $x_i, x_{i+1}, \dots, x_j$ are consecutive integers in increasing order, so that $x_j = x_i + j - i$; a similar statement holds for y_j.

We prove a series of useful facts.

Lemma 3.6. *Suppose $k \in \{i+1, \dots, i+r\}$, and suppose that for every $j \in \{i+1, \dots, k-1\} \setminus \mathrm{D}(X,Y)$:*

- ◇ *x_j is a space in $\mathrm{Ab}(X)$ or μ_X has no addable $(j-i)$-hook at runner i, and*
- ◇ *y_j is a bead in $\mathrm{Ab}(Y)$ or λ_Y has no removable $(j-i)$-hook at runner j.*

Then position $y_j + i - j$ is a bead in $\mathrm{Ab}(Y)$ and position $x_j + i - j$ is a space in $\mathrm{Ab}(X)$, for every $j \in \{i+1, \dots, k\}$.

Proof. We only prove the first part of the statement; the second follows by a perfectly dual argument.

Consider the node $\boxed{j}$ in Y. Let $t \in \mathbb{Z}$ be such that $y_j = y_i - i + j + tp$. If $t = 0$, then position $y_j + i - j$ is a bead since y_i is by assumption. If $t \neq 0$, then (from the observation just before the lemma) $\boxed{i}$ and $\boxed{j}$ lie in different connected components of Y. Let $l \in \{i+2, \dots, j\}$ be such that $\boxed{l}$ is the foot node of the connected component of Y containing $\boxed{j}$. This implies that $y_l = y_j + l - j$, so that $y_j + i - j = y_l + i - l$. Moreover, $\boxed{l{-}1}$ is the hand node of a connected component of Y, so that y_{l-1} is a right half-bead in $\mathrm{Ab}(Y)$ and $l - 1 \notin \mathrm{D}(X, Y)$. The hypotheses of the lemma then imply that λ_Y has no removable $(l-1-i)$-hooks at runner $l-1$, so position $y_l + i - l$ must be a bead in $\mathrm{Ab}(Y)$ because $y_l - 1$ is a left half-bead. □

Lemma 3.7. *Let k be as in Lemma 3.6 and let $j \in \{i+1, \dots, k-1\} \setminus \mathrm{D}(X, Y)$.*

1. *If x_j is a space in $\mathrm{Ab}(X)$, then μ_X has an addable $(j-i)$-hook at runner i;*
2. *If y_j is a bead in $\mathrm{Ab}(Y)$, then λ_Y has a removable $(j-i)$-hook at runner j.*

Proof. We only show (1); a dual argument then applies to (2). The hypothesis that $j \notin \mathrm{D}(X, Y)$ implies that y_j is a space or a right half-bead in $\mathrm{Ab}(Y)$. On the one hand, the hypothesis on k means that λ_Y has no removable $(j-i)$-hooks at runner j. On the other hand, by Lemma 3.6, position $y_j + i - j$ is a bead in $\mathrm{Ab}(Y)$ meaning that λ_Y has an addable $(j-i)$-hook at runner i. By Proposition 2.2, it follows that λ_X has an addable $(j-i)$-hook at runner i. This hook is not at position x_i because this is a space by assumption, hence it is an addable $(j-i)$-hook at runner i of μ_X as well. □

We now give the main result of this section.

Theorem 3.8. *Let $X, Y \in \mathcal{B}$ such that $\mathrm{d}(X, Y) > 0$. Then there is $Z \in \mathcal{B}$ such that $\mathrm{d}(X, Z) < \mathrm{d}(X, Y)$ and $\mathrm{d}(Y, Z) < \mathrm{d}(X, Y)$.*

Proof. Let all the setting be as above and let $k \in \{i+1, \dots, i+r\}$ be maximal subject to the hypotheses of Lemma 3.6. Then we claim that $k \notin \mathrm{D}(X, Y)$ and either

- ◇ x_k is a bead or a right half-bead in $\mathrm{Ab}(\mu_X \setminus \lambda_X)$ and μ_X has an addable $(k-i)$-hook at runner i

or

- ◇ y_k is a space or a right half-bead in $\mathrm{Ab}(\mu_Y \setminus \lambda_Y)$ and λ_Y has a removable $(k-i)$-hook at runner k.

If $k < i + r$ then this follows from the assumption that k is maximal subject to the hypotheses of Lemma 3.6, so suppose instead that $k = i + r$. The node $\boxed{i{+}r}$ is the hand node of a connected component of both X and Y because $i + r + 1 \notin \mathcal{C}$. Then positions x_{i+r} and y_{i+r} are right half-beads in $\mathrm{Ab}(X)$ and $\mathrm{Ab}(Y)$, respectively, and in particular $i + r \notin \mathrm{D}(X, Y)$. Moreover, μ_X has a removable r-hook at runner $i + r$ since position x_{i+r} is a right half-bead and $x_{i+r} - r$ is a space by Lemma 3.6. By Proposition 2.2 it follows that μ_X has an addable r-hook at runner i or μ_Y has a removable r-hook at runner $i + r$. In the latter case, the removable hook does not occur at position y_{i+r} because position $y_{i+r} - r$ is a bead by Lemma 3.6, then it is a removable r-hook at runner $i + r$ of λ_Y as well. Hence, $i + r$ satisfies at least one of the above conditions.

We show that the required p-shape exists by modifying $\mathrm{Ab}(X)$ or $\mathrm{Ab}(Y)$, according to which of the above conditions k satisfies. We show how to proceed in the first case, with the other case following by analogous and dual arguments. So we assume that x_k is a bead or a right half-bead in $\mathrm{Ab}(X)$ and μ_X has an addable $(k-i)$-hook at runner i.

The strategy is to operate on the abacus display $\mathrm{Ab}(X)$ by performing a series of bead moves as shown in Figure 1. Note that the rows of the abacus in which the given positions appear are illustrative only; these rows could easily appear in a different order in practice. The abacus is arranged such that i labels the leftmost runner and k the rightmost one.

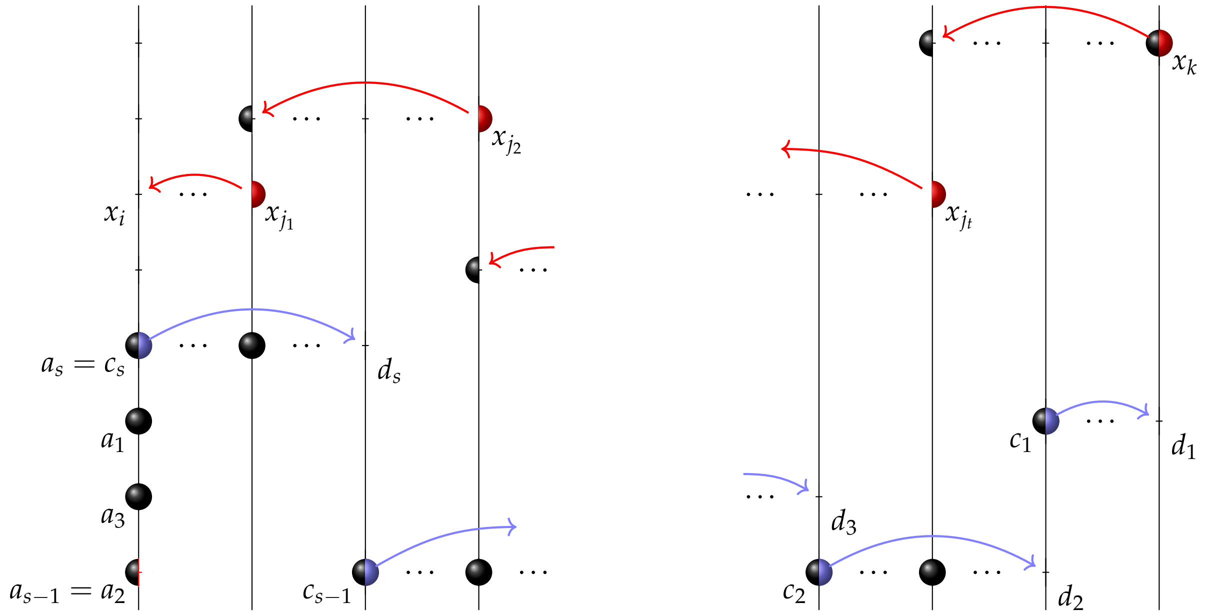


Figure 1

In terms of the Young diagram, what we are going to do is to remove all the nodes $\boxed{i{+}1}, \ldots, \boxed{k}$ from μ_X, and to put these nodes back in new positions.

We work on $\mathrm{Ab}(X)$. By assumption there exists $a_1 \equiv i \pmod p$ such that μ_X has an addable $(k-i)$-hook at a bead position a_1. Let $1 \leqslant k_1 \leqslant k-i$ be the least integer such that $k-k_1 \notin \mathrm{D}(X,Y)$, position x_{k-k_1} is a space and $a_1+k-i-k_1$ is a bead. By Lemma 3.7, if $k_1 < k-i$, then there exists an integer $a_2 \equiv i \pmod p$ such that μ_X has an addable $(k-i-k_1)$-hook at position a_2. Then we repeat the above process, letting $1 \leqslant k_2 \leqslant k-i-k_1$ be minimal such that $k-k_1-k_2 \notin \mathrm{D}(X,Y)$, $x_{k-k_1-k_2}$ is a space and $a_2+k-i-k_1-k_2$ is a bead. It is clear that this procedure always terminates at a step s such that $i = k - \sum_{1\leqslant u\leqslant s} k_u$. Summarising, we have found integers $a_1, \ldots, a_s \equiv i \pmod p$ and $k_1, \ldots, k_s$ such that, for every $1 \leqslant v \leqslant s$, in $\mathrm{Ab}(X)$, there is a bead at position $c_v := a_v + k - i - \sum_{1\leqslant u\leqslant v} k_u$ and a space at position $d_v := a_v + k - i - \sum_{1\leqslant u\leqslant v-1} k_u$. Let $i+1 \leqslant j_1 < \cdots < j_t \leqslant k$ be those residues such that $\boxed{j_1}, \ldots, \boxed{j_t}$ are hand nodes of connected components of X.

We now construct a new abacus configuration by modifying $\mathrm{Ab}(X)$. Our aim is to modify μ_X while keeping λ_X the same, so each move only moves the right half of a bead. We perform the following moves (where the colours correspond to the coloured arrows in Figure 1):

- move the right half-bead (or the right half of the bead) at x_k to the left half-bead at $x_{j_t+1} - 1$,
- for every $2 \leqslant v \leqslant t$, move the right half-bead at x_{j_v} to the left half-bead at $x_{j_{v-1}+1} - 1$,
- move the right half-bead at x_{j_1} to the space at x_i,
- for every $1 \leqslant v \leqslant s$, move the right half of the bead at c_v to the space at d_v.

If μ_Z is the partition of m defined by this new abacus configuration, then by construction, $\mu_Z \setminus \lambda_X$ is a skew-partition in the combinatorial block B (we have replaced each removed node with a node with the same residue). Let Z be the p-shape of this skew-partition. We will show that in most cases, Z is the p-shape desired in the theorem. For every $j \in \mathcal{C}$, let z_j be the position of the node $\boxed{j}$ in

$\mathrm{Ab}(\mu_Z \setminus \lambda_X)$. Note that for $j \notin \{i+1,\dots,k\}$, then $z_j = x_j$. Also, $\{z_{i+1},\dots,z_k\}$ is the union of intervals $[c_1,d_1],\dots,[c_s-1,d_s]$, e.g. $z_k = d_1$.

We firstly note that $i \notin \mathrm{D}(X,Z) \cup \mathrm{D}(Y,Z)$ because z_i is a right half-bead in $\mathrm{Ab}(\mu_Z \setminus \lambda_X)$ by construction. If $j \in \{i+1,\dots,k-1\}$ with $j \notin \mathrm{D}(X,Y)$, then $j \notin \mathrm{D}(X,Z)$. Indeed, if x_j is a bead $\mathrm{Ab}(X)$, then by the definition of k, μ_X has no addable $(j-i)$-hooks at runner i, so that z_j is a bead in $\mathrm{Ab}(\mu_Z \setminus \lambda_X)$; if x_j is a space in $\mathrm{Ab}(X)$, then the choice of $k_1,\dots,k_s$ means that z_j is a space or a right half-bead in $\mathrm{Ab}(\mu_Z \setminus \lambda_X)$. Similarly it can be deduced that $j \notin \mathrm{D}(Y,Z)$.

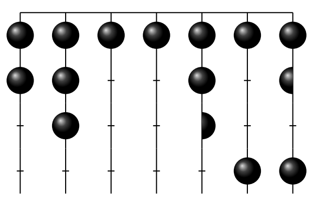
Furthermore, $k \notin \mathrm{D}(X,Z)$ because x_k and z_k are beads in the respective abacus configurations. It follows that $\mathrm{d}(X,Z) < \mathrm{d}(X,Y)$. Regarding to Y, it can happen that $k \in D(Y,Z)$: this situation occurs if and only if y_k is a space in $\mathrm{Ab}(Y)$, x_k is right half-bead in $\mathrm{Ab}(X)$ and $a_1 = x_{k+1} - k + i - 1$. This condition can be rephrased in terms of Young diagrams, saying that $\boxed{k}$ is on the left of $\boxed{k{+}1}$ in Y, and during the process from μ_X to μ_Z, the node $\boxed{k}$ – that is the hand node of a connected component of X – has joined below the node $\boxed{k{+}1}$. If this is note the case, then $\mathrm{d}(Y,Z) < \mathrm{d}(X,Y)$ and Z is the required p-shape.

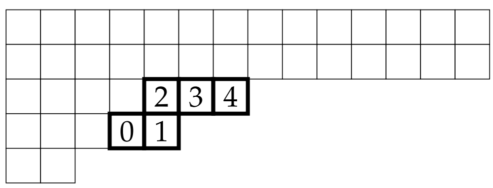
Assume for the rest of the proof that position y_k is a space in $\mathrm{Ab}(Y)$, position x_k is a right half-bead in $\mathrm{Ab}(X)$ and $a_1 = x_{k+1} - k + i - 1$. We can also assume that the only addable $(k-i)$-hook at runner i of μ_X is at position a_1. Indeed, if μ_X had another such an addable hook, then the above process of modification of μ_X could be initiated from this addable hook, resulting a different partition $\mu_Z \vdash m$ that works for our purposes. By Lemma 3.6, position $x_k + i - k$ is a space in $\mathrm{Ab}(X)$, therefore μ_X has a removable $(k-i)$-hook at runner k (at position x_k). Again by Lemma 3.6, position $y_k + i - k$ is a bead in $\mathrm{Ab}(Y)$, therefore μ_Y has an addable $(k-i)$-hook at runner i (at position $y_k + i - k$). The fact that μ_X has a removable $(k-i)$-hook at runner k and exactly one addable $(k-i)$-hook at runner i, together with Proposition 2.2, implies that μ_Y has a removable $(k-i)$-hook at runner k. This is not at position y_k (because y_k is a space), then it is a removable hook of λ_Y as well. Now we can perform a dual version of the bead moves above operating on $\mathrm{Ab}(Y)$ and constructing another skew-partition in B. In terms of young diagrams, we add all the nodes $\boxed{i{+}1},\dots,\boxed{k}$ to λ_Y and we remove these nodes from other positions. Since the construction is perfectly dual, we only show an illustative drawing (Figure 2) that gives an idea of the bead moves. (Again, the labels of the shown runners go from i to k.)

By construction, we obtain a partition $\lambda_W \vdash l$ such that $\mu_Y \setminus \lambda_W \in B$. We let W be the p-shape of $\mu_Y \setminus \lambda_W$. Similar to the above with Z, it can be shown that $i \notin \mathrm{D}(X,W) \cup \mathrm{D}(Y,W)$ and for every $j \in \{i+1,\dots,k-1\}$ with $j \notin \mathrm{D}(X,Y)$, then $j \notin \mathrm{D}(X,W) \cup \mathrm{D}(Y,W)$. Also, $k \notin \mathrm{D}(Y,W)$ and hence $\mathrm{d}(Y,W) < \mathrm{d}(X,Y)$. Finally, by the assumption that x_k is a right half-bead in $\mathrm{Ab}(X)$, it follows that $k \notin \mathrm{D}(X,W)$. This implies that $\mathrm{d}(X,W) < \mathrm{d}(X,Y)$ and therefore that W is the required p-shape. □

Example. Let $m = 42$, $l = 37$ and $p = 7$. The skew-partitions $\mu_X \setminus \lambda_X = (14^2,7,5,2) \setminus (14^2,4,3,2)$ and $\mu_Y \setminus \lambda_Y = (9,8,7,5^2,4^2) \setminus (9,8,4^5)$ have size $n = m - l = 5$ and belong to the same combinatorial block of $\mathcal{C}^k_{37,42}$ having 7-content $\mathcal{C} = \{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$. Indeed, $\mathrm{core}_7(\mu_X) = \mathrm{core}_7(\mu_Y) = (5,3,2^3)$ and $\mathrm{core}_7(\lambda_X) = \mathrm{core}_7(\lambda_Y) = (7,3,2^3)$. Let X and Y be the p-shapes of $\mu_X \setminus \lambda_X$ and $\mu_Y \setminus \lambda_Y$, respectively. The Young diagrams and the abacus configurations (with charge 14) are shown below.

$\mu_X \setminus \lambda_X$





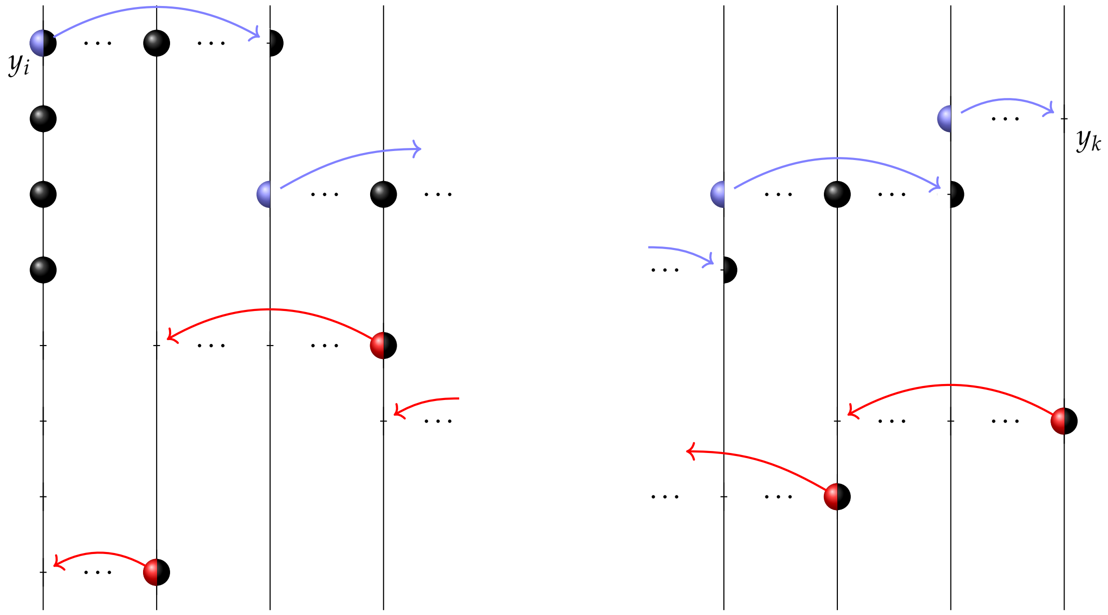
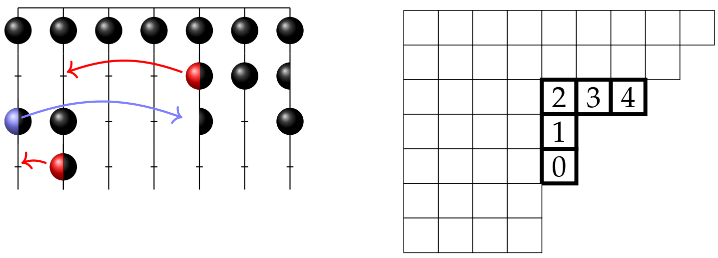


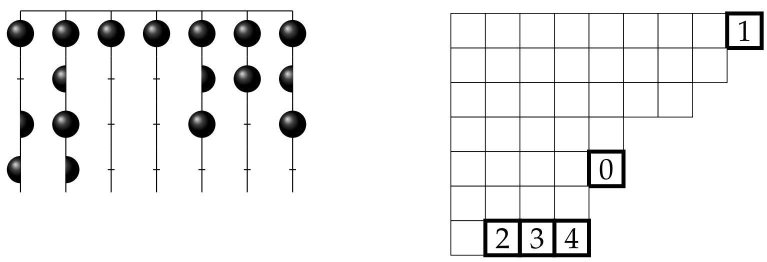
Figure 2

$\mu_Y \setminus \lambda_Y$



We see that $\mathrm{D}(X,Y) = \{\bar{0}\}$, so that $\mathrm{d}(X,Y) = 1$. In particular, position $x_0 = y_0 = 14$ is a space in $\mathrm{Ab}(X)$ and a bead in $\mathrm{Ab}(Y)$. We have that $k = 4$ and the construction in Theorem 3.8 comprises the bead moves shown in the abacus configuration $\mathrm{Ab}(Y)$ above (we are modifying λ_Y but not μ_Y, so we move the left half of each bead, as indicated by the colours). We find a partition $\lambda_Z = (8^2, 7, 5, 4^2, 1) \vdash 37$ such that $\mu_Y \setminus \lambda_Z$ has 7-shape Z such that $\mathrm{d}(X,Z) = \mathrm{d}(Y,Z) = 0$ as can be easily deduced from the Young diagrams.

$\mu_Y \setminus \lambda_Z$



We immediately deduce the following corollary, using induction on $\mathrm{d}(X,Y)$.

Corollary 3.9. *Suppose $X, Y \in \mathcal{B}$. Then X and Y are p-linked.*

We deduce as a corollary the main result of this section.

Corollary 3.10. *Conjecture 2.5 holds for generalised ribbon blocks.*

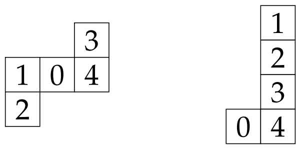
Proof. Let B be a generalised ribbon block as above. It follows from Corollary 3.9 and the implication (3⇒4) in Theorem 3.5 that the matrix D_B is connected. As explained in the discussion following Theorem 3.5, this implies that the decomposition matrix of B is connected, so that B is a single p-block of $\mathcal{C}^k_{l,m}$. □

3.2 Belt blocks

For the rest of this section suppose that $p = n$. Our aim is to establish Conjecture 2.5 for a *belt block* B, i.e. a combinatorial block of $\mathcal{C}^k_{l,m}$ whose central character equals $\mathbb{Z}/p\mathbb{Z} = \{\bar{0}, \dots, \overline{p-1}\}$.

As in Section 3.1, we first replace B (again regarded as a subset of $\mathcal{P}_{l,m}$) with its set $\mathcal{B}$ of p-shapes. We keep going with the same notation established in the preface of the chapter talking about the connected components of a p-shape. Now the maximal p-shapes are ribbons. Unfortunately in this situation we cannot mimic at all the arguments we have used for ribbon blocks. Indeed, even if it is still true that the formal character of a disconnected p-shape is the sum of the formal characters of its refinements, a basic step of the reasoning in Section 3.1, namely Lemma 3.1, is no longer true in this setting. This means that, in general, different ribbons may share composition factors.

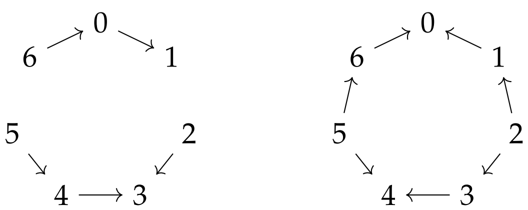
Example. Let $p = 5$. The formal characters of the following two ribbons share two summands: $(1,2,0,3,4)$ and $(1,2,3,0,4)$.



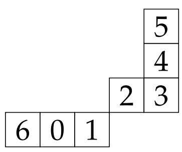
To fix this inconvenience, we need to introduce a functional 'refinement' of the set of ribbons. To do this, we begin with the introduction of a graph analogous to that in Section 3.1. We define a *(circular) arrow graph* to be a directed graph on the set $\mathbb{Z}/p\mathbb{Z}$ such that every arrow has the form $i \to i+1$ or $i+1 \to i$ for some i and the graph is not a directed cycle. An arrow graph is a *belt* if for every i there is either an arrow $i \to i+1$ or an arrow $i+1 \to i$.

We will draw arrow graphs with $\bar{0}, \bar{1}, \dots, \overline{p-1}$ in clockwise order, omitting bars. So we may say that the arrows of an arrow graph are all oriented clockwise to mean that they all have the form $i \to i+1$.

Example. Let $p = 7$. The following are two arrow graphs, the one on the right being a belt.



We observe that every p-shape X in a belt block is uniquely associated to an arrow graph Γ_X through the same procedure explained in Section 3.1. For example, the arrow graph on the left of the above example corresponds to the p-shape below.



Note that in this case the procedure does not give rise to a bijection between all possible p-shapes and all possible arrow graphs. In particular, no p-shape corresponds to a belt. If Γ is an arrow graph, we define its formal character $\mathrm{ch}(\Gamma)$ to be the formal sum of all permutations $\mathbf{b} = (b_0, \ldots, b_{p-1})$ of $\mathbb{Z}/p\mathbb{Z}$ with the property that i appears before [resp. after] $i+1$ for every arrow $i \to i+1$ [resp. $i+1 \to i$]. Note that if X is a p-shape, then $\mathrm{ch}(\Gamma_X) = \mathrm{ch}(X)$.

Our next task is to build for every belt Γ a $\mathcal{H}_p^k$-module whose formal character is $\mathrm{ch}(\Gamma)$. Our construction is inspired by Young's orthogonal form, and as far as we can tell it is new. We take a vector space M_Γ with basis $\{e_\mathbf{b}\}$ labelled by the different summands of $\mathrm{ch}(\Gamma)$. If $\mathbf{b} = (b_0, \ldots, b_{p-1})$ is a summand of $\mathrm{ch}(\Gamma)$, then for each $k \in \{1, \ldots, p-1\}$ we define $\sigma_k \mathbf{b} = (b_0, \ldots, b_{k+1}, b_k, \ldots, b_{p-1})$. We define an action of $\mathcal{H}_p^k$ on M_Γ as follows. Fix a summand $\mathbf{a}$ of $\mathrm{ch}(\Gamma)$. We set

$$z_k \, \mathrm{e}_\mathbf{b} = b_k \, \mathrm{e}_\mathbf{b}$$

and

$$s_k \, \mathrm{e}_\mathbf{b} = \begin{cases} \frac{1}{b_{k+1}-b_k} \, \mathrm{e}_\mathbf{b} + \mathrm{e}_{\sigma_k \mathbf{b}} & \text{if } b_k \text{ appears } \textit{after } b_{k+1} \text{ in } \mathbf{a}, \\ \frac{1}{b_{k+1}-b_k} \, \mathrm{e}_\mathbf{b} + \left(1 - \frac{1}{(b_{k+1}-b_k)^2}\right) \mathrm{e}_{\sigma_k \mathbf{b}} & \text{if } b_k \text{ appears } \textit{before } b_{k+1} \text{ in } \mathbf{a}. \end{cases}$$

Theorem 3.11. *With the above rule M_Γ is a $\mathcal{H}_p^k$-module with formal character* $\mathrm{ch}(\Gamma)$.

Proof. We just need to prove the first statement; the second part is then obvious from the definition of the formal character of a $\mathcal{H}_p^k$-module.

We prove that M_Γ is an $\mathcal{H}_p^k$-module by showing that the above defined rules respect the defining relations of $\mathcal{H}_p^k$ shown in Section 2.6. Relations (1) and (3) are trivially satisfied and need no further investigations. Therefore we only consider the *braid* relations in (2) and the relations between the polynomial and the Coxeter generators in (4).

We start from the braid relations. These divide into three types depending on the integers $m_{i,j}$ such that $(s_i s_j)^{m_{i,j}} = 1$. For each relation several cases arise depending on the positions of the involved entries of $\mathbf{b}$ in the fixed summand $\mathbf{a}$ of $\mathrm{ch}(\Gamma)$. We only analyse one case for each relation, the others being similar.

- ◇ $s_k^2 = 1$. Suppose that b_k appears before b_{k+1} in $\mathbf{a}$. We have that

$$\begin{aligned} s_k \mathrm{e}_\mathbf{b} &= \frac{1}{b_{k+1} - b_k} \mathrm{e}_\mathbf{b} + \left(1 - \frac{1}{(b_{k+1} - b_k)^2}\right) \mathrm{e}_{\sigma_k \mathbf{b}} \quad \Rightarrow \\ s_k^2 \mathrm{e}_\mathbf{b} &= \frac{1}{(b_{k+1} - b_k)^2} \mathrm{e}_\mathbf{b} + \frac{1}{b_{k+1} - b_k} \left(1 - \frac{1}{(b_{k+1} - b_k)^2}\right) \mathrm{e}_{\sigma_k \mathbf{b}} \\ &\quad - \frac{1}{b_{k+1} - b_k} \left(1 - \frac{1}{(b_{k+1} - b_k)^2}\right) \mathrm{e}_{\sigma_k \mathbf{b}} + \left(1 - \frac{1}{(b_{k+1} - b_k)^2}\right) \mathrm{e}_\mathbf{b} \\ &= \mathrm{e}_\mathbf{b}. \end{aligned}$$

- ◇ $s_j s_k = s_k s_j$ with $|k - j| > 1$. Suppose that b_k appears after b_{k+1} and that b_j appears before b_{j+1} in $\mathbf{a}$. For the left-hand side we have

$$\begin{aligned} s_k \mathrm{e}_\mathbf{b} &= \frac{1}{b_{k+1} - b_k} \mathrm{e}_\mathbf{b} + \mathrm{e}_{\sigma_k \mathbf{b}} \quad \Rightarrow \\ s_j s_k \mathrm{e}_\mathbf{b} &= \frac{1}{(b_{k+1} - b_k)(b_{j+1} - b_j)} \mathrm{e}_\mathbf{b} + \frac{1}{b_{k+1} - b_k} \left(1 - \frac{1}{(b_{j+1} - b_j)^2}\right) \mathrm{e}_{\sigma_j \mathbf{b}} \\ &\quad + \frac{1}{b_{j+1} - b_j} \mathrm{e}_{\sigma_k \mathbf{b}} + \left(1 - \frac{1}{(b_{j+1} - b_j)^2}\right) \mathrm{e}_{\sigma_j \sigma_k \mathbf{b}}, \end{aligned}$$

while for the right-hand side we have

$$\begin{aligned}
s_j \mathrm{e}_{\mathbf{b}} &= \frac{1}{b_{j+1}-b_j}\mathrm{e}_{\mathbf{b}} + \left(1-\frac{1}{(b_{j+1}-b_j)^2}\right)\mathrm{e}_{\sigma_j\mathbf{b}} \quad \Rightarrow \\
s_k s_j \mathrm{e}_{\mathbf{b}} &= \frac{1}{(b_{k+1}-b_k)(b_{j+1}-b_j)}\mathrm{e}_{\mathbf{b}} + \frac{1}{b_{j+1}-b_j}\mathrm{e}_{\sigma_k\mathbf{b}} \\
&\quad + \frac{1}{b_{k+1}-b_k}\left(1-\frac{1}{(b_{j+1}-b_j)^2}\right)\mathrm{e}_{\sigma_j\mathbf{b}} + \left(1-\frac{1}{(b_{j+1}-b_j)^2}\right)\mathrm{e}_{\sigma_k\sigma_j\mathbf{b}}.
\end{aligned}$$

Comparing the coefficients of the different basis elements we easily gain the proof.

◇ $s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}$. Suppose that b_{k+1} precedes b_{k+2} which precedes b_k in $\mathbf{a}$. The left-hand and right-hand sides are as follows

$$\begin{aligned}
s_k \mathrm{e}_{\mathbf{b}} &= \frac{1}{b_{k+1}-b_k}\mathrm{e}_{\mathbf{b}} + \mathrm{e}_{\sigma_k\mathbf{b}} \quad \Rightarrow \\
s_{k+1} s_k \mathrm{e}_{\mathbf{b}} &= \frac{1}{(b_{k+1}-b_k)(b_{k+2}-b_{k+1})}\mathrm{e}_{\mathbf{b}} + \frac{1}{b_{k+1}-b_k}\left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_{k+1}\mathbf{b}} \\
&\quad + \frac{1}{b_{k+2}-b_k}\mathrm{e}_{\sigma_k\mathbf{b}} + \mathrm{e}_{\sigma_{k+1}\sigma_k\mathbf{b}} \quad \Rightarrow \\
s_k s_{k+1} s_k \mathrm{e}_{\mathbf{b}} &= \frac{1}{(b_{k+1}-b_k)^2(b_{k+2}-b_{k+1})}\mathrm{e}_{\mathbf{b}} + \frac{1}{(b_{k+1}-b_k)(b_{k+2}-b_{k+1})}\mathrm{e}_{\sigma_k\mathbf{b}} \\
&\quad + \frac{1}{(b_{k+1}-b_k)(b_{k+2}-b_k)}\left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_{k+1}\mathbf{b}} \\
&\quad + \frac{1}{b_{k+1}-b_k}\left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_k\sigma_{k+1}\mathbf{b}} \\
&\quad - \frac{1}{(b_{k+2}-b_k)(b_{k+1}-b_k)}\mathrm{e}_{\sigma_k\mathbf{b}} + \frac{1}{b_{k+2}-b_k}\left(1-\frac{1}{(b_{k+1}-b_k)^2}\right)\mathrm{e}_{\mathbf{b}} \\
&\quad + \frac{1}{b_{k+2}-b_{k+1}}\mathrm{e}_{\sigma_{k+1}\sigma_k\mathbf{b}} + \left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_k\sigma_{k+1}\sigma_k\mathbf{b}}
\end{aligned}$$

and

$$
\begin{aligned}
s_{k+1}\mathrm{e}_{\mathbf{b}} &= \frac{1}{b_{k+2}-b_{k+1}}\mathrm{e}_{\mathbf{b}} + \left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_{k+1}\mathbf{b}} \quad \Rightarrow \\
s_k s_{k+1}\mathrm{e}_{\mathbf{b}} &= \frac{1}{(b_{k+2}-b_{k+1})(b_{k+1}-b_k)}\mathrm{e}_{\mathbf{b}} + \frac{1}{b_{k+2}-b_{k+1}}\mathrm{e}_{\sigma_k\mathbf{b}} \\
&\quad + \frac{1}{b_{k+2}-b_k}\left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_{k+1}\mathbf{b}} + \left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_k\sigma_{k+1}\mathbf{b}} \quad \Rightarrow \\
s_{k+1}s_k s_{k+1}\mathrm{e}_{\mathbf{b}} &= \frac{1}{(b_{k+2}-b_{k+1})^2(b_{k+1}-b_k)}\mathrm{e}_{\mathbf{b}} \\
&\quad + \frac{1}{(b_{k+2}-b_{k+1})(b_{k+1}-b_k)}\left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_{k+1}\mathbf{b}} \\
&\quad + \frac{1}{(b_{k+2}-b_k)(b_{k+2}-b_{k+1})}\mathrm{e}_{\sigma_k\mathbf{b}} + \frac{1}{b_{k+2}-b_{k+1}}\mathrm{e}_{\sigma_{k+1}\sigma_k\mathbf{b}} \\
&\quad - \frac{1}{(b_{k+2}-b_k)(b_{k+2}-b_{k+1})}\left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_{k+1}\mathbf{b}} \\
&\quad + \frac{1}{b_{k+2}-b_k}\left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\mathbf{b}} \\
&\quad + \frac{1}{b_{k+1}-b_k}\left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_k\sigma_{k+1}\mathbf{b}} + \left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_{k+1}\sigma_k\sigma_{k+1}\mathbf{b}}.
\end{aligned}
$$

The coefficients of $\mathrm{e}_{\sigma_k\sigma_{k+1}\mathbf{b}}, \mathrm{e}_{\sigma_{k+1}\sigma_k\mathbf{b}}$ and $\mathrm{e}_{\sigma_k\sigma_{k+1}\sigma_k\mathbf{b}} = \mathrm{e}_{\sigma_{k+1}\sigma_k\sigma_{k+1}\mathbf{b}}$ are the same on both sides. The coefficients of $\mathrm{e}_{\mathbf{b}}, \mathrm{e}_{\sigma_k\mathbf{b}}$ and $\mathrm{e}_{\sigma_{k+1}\mathbf{b}}$ turn out to be equal from easy calculations comparing the two above formulas.

We finally focus on the relation (4): $s_k z_k = z_{k+1}s_k - 1$. Suppose that b_k precedes b_{k+1} in $\mathbf{a}$. We evaluate the action of both sides of (4) on $\mathrm{e}_{\mathbf{b}}$. On the left-hand we have

$$
\begin{aligned}
z_k\mathrm{e}_{\mathbf{b}} &= b_k\mathrm{e}_{\mathbf{b}} \quad \Rightarrow \\
s_k z_k \mathrm{e}_b &= \frac{b_k}{b_{k+1}-b_k}\mathrm{e}_{\mathbf{b}} + b_k\left(1-\frac{1}{(b_{k+1}-b_k)^2}\right)\mathrm{e}_{\sigma_k\mathbf{b}},
\end{aligned}
$$

while on the right we have

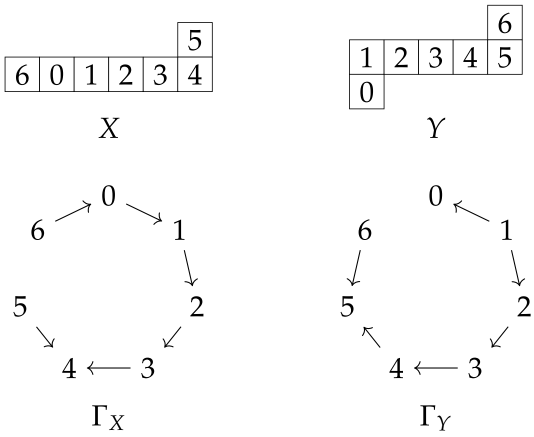
$$
\begin{aligned}
s_k\mathrm{e}_{\mathbf{b}} &= \frac{1}{b_{k+1}-b_k}\mathrm{e}_{\mathbf{b}} + \left(1-\frac{1}{(b_{k+1}-b_k)^2}\right)\mathrm{e}_{\sigma_k\mathbf{b}} \quad \Rightarrow \\
(z_{k+1}s_k - 1)\mathrm{e}_{\mathbf{b}} &= \frac{b_{k+1}}{b_{k+1}-b_k}\mathrm{e}_{\mathbf{b}} + b_k\left(1-\frac{1}{(b_{k+1}-b_k)^2}\right)\mathrm{e}_{\sigma_k\mathbf{b}} - \mathrm{e}_{\mathbf{b}}.
\end{aligned}
$$

It is now straightforward to check that the coefficients of $\mathrm{e}_{\mathbf{b}}$ and $\mathrm{e}_{\sigma_k\mathbf{b}}$ are equal on both sides. This ends the proof. □

Theorem 3.11 strengthens the above discussion and shows that *every* arrow graph corresponds to a $\mathcal{H}_p^k$-module. In particular, we can talk about the composition factors of a belt, meaning the composition factors of the corresponding $\mathcal{H}_p^k$-module. Later on we may look at the set of p-shapes $\mathcal{B}$ as a subset of the set of all possible arrow graphs.

As in Section 3.1 we rely on the notion of distance between p-shapes. Given p-shapes X and Y, we define $\mathrm{D}(X,Y)$ and $\mathrm{d}(X,Y)$ as in Section 3.1.

Example. Let $p = 7$, and let X and Y be the p-shapes below. From the corresponding arrow graphs, we see that $\mathrm{D}(X,Y) = \{\bar{0},\bar{4}\}$, and therefore $\mathrm{d}(X,Y) = 2$.



We can now state a variant of Lemma 3.1.

Lemma 3.12. *Consider* $\mathbf{a} = (a_0, \ldots, a_{p-1})$ *where* $a_0, \ldots, a_{p-1}$ *equal* $\bar{0}, \ldots, \overline{p-1}$ *in some order. Then* $\mathbf{a}$ *appears in the formal character of exactly one belt.*

Proof. This result follows similarly to Lemma 3.1. Take an empty arrow graph and fill its edges according to the relative position between the entries of $\mathbf{a}$ until a belt X is reached. By construction $\mathbf{a}$ is a term of the formal character $\mathrm{ch}(X)$. If a different belt Y is considered, then $\mathrm{D}(X,Y) \neq \emptyset$ and repeating the last argument in the proof of Lemma 3.1, we see that $\mathrm{ch}(X)$ and $\mathrm{ch}(Y)$ have no terms in common. □

Corollary 3.13. *Different belts have no composition factors in common.*

At this point we can mimic the arguments in Section 3.1 using belts in place of maximal p-shapes.

Definition. Let X be a p-shape. A belt Y is a *refinement* of X if Y can be obtained by adding edges to X.

Corollary 3.14. *Let X be a p-shape. The formal character of X is the sum of the formal characters of the refinements of X.*

Proof. Given Y a refinement of X, every term of $\mathrm{ch}(Y)$ is also a term of $\mathrm{ch}(X)$ (just because X is obtained forgetting some arrows of Y). Conversely, each term of $\mathrm{ch}(X)$ is a term of the formal character of a unique refinement of X: for every j and $j+1 \pmod p$ that are not linked by an arrow in X, the relative order of these entries in the chosen term of $\mathrm{ch}(X)$ indicate how they need to be linked to find the desired belt. □

As in Corollary 3.4, [K, Theorem 5.3.1] ensures the following.

Corollary 3.15. *If X is a p-shape, then the composition factors of X are the composition factors of its refinements.*

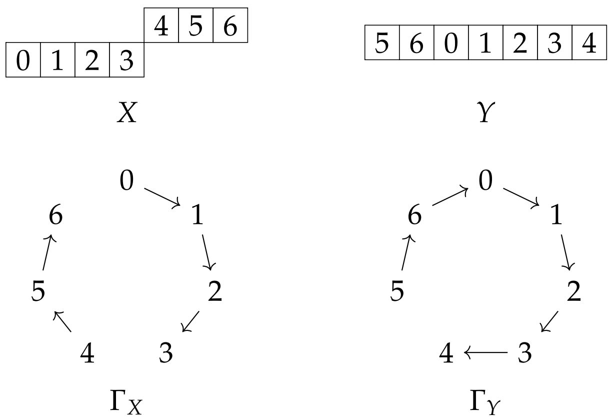
We are know ready to establish a result which gives us a way to recognise when two Specht $\mathcal{C}^k_{l,m}$-modules share a composition factor.

Theorem 3.16. *Let X and Y be p-shapes. The following are equivalent:*

1. *X and Y have a composition factor in common;*
2. *The formal characters of X and Y have a term in common;*
3. $\mathrm{d}(X,Y) = 0$ *and* $\Gamma_X \cup \Gamma_Y$ *is not a directed cycle;*
4. *There is a belt which is a refinement of both X and Y.*

Remark. Before proving Theorem 3.16 we dwell a moment on the meaning of condition (3). Since $\mathrm{d}(X,Y) = 0$, the overlapping $\Gamma_X \cup \Gamma_Y$ is an admissible arrow graph. We have that $\Gamma_X \cup \Gamma_Y$ is a directed cycle if and only if all arrows in Γ_X and Γ_Y are anticlockwise (or all clockwise) and there is no $k \in \mathbb{Z}/p\mathbb{Z}$ such that $\boxed{k}$ is simultaneously the foot node of a connected component of X and a connected component of Y.

Example. Let $p = 7$, and let X and Y be the p-shapes given below. All arrows in Γ_X and Γ_Y are oriented clockwise (because neither X nor Y has a node immediately above another). Moreover, X and Y do not have foot nodes of the same residue. So $\Gamma_X \cup \Gamma_Y$ is a directed cycle. As a consequence, the formal characters of X and Y have no term in common, so X and Y do not share a composition factor.



Proof of Theorem 3.16.

(1⇒ 2) This is trivial because the formal character of a $\mathcal{H}_p^k$-module is the sum of the formal characters of its composition factors.

(2⇒ 3) Suppose that $\mathrm{ch}(X)$ and $\mathrm{ch}(Y)$ have a term in common. Then it immediately follows that $\mathrm{d}(X,Y) = 0$. Suppose for a contradiction that $\Gamma_X \cup \Gamma_Y$ is a clockwise directed cycle and set

$$F_X := \{k \in \mathcal{C}(B) \mid \boxed{k} \text{ is the foot node of a connected component of } X\}$$

and

$$F_Y := \{k \in \mathcal{C}(B) \mid \boxed{k} \text{ is the foot node of a connected component of } Y\}.$$

(For example, for the 7-shapes in the example above, $F_X = \{\bar{0}, \bar{4}\}$ and $F_Y = \{\bar{5}\}$.) The remark above shows that $F_X \cap F_Y = \emptyset$. Now since every term of $\mathrm{ch}(X)$ has an element of F_X as first component and every term of $\mathrm{ch}(Y)$ has an element of F_Y as first component, it follows that $\mathrm{ch}(X)$ and $\mathrm{ch}(Y)$ have no terms in common and this contradicts the hypothesis. (If the arrows in $\Gamma_X \cup \Gamma_Y$ are all anticlockwise, the proof follows similarly replacing "foot" with "hand" in the definitions of F_X and F_Y above.)

(3⇒ 4) By the hypothesis the union $\Gamma_X \cup \Gamma_Y$ is an arrow graph. This can be refined to a belt by adding in arrows if necessary (ensuring that these arrows are not all oriented the same way as all the existing arrows). This belt is a refinement of both X and Y.

(4⇒ 1) This follows from Corollary 3.15. □

In fact, we can prove that the modules M_Γ are simple. We do this next (although we do not actually need it for our main result).

Corollary 3.17. *Let* Γ *be a belt. Then the* $\mathcal{H}_p^k$*-module* M_Γ *is simple.*

Proof. Let $\mathbf{B}$ be the set of all permutations $\mathbf{b}$ appearing in $\mathrm{ch}(\Gamma)$. Given $\mathbf{c} \in \mathbf{B}$, define

$$z_{\mathbf{c}} = \prod_{k=1}^{p} \prod_{\substack{i \in \mathbb{Z}/p\mathbb{Z} \\ i \neq \mathbf{c}_k}} (z_k - i).$$

It is easy to see that $z_{\mathbf{c}}\mathrm{e}_{\mathbf{c}} \neq 0$, while $z_{\mathbf{c}}\mathrm{e}_{\mathbf{b}} = 0$ for every $\mathbf{b} \neq \mathbf{c}$.

Now suppose N is a non-zero submodule of M_Γ. Take a non-zero element $n = \sum_{\mathbf{b}\in\mathbf{B}} \lambda_{\mathbf{b}}\mathrm{e}_{\mathbf{b}}$ of N, and choose $\mathbf{c}$ such that the coefficient $\lambda_{\mathbf{c}}$ is non-zero. Then $z_{\mathbf{c}}n$ is a non-zero scalar multiple of $\mathrm{e}_{\mathbf{c}}$, so $\mathrm{e}_{\mathbf{c}} \in N$.

Now suppose $\mathbf{d}$ is another element of $\mathbf{B}$, and write $\mathbf{d} = \sigma_{k_1} \dots \sigma_{k_r}\mathbf{c}$ with r as small as possible. Note then that all the intermediate terms $\sigma_{k_j} \dots \sigma_{k_r}\mathbf{c}$ will also lie in $\mathbf{B}$. Now the formula defining the action of s_k on M_Γ shows that $\mathrm{e}_{\mathbf{d}}$ appears with non-zero coefficient in $s_{k_1} \dots s_{k_r}\mathrm{e}_{\mathbf{c}}$. Hence $z_{\mathbf{d}}s_{k_1} \dots s_{k_r}\mathrm{e}_{\mathbf{c}}$ is a non-zero multiple of $\mathrm{e}_{\mathbf{d}}$ lying in N, so that $\mathrm{e}_{\mathbf{d}} \in N$. So $N = M_\Gamma$. □

Now we construct an approximation to the decomposition matrix as in Section 3.1: we construct the matrix

$$D_B = (d_{X,Y})_{X \in \mathcal{B}, Y \in \mathrm{Belt}},$$

where $d_{X,Y} = 1$ if Y is a refinement of X, and 0 otherwise. As in Section 3.1, showing that D_B is a connected matrix is sufficient to show that B is a block.

Definition. Let X, Y be p-shapes in $\mathcal{B}$. We say that X and Y are *p-linked* if there exists a series of p-shapes $X = X_1, X_2, \dots, X_{t-1}, X_t = Y$ in $\mathcal{B}$ such that X_j and X_{j+1} satisfy *any* of the equivalent conditions in Theorem 3.16, for every $1 \leqslant j \leqslant t-1$.

We now move to the heart of this section. First of all we show that we can reduce to the case where two p-shapes in a belt block have distance zero.

Proposition 3.18. *Let* X *and* Y *be* p*-shapes in* $\mathcal{B}$ *with* $\mathrm{d}(X,Y) > 0$*. There exists a* p*-shape* $Z \in \mathcal{B}$ *such that* $\mathrm{d}(X,Z) < \mathrm{d}(X,Y)$ *and* $\mathrm{d}(Y,Z) < \mathrm{d}(X,Y)$*.*

Proof. Fix $i \in \mathrm{D}(X,Y)$. We use all the notation introduced for proving Theorem 3.8. Let $\mu_X \setminus \lambda_X$ and $\mu_Y \setminus \lambda_Y$ be skew-partitions in B having p-shapes X and Y, respectively. Assume without loss of generality that x_i is a space in $\mathrm{Ab}(X)$ and y_i is a bead in $\mathrm{Ab}(Y)$. Let $k \in \{i+1, \dots, i+p-1\}$ be the integer defined in Lemma 3.6. If k exists, then we can follow the proof of Theorem 3.8 to modify μ_X or λ_Y depending on the situation, and to show the existence of a skew-partition in B with an appropriate p-shape. If k does not exist, it follows that for every $j \in \{i+1, \dots, i+p-1\}$ with $j \notin \mathrm{D}(X,Y)$,

- ◇ if x_j is a bead or a right half-bead in $\mathrm{Ab}(X)$, then μ_X has no addable $(k-i)$-hooks at runner i,

and

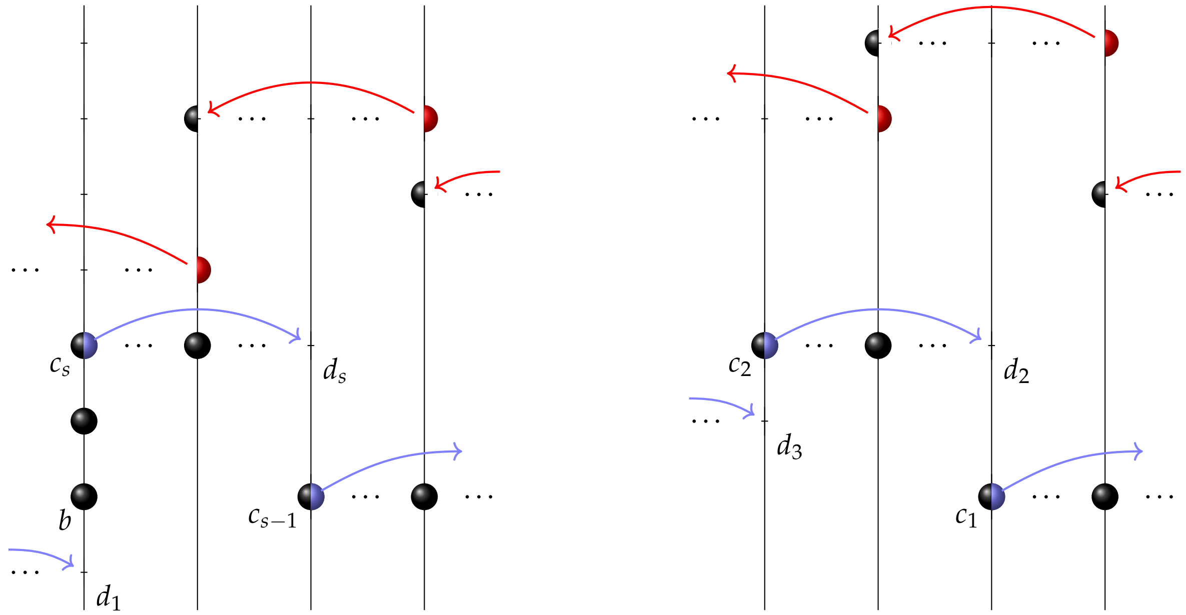


Figure 3

◇ if y_k is a space or a right half-bead in $\mathrm{Ab}(Y)$, then λ_Y has no removable $(k-i)$-hooks at runner k.

We perform a modification of μ_X that comprises deleting *all* the nodes in $\mu_X \setminus \lambda_X$ and adding them in different positions. The illustrative abacus in Figure 3 illustrates the bead moves we are going to do (we arrange the runners such that runner i is the leftmost).

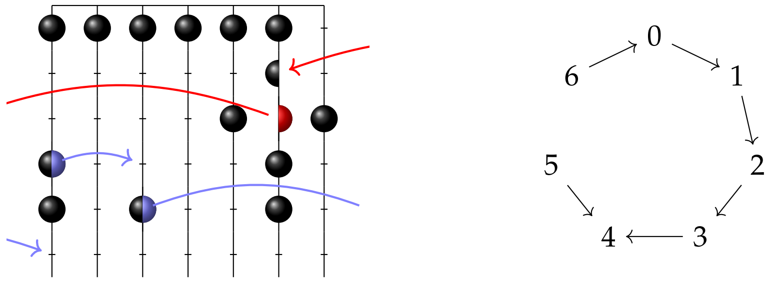
The partition μ_X has an addable p-hook at runner i, this is at the largest bead position of runner i of $\mathrm{Ab}(\mu_X)$, say $b \equiv i \pmod p$. We can now emulate the construction in the proof of Theorem 3.8 to find integers $k_1, \dots, k_s$ such that $\sum_{1 \leqslant u \leqslant s} k_u = p$, and positions $c_1, \dots, c_s, d_1, \dots, d_s$ of the abacus display such that $c_1, \dots, c_s$ are beads and $d_1, \dots, d_s$ are spaces in $\mathrm{Ab}(X)$. We find a partition $\mu_Z \vdash m$ obtained from μ_X removing the p nodes in $\mu_X \setminus \lambda_X$ and moving the right half of the bead at c_v to the space at d_v for all v. If Z is the p-shape of $\mu_Z \setminus \lambda_X$, as in Theorem 3.8 the hypothesis on k implies that Z is the desired p-shape. □

Example. Let $m = 119$, $l = 112$ and $p = n = 7$. Consider the skew-partitions

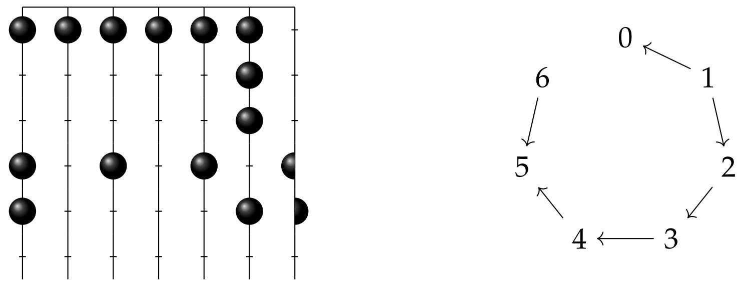
$$\mu_X \setminus \lambda_X = (20, 18, 17, 16, 12^4) \setminus (20, 18, 17, 16, 12^2, 11, 6),$$
$$\mu_Y \setminus \lambda_Y = (21^2, 17, 15, 14, 13, 12, 6) \setminus (20, 16^2, 15, 14, 13, 12, 6)$$

in $\mathcal{P}_{112,119}$ having 7-shapes X and Y, respectively. These belong to the same combinatorial block of $\mathcal{C}^k_{112,119}$ as can be easily checked (all the partitions involved have 7-core $(13, 7, 3, 2^2, 1) \vdash 28$). Below we show the abacus configurations with charge 21 and the arrow graphs of X and Y.

X

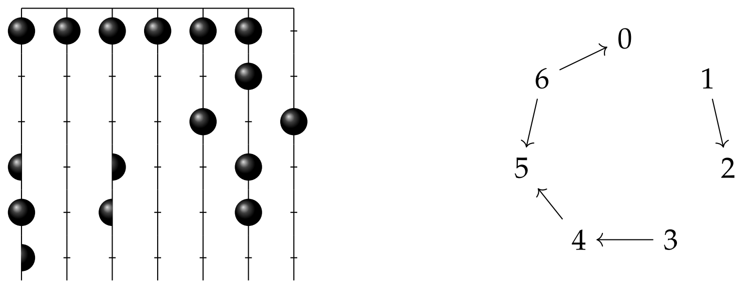


Y



We see that $\mathrm{D}(X,Y) = \{\bar{0},\bar{4}\}$ and hence $\mathrm{d}(X,Y) = 2$. Let $i = \bar{0}$. It is easy to see that the integer k does not exists. Following Theorem 3.8 and Proposition 3.18, we perform the modification of μ_X as shown in the abacus above. We obtain the partition $\mu_Z = (22,21,17,16,14,12,11,6) \vdash 119$. Denoting by Z the 7-shape of $\mu_Z \setminus \lambda_X$, we see that $\mathrm{d}(X,Z) = 1$ and $\mathrm{d}(Y,Z) = 0$.

Z



Now we use Proposition 3.18 to show that the combinatorial block B is actually a p-block of $\mathcal{C}^k_{l,m}$. This is slightly more complicated than in Section 3.1, because it is no longer the case that two p-shapes at distance 0 are p-linked. Take two p-shapes $X, Y \in \mathcal{B}$ such that $\mathrm{d}(X,Y) = 0$. By Theorem 3.16, X and Y are p-linked if $\Gamma_X \cup \Gamma_Y$ is not a directed cycle. Hence from now on we suppose that $\Gamma_X \cup \Gamma_Y$ is a directed cycle and we assume without loss of generality that all its arrows are oriented clockwise. The aim of the rest of this section is to prove that X and Y are p-linked.

First of all we establish some notation. Every skew-partition $\mu \setminus \lambda$ in B satisfies the hypothesis of Proposition 2.4 and hence $\mathrm{core}_p(\mu) = \mathrm{core}_p(\lambda)$. So there is a p-core γ and an integer w such that

$$\mathrm{core}_p(\mu) = \mathrm{core}_p(\lambda) = \gamma, \qquad \mathrm{weight}_p(\lambda) = w, \quad \mathrm{weight}_p(\mu) = w + 1$$

for every skew-partition $\mu \setminus \lambda$ in B. Let $c \in \mathbb{Z}/p\mathbb{Z}$ be the p-residue of the right-most node in the first part of γ, i.e. $c = \overline{\gamma_1 - 1}$. If γ is the empty partition, we set $c := \overline{p-1}$.

Let L be the p-block of S_l whose partitions all have p-core γ and p-weight w. For each $i \in \mathbb{Z}/p\mathbb{Z}$, let

$$L_i = \{\lambda \in L \mid \overline{\lambda_1 - 1} = i\}.$$

We remark that some L_i can be empty and that

$$L = \bigsqcup_{i \in \mathbb{Z}/p\mathbb{Z}} L_i.$$

For every $\lambda \in L$, let O_λ be the set of partitions $\mu \vdash m$ such that $\mu \setminus \lambda \in B$ and all the arrows of the p-shape of $\mu \setminus \lambda$ are clockwise (i.e. $\mu \setminus \lambda$ does not contain two nodes in the same column).

Remark. Suppose $\lambda_X, \lambda_Y \in L$, $\mu_X \in O_{\lambda_X}$ and $\mu_Y \in O_{\lambda_Y}$, and let X and Y be the p-shapes of $\mu_X \setminus \lambda_X$ and $\mu_Y \setminus \lambda_Y$ respectively. Then $\mathrm{d}(X,Y) = 0$, because all edges are oriented clockwise. Moreover, every pair X, Y such that $\Gamma_X \cup \Gamma_Y$ is a clockwise directed cycle arises in this way.

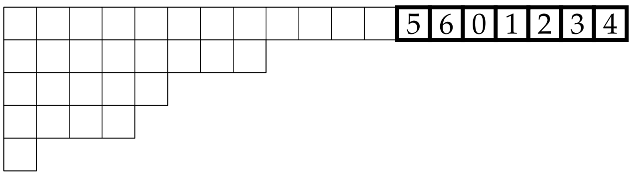
For every $i \in \mathbb{Z}/p\mathbb{Z}$, let X_i be the (ribbon) p-shape

$$\boxed{i{+}1}\,\boxed{\;\cdots\cdots\;}\,\boxed{i{-}1}\,\boxed{\;i\;}\,.$$

For example, if $p = 7$ and $i = 4$, X_4 is the p-shape Y in the example in Theorem 3.16.

Let $\lambda \in L_i$ and let $\mu_\lambda = (\lambda_1 + p, \lambda_2, \ldots, \lambda_{l(\lambda)})$. Then the abacus configuration for μ_λ is obtained from the abacus configuration for λ by moving the last bead down one position. It is easy to see that $\mu_\lambda \in O_\lambda$ and $\mu_\lambda \setminus \lambda$ has p-shape X_i.

Example. Let $m = 37$, $l = 30$ and $p = n = 7$. Consider the partition $\lambda = (12, 8, 5, 4, 1) \vdash 30$. Then $\mu_\lambda = (19, 8, 5, 4, 1)$ and $\mu_\lambda \setminus \lambda$ has 7-shape X_4.



$\mu_\lambda \setminus \lambda$

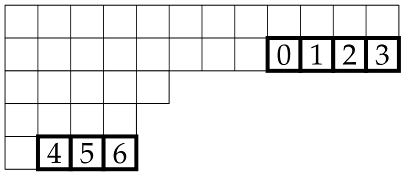
Now take $\mu \in O_\lambda$ with $\mu \neq \mu_\lambda$, and call X the p-shape of $\mu \setminus \lambda$.

Proposition 3.19. *X and X_i are p-linked.*

Proof. If a connected component of X has hand node $\boxed{i}$ then we are done because $\mathrm{ch}(X_i)$ and $\mathrm{ch}(X)$ share the term $(i+1, \ldots, i)$. Suppose that this is not the case, and let $j \in \mathbb{Z}/p\mathbb{Z}$ be such that $\boxed{j}$ is the hand node of the connected component of X containing $\boxed{i}$. We firstly observe that $\mu_1 = \lambda_1$ because otherwise the nodes $(1, \lambda_1 + 1), \ldots, (1, \mu_1)$ of μ would form a connected component of X with $\boxed{i{+}1}$ as foot node and this would imply that X has a connected component whose hand node is $\boxed{i}$.

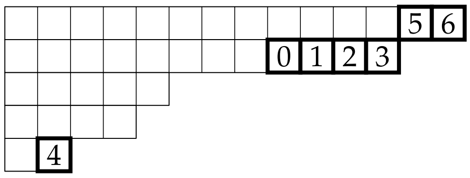
Now we can modify μ by removing the nodes $\boxed{i{+}1}, \ldots, \boxed{j}$ and putting them back at the end of the first row. Let $\tilde{\mu} \vdash m$ be the resulting partition. By construction $\mathrm{core}_p(\tilde{\mu}) = \mathrm{core}_p(\mu)$, then $\tilde{\mu} \setminus \lambda$ is in B. If $\tilde{X}$ denotes the p-shape of $\tilde{\mu} \setminus \lambda$, then $\tilde{X}$ is p-linked with both X_i and X: $\mathrm{ch}(X)$ and $\mathrm{ch}(\tilde{X})$ share the term $(i+1, i+2, \ldots, i)$ while $\mathrm{ch}(X_i)$ and $\mathrm{ch}(\tilde{X})$ share $(j+1, j+2, \ldots, j)$. So X and X_i are p-linked. $\square$

Example. Let m, l, p, n and λ be as in the previous example. Let $\mu = (12^2, 5, 4^2) \in O_\lambda$. The skew-partition $\mu \setminus \lambda$ belongs to the combinatorial block B of $\mathcal{C}^k_{30,37}$ characterised by the 7-core $\gamma := \mathrm{core}_7(B) = (1^2) \vdash 2$. Call X the 7-shape of $\mu \setminus \lambda$.



$\mu \setminus \lambda$

Retracing the notation of Proposition 3.19, we have that $i = \bar{4}$ (because $\lambda_1 - 1 = 11 \equiv 4 \pmod 7$ and then $\lambda \in L_4$) and $j = \bar{6}$ because the node $\boxed{4}$ belongs to the connected component of X having hand node $\boxed{6}$. So we find $\tilde{\mu}$ by moving the nodes $\boxed{5}\boxed{6}$ to the end of the first row, giving $\tilde{\mu} = (14, 12, 5, 4, 2) \vdash 37$.



$\tilde{\mu} \setminus \lambda$

Now the 7-shape $\tilde{X}$ of $\tilde{\mu} \setminus \lambda$, is 7-linked with both X and X_4.

Remark. If $\lambda_1, \lambda_2 \in L_i$, then $\mu_{\lambda_1} \setminus \lambda_1$ and $\mu_{\lambda_2} \setminus \lambda_2$ have the same p-shape X_i. By the previous proposition we deduce that, for every $\mu_1 \in O_{\lambda_1}$ and $\mu_2 \in O_{\lambda_2}$, the p-shapes of $\mu_1 \setminus \lambda_1$ and $\mu_2 \setminus \lambda_2$ are p-linked.

We are now very close to the end of this section. The next result will link p-shapes X_i, X_j for all pairs of $i, j \in \mathbb{Z}/p\mathbb{Z}$. Recall that $c = \overline{\gamma_1 - 1}$, and observe that L_c is non-empty: for example, it contains the partition $\bar{\gamma} = (\gamma_1 + wp, \gamma_2, \gamma_3, \dots)$. In particular this shows that B contains at least one skew-partition with p-shape X_c.

Proposition 3.20. *Let $i \in \mathbb{Z}/p\mathbb{Z}$ such that $L_i \neq \emptyset$. Then X_i and X_c are p-linked.*

Proof. Let $\lambda \in L_i$ and consider the skew-partitions $\mu_\lambda \setminus \lambda$ and $\mu_{\bar{\gamma}} \setminus \bar{\gamma}$ in B having p-shapes X_i and X_c, respectively. For convenience we suppose that $c = \bar{0}$, though the proof can easily be modified for other values of c.

If $i = \bar{0}$, the result follows by the previous remark.

Suppose that $i \neq \bar{0}$. First of all we observe that in the abacus configuration of γ, the leftmost runner, i.e. runner 0, has more beads than any other runner: this is because the last bead on the abacus occurs on this runner and (because γ is a p-core) the beads are at the topmost positions of their runners. So, following the notation of Section 2.5, we have that for every $j \in \mathbb{Z}/p\mathbb{Z} \setminus \{\bar{0}\}$,

$$b_0(\gamma) > b_j(\gamma). \tag{1}$$

The abacus configuration of λ is obtained by lowering down beads in $\mathrm{Ab}(\gamma)$ a total amount of w times. Now recall the p-quotient $(\lambda^{(0)}, \dots, \lambda^{(p-1)})$ of λ. The fact that $\lambda \in L_i$ implies that

$$\lambda^{(i)}{}_1 \geqslant \lambda^{(0)}{}_1 + b_0(\lambda) - b_i(\lambda), \tag{2}$$

where $\lambda^{(i)}{}_1$ and $\lambda^{(0)}{}_1$ denote the number of spaces above the lowest beads in the i-th and 0-th runners of $\mathrm{Ab}(\lambda)$, respectively. Recalling that $b_j(\lambda) = b_j(\gamma)$ for all $j \in \mathbb{Z}/p\mathbb{Z}$, we combine (1) and (2) getting

$$\lambda^{(i)}{}_1 > \lambda^{(0)}{}_1. \tag{3}$$

By construction, if $(\mu_\lambda^{(0)}, \dots, \mu_\lambda^{(p-1)})$ denotes the p-quotient of μ_λ, then $(\mu_\lambda^{(j)})_1 = \lambda^{(j)}{}_1$ for every $j \in \mathbb{Z}/p\mathbb{Z} \setminus \{i\}$ and

$$(\mu_\lambda^{(i)})_1 = \lambda^{(i)}{}_1 + 1. \tag{4}$$

Comparing (3) and (4) we find that

$$(\mu_\lambda^{(i)})_1 > (\mu_\lambda^{(0)})_1 + 1.$$

This inequality implies that μ_λ has at least two addable i-hooks at runner 0. We consider one of them, say at position $a_1 \equiv 0 \pmod p$, avoiding the one that affects the space immediately above the lowest bead in the i-th runner of $\mathrm{Ab}(\mu_\lambda)$, if necessary. Now let $1 \leqslant k_1 \leqslant i$ be the least integer such that $a_1 + i - k_1$ is a bead. If $k_1 < i$, by (1) and the fact that $b_j(\mu_\lambda) = b_j(\gamma)$ for all $j \in \mathbb{Z}/p\mathbb{Z}$, μ_λ has an addable $(i - k_1)$-hook at runner 0, say at position a_2. Then we start again the process from position a_2, letting k_2 be minimal such that position $a_2 + i - k_1 - k_2$ is a bead. It is clear that this procedure terminates at a step s such that $i = \sum_{1 \leqslant u \leqslant s} k_u$. Then for $1 \leqslant v \leqslant s$ we have positions $c_v := a_v + i - \sum_{1 \leqslant u \leqslant v} k_u$ and $d_v := a_v + i - \sum_{1 \leqslant u \leqslant v-1} k_u$ such that c_v is a bead and $d_1, \dots, d_v$ are all spaces in $\mathrm{Ab}(\mu_\lambda)$. Then we modify $\mathrm{Ab}(\mu_\lambda \setminus \lambda)$ as follows (we denote C the charge of the abacus display):

- ◇ move the right half-bead at $x := \lambda_1 + p - 1 + C$ to the space at $x - i$,
- ◇ for every $1 \leqslant v \leqslant s$, move the right half of the bead at c_v to the space d_v.

An illustrative abacus is drawn in Figure 4 (only runners from 0 to i are shown).

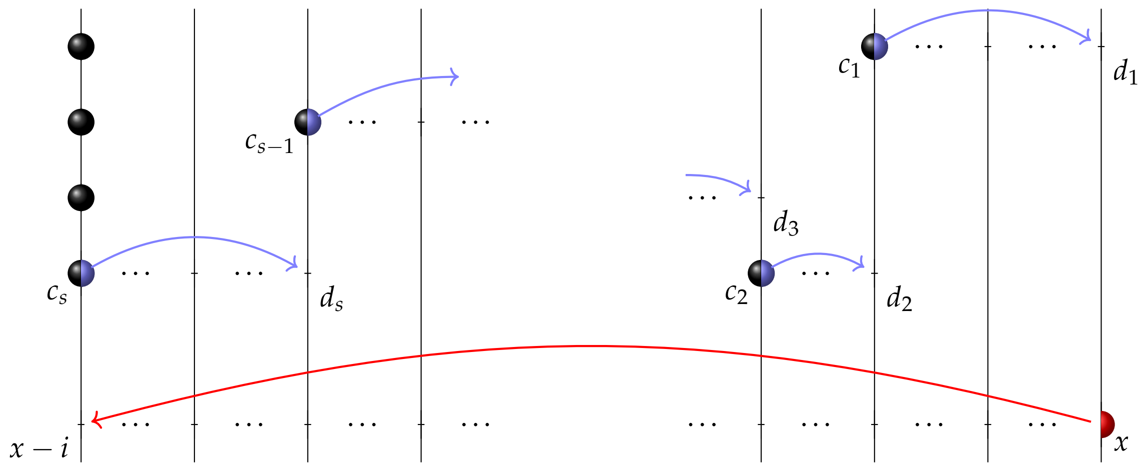
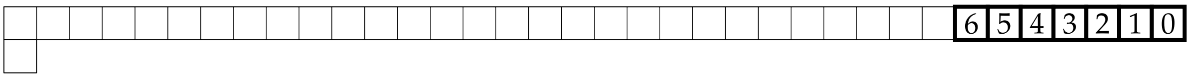


Figure 4

The resulting partition $\hat{\mu} \vdash m$ is such that $\hat{\mu} \setminus \lambda \in B$ and has p-shape $\hat{X}$ that is p-linked with both X_i and X_c: the formal characters $\mathrm{ch}(X_i)$ and $\mathrm{ch}(\hat{X})$ share the term $(i+1, \dots, i)$ while $\mathrm{ch}(X_c)$ and $\mathrm{ch}(\hat{X})$ share the term $(\overline{1}, \overline{2}, \dots, \overline{p-1}, \overline{0})$. This ends the proof.

□

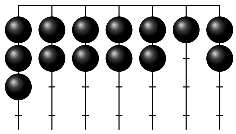
Example. Let B the combinatorial block of $\mathcal{C}^k_{30,37}$ from the previous examples. As already pointed out, following the notation, $\gamma = (1^2)$ and then $c = \gamma_1 - 1 \equiv 0 \pmod 7$. We see that $\bar{\gamma} = (29, 1)$ and $\mu_{\bar{\gamma}} = (36, 1)$.



$\mu_{\bar{\gamma}} \setminus \bar{\gamma}$

We show how to modify the skew-partition μ_λ from the previous example, to find a 7-shape that is 7-linked with both X_4 and X_0.

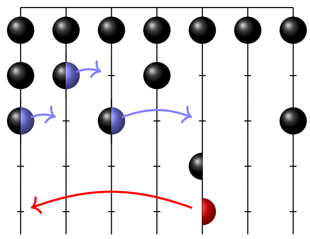
We first draw the abacus configuration of γ (all abacus configurations in the example have charge $C = 14$).



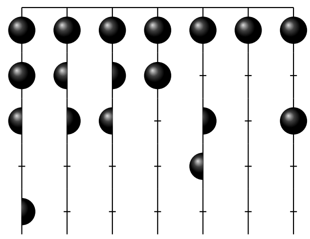
γ

Note that as remarked in the proof of Proposition 3.20, the number of beads in runner 0 of γ, $b_0(\gamma) = 3$, is bigger than the number of beads in any other runner of $\mathrm{Ab}(\gamma)$.

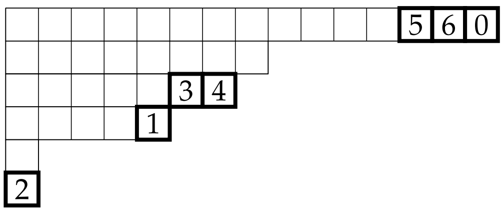
Our proof points out that μ_λ has at least two addable 4-hooks at runner 0, actually these are at positions 7 and 14. Choosing $a_1 = 14$ and following the construction in the proof of Proposition 3.20, we modify $\mathrm{Ab}(\mu_\lambda \setminus \lambda)$ as shown in the picture. The resulting skew-partition $\hat{\mu} \setminus \lambda = (15, 8, 7, 5, 1^2) \setminus (12, 8, 5, 4, 1)$ has the desired 7-shape.



$\mu_\lambda \setminus \lambda$



$\hat{\mu} \setminus \lambda$



$\hat{\mu} \setminus \lambda$

We have finally gained the key result of this section.

Corollary 3.21. *Suppose $X, Y \in \mathcal{B}$. Then X and Y are p-linked.*

Proof. By Proposition 3.18, we can reduce to the case when $\mathrm{d}(X, Y) = 0$. Moreover, Theorem 3.16 tells us that we are done if $\Gamma_X \cup \Gamma_Y$ is not an oriented cycle. In the opposite case, the description of the remaining cases together with Propositions 3.19 and 3.20 ends the proof. □

Now, Corollary 3.21 and Theorem 3.16 implies that the decomposition matrix D_B of the belt block B is connected. As remarked along the treatment, this is enough to state our main result.

Corollary 3.22. *Conjecture 2.5 holds for belt blocks.*

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