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Reducible Specht modules

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Abstract

James and Mathas conjecture a criterion for the Specht module S^λ for the symmetric group to be irreducible over a field of prime characteristic. We extend a result of Lyle to prove this conjecture in one direction; our techniques are elementary.

1 Introduction

In the representation theory of finite groups, it is useful to know which ordinary irreducible representations remain irreducible modulo a prime p. For the symmetric group $\mathfrak{S}_n$, this amounts to determining which Specht modules are irreducible over a field of characteristic p. James and Mathas have solved this problem in the case $p = 2$ [4] and put forward a conjecture for the case where p is odd, as follows.

Conjecture 1.1. [6, Conjecture 5.47]

If p is an odd prime, then the Specht module $S^\lambda_{\mathbb{F}_p}$ is reducible if and only if the Young diagram $[\lambda]$ contains nodes (a, b), (a, y) and (x, b) such that

$$\nu_p(h(a, b)) > 0$$

and

$$\nu_p(h(x, b)) \neq \nu_p(h(a, b)) \neq \nu_p(h(a, y)).$$

Carter's Criterion [6, Proposition 5.40] says that Conjecture 1.1 holds in the case where λ is p-regular; combining this with [3, Theorem 8.15], we find that the conjecture also holds when λ is p-restricted. In [5], Lyle proves a major part of the 'if' half of Conjecture 1.1.

Theorem 1.2. [5, Theorem 2.16]

Suppose that p is an odd prime, and that the Young diagram $[\lambda]$ contains nodes (a, b), (x, b) and (a, y) such that

$$p \mid h(a, b), \qquad p \nmid h(x, b), \qquad p \nmid h(a, y).$$

Then the Specht module $S^\lambda_{\mathbb{F}_p}$ is reducible.

The purpose of this paper is to use Theorem 1.2 to complete the proof of the 'if' part of Conjecture 1.1. We begin by summarizing the notation and results which we shall use.

The essential reference for the representation theory of the symmetric groups remains James's book [3]. Let n be a non-negative integer. For any partition λ of n, one defines a *Specht module* S^λ_F for any field F. If F has infinite characteristic, then S^λ_F is irreducible, and the modules S^λ_F afford a complete set of irreducible representations of $F\mathfrak{S}_n$ as λ ranges over the set of partitions of n. If F has prime characteristic p, then the S^λ_F are not generally irreducible; however, if λ is p-regular (that is, if it does not have p equal parts), then S^λ_F has an irreducible cosocle D^λ_F, and the modules D^λ_F afford a complete set of irreducible modules for $F\mathfrak{S}_n$ as λ ranges over the set of p-regular partitions of n. Moreover, for any partition μ, the composition factors of S^μ_F are all of the form D^λ_F with $\lambda \trianglerighteq \mu$, where $\trianglerighteq$ is the usual dominance order on partitions.

From now on, we fix an odd prime p and a field F of characteristic p, and write S^λ and D^λ for S^λ_F and D^λ_F. As usual, we let λ' denote the partition conjugate to λ, so that

$$\lambda'_i = |\{j \mid \lambda_j \geqslant i\}|,$$

and we define the *Young diagram* of λ (whose elements we call *nodes*) to be

$$[\lambda] = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid j \leqslant \lambda_i\}.$$

For any node (i, j) of $[\lambda]$, we define $h(i, j)$ to be the (i, j) hook length, i.e. the number

$$\lambda_i - j + \lambda'_j - i + 1$$

of nodes directly to the right of or directly below (i, j). For $r \geqslant 1$, define the rth *ladder* to be the set

$$\{(i, j) \mid i + (p-1)j = p - 1 + r\} \subset \mathbb{N} \times \mathbb{N},$$

and define the *p-regularisation* λ^{R} of λ to be the partition whose Young diagram is obtained by moving all the nodes of $[\lambda]$ as far as possible up their ladders, i.e. so that if (i, j) is a node of $[\lambda^{\mathrm{R}}]$ with $i \geqslant p$, then $(i - p + 1, j + 1)$ is also a node. Then we have the following.

Theorem 1.3. [2, Theorem A]
Let λ be a partition of n. Then:

1. *λ^{R} is a p-regular partition of n;*
2. *$D^{\lambda^{\mathrm{R}}}$ occurs as a composition factor of S^λ with multiplicity 1;*
3. *any other composition factor of S^λ is of the form D^μ with $\mu \rhd \lambda^{\mathrm{R}}$.*

Finally, we shall use the following theorem concerning homomorphisms between Specht modules, due to the author and Lyle.

Theorem 1.4. [1, Theorem 2.3]
Suppose that λ and μ are partitions of n, and that $\lambda'_1 = \mu'_1$. Define $\hat{\lambda}$ and $\hat{\mu}$ by

$$\hat{\lambda}_i = \max(\lambda_i - 1, 0), \qquad \hat{\mu}_i = \max(\mu_i - 1, 0).$$

Then

$$\dim_F \mathrm{Hom}_{F\mathfrak{S}_n}(S^\lambda, S^\mu) = \dim_F \mathrm{Hom}_{F\mathfrak{S}_n}(S^{\hat{\lambda}}, S^{\hat{\mu}}).$$

2 A proof of the 'if' part of Conjecture 1.1

We begin by examining the connection between ladders and dominance. Given a partition λ, define $l_r(\lambda)$ to be the number of nodes of $[\lambda]$ in the rth ladder. If λ and μ are partitions of the same integer, write $\mu \geqslant \lambda$ if the largest r for which $l_r(\mu) \neq l_r(\lambda)$ satisfies $l_r(\mu) > l_r(\lambda)$. [2, 1.2] shows that $\geqslant$ is a total order on the p-regular partitions of n.

Lemma 2.1. *Suppose λ and μ are p-regular partitions. If $\mu > \lambda$, then $\lambda \not\trianglerighteq \mu$.*

Proof. Let r be maximal such that $l_r(\mu) \neq l_r(\lambda)$, and suppose that the first node of the rth ladder (that is, the one with smallest first coordinate) which appears in $[\mu]$ but not in $[\lambda]$ is (a, b). Since μ is p-regular, the nodes $(a - p + 1, b + 1), (a - 2p + 2, b + 2), \ldots$ are in $[\mu]$, and hence any node (c, d) lying in ladder s with $c \leqslant a$ and $s \leqslant r$ must lie in $[\mu]$ (since it lies above and to the left of one of these nodes). Since $[\lambda]$ and $[\mu]$ have identical intersections with ladders $r + 1, r + 2, \ldots$, we must therefore have $\mu_i \geqslant \lambda_i$ for $i = 1, \ldots, a - 1$, and since $[\mu] \ni (a, b) \notin [\lambda]$, we have $\mu_a > \lambda_a$. Thus

$$\mu_1 + \cdots + \mu_a > \lambda_1 + \cdots + \lambda_a$$

and so $\lambda \not\trianglerighteq \mu$. □

Now we examine the partitions which satisfy the criteria of Conjecture 1.1 but not Theorem 1.2.

Lemma 2.2. *Suppose that λ does not satisfy the criteria of Theorem 1.2, i.e. whenever there is a node (a, b) in $[\lambda]$ such that $p \mid h(a, b)$, then either*

$$p \mid h(a, y) \text{ for all nodes } (a, y) \text{ of } [\lambda]$$

or

$$p \mid h(x, b) \text{ for all nodes } (x, b) \text{ of } [\lambda].$$

Then there exist $k, l \geqslant 0$ such that:

- *$(k + 1, l + 1)$ is not a node of $[\lambda]$;*
- *if (a, b) is a node of $[\lambda]$ with $a \leqslant k$ and $b \leqslant l$, then $p \nmid h(a, b)$;*

- *if* (a,b) *and* (c,b) *are nodes of* $[\lambda]$ *with* $b > l$*, then* $p \mid h(a,b)$ *if and only if* $p \mid h(c,b)$*;*
- *if* (a,b) *and* (a,d) *are nodes of* $[\lambda]$ *with* $a > k$*, then* $p \mid h(a,b)$ *if and only if* $p \mid h(a,d)$*.*

Proof. Suppose (a,b) is a node with $p \mid h(a,b)$; we claim it cannot be the case that both

$$p \mid h(a,y) \text{ for all nodes } (a,y) \text{ of } [\lambda]$$

and

$$p \mid h(x,b) \text{ for all nodes } (x,b) \text{ of } [\lambda];$$

if this is the case, then $p \mid h(a,\lambda_a)$, which means that $\lambda_a = \lambda_{a+1} = \cdots = \lambda_{a+p-1}$; in particular, $(a+1,b)$ is a node of $[\lambda]$ and we have $h(a+1,b) = h(a,b) - 1$, and so $h(a,b)$ and $h(a+1,b)$ cannot both be divisible by p.

Now let

$$\mathcal{A} = \{a \mid \lambda_a > 0 \text{ and } p \mid h(a,b)\ \forall\ (a,b) \in [\lambda]\}$$

and

$$\mathcal{B} = \{b \mid \lambda'_b > 0 \text{ and } p \mid h(a,b)\ \forall\ (a,b) \in [\lambda]\}.$$

If $\mathcal{A}$ is empty, then we may take $(k,l) = (\lambda'_1, 0)$, while if $\mathcal{B}$ is empty, then we take $(k,l) = (0,\lambda_1)$. If both are non-empty, then let α and β be their respective least elements. Then by the last paragraph (α,β) is not a node of $[\lambda]$, and we may take $(k,l) = (\alpha-1,\beta-1)$. □

Proof of Conjecture 1.1 ('if' part). Suppose λ satisfies the criteria of Lemma 2.2, and let (k,l) be given by the conclusion of that result. If λ satisfies the criteria of Conjecture 1.1, then either there exist (a,b) and (a,d) in $[\lambda]$ such that $\nu_p(h(a,b))$ and $\nu_p(h(a,d))$ are positive and unequal, or there exist (a,b) and (c,b) in $[\lambda]$ such that $\nu_p(h(a,b))$ and $\nu_p(h(c,b))$ are positive and unequal. It suffices for our purposes to assume the latter, since by [3, Theorem 8.15], S^λ is reducible if and only if $S^{\lambda'}$ is reducible. Define the partition $\tilde{\lambda}$ by removing the first l columns of $[\lambda]$, i.e.

$$\tilde{\lambda}_i = \max(\lambda_i - l, 0).$$

Then $\tilde{\lambda}$ is p-regular, since if $\lambda_i = \lambda_{i+p-1} \neq \lambda_{i+p}$, then p divides $h(i,\lambda_i)$ but not $h(i+p-1,\lambda_i)$. Also, $[\tilde{\lambda}]$ contains nodes $(a,b-l)$ and $(c,b-l)$ such that $\nu_p(h(a,b-l))$ and $\nu_p(h(c,b-l))$ are unequal. Hence, by Carter's Criterion, $S^{\tilde{\lambda}}$ is reducible; thus there is some p-regular partition $\tilde{\nu} \rhd \tilde{\lambda}$ such that

$$\mathrm{Hom}_{F\Sigma_n}(S^{\tilde{\nu}}, S^{\tilde{\lambda}}) \neq 0.$$

We define a partition ν by adding the first l columns of λ to $\tilde{\nu}$, i.e.

$$\nu_i = \begin{cases} \tilde{\nu}_i + l & (\lambda_i \geqslant l) \\ \lambda_i & (\lambda_i \leqslant l). \end{cases}$$

By Theorem 1.4 we have

$$\mathrm{Hom}_{F\mathfrak{S}_n}(S^\nu, S^\lambda) \neq 0,$$

so some composition factor D^μ of S^ν is also a composition factor of S^λ. It remains to show that $\mu \neq \lambda^{\mathrm{R}}$.

We have $\tilde{\nu} \rhd \tilde{\lambda}$, and so $\tilde{\nu} \succ \tilde{\lambda}$ by Lemma 2.1. Adding columns clearly does not affect $\succcurlyeq$, and so $\nu \succ \lambda$. Regularizing does not affect $\succcurlyeq$ either, so $\nu^{\mathrm{R}} \succ \lambda^{\mathrm{R}}$, whence $\lambda^{\mathrm{R}} \not\trianglerighteq \nu^{\mathrm{R}}$, by Lemma 2.1. But by Theorem 1.3 we have $\mu \trianglerighteq \nu^{\mathrm{R}}$, so $\mu \neq \lambda^{\mathrm{R}}$. □

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