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Algebr. Represent. Theory **8** (2005) 415–26.

http://dx.doi.org/10.1007/s10468-005-8146-3

q-Schur subalgebras

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2000 Mathematics subject classification: 20C30, 20C33

Abstract

In [5], the author and Martin constructed embeddings of Schur algebras $S(2,r) \hookrightarrow S(2,R)$. Here, we generalise to the q-Schur algebras $S_q(2,r)$.

1 Introduction

Let F be a field. The *Schur algebra* $S(n,r)$ over F is a finite-dimensional which was introduced by Schur; if F is infinite, the module category of $S(n,r)$ is equivalent to the category of r-homogeneous polynomial representations of $\mathrm{GL}_n(F)$ over F. In [8] and [9], Henke observed various repeating patterns in the decomposition matrices for the Schur algebras $S(2,r)$, and proved the existence of algebra embeddings $S(2,r) \hookrightarrow S(2,R)$ for various r and R depending on the characteristic of F, which explain translational symmetry of the decomposition matrices. In [5], the author and Martin constructed these embeddings explicitly, and also found embeddings $S(2,r) \hookrightarrow S(2,pr)$ (when $p = \mathrm{char}(F)$ is positive) which correspond to dilations of the decomposition matrices. In [4], the author constructed yet more embeddings and showed that these could be used to recover the decomposition matrix of $S(2,r)$.

Now let q be any element of F. The *q-Schur algebra* $S_q(n,r)$ is a deformation of $S(n,r)$, introduced by Dipper and James [3]. In the case where q is a prime power not divisible by the characteristic of F, the q-Schur algebra describes the unipotent representations of the finite general linear group $\mathrm{GL}_n(q)$ over F.

In this paper, we generalise the results of [5] to the q-Schur algebras $S_q(2,r)$. Specifically, we exhibit embeddings $S_q(2,r) \hookrightarrow S_q(2,R)$ for various r and R depending on the characteristic of F and on the integer

$$e = \min\{i > 0 \mid 1 + q + \cdots + q^{i-1} = 0\}$$

when the latter is finite, and we construct an embedding of the classical Schur algebra $S(2,r)$ into the q-Schur algebra $S_q(2,er)$. Although the construction is simply a q-analogue of the results in [5], the proof is rather different, and a $q = 1$ specialisation of the proof here affords a much quicker proof of the embedding in [5]. Additionally, by stating our results in terms of codeterminants, we are able to conjecture a generalisation to the q-Schur algebras $S_q(n,r)$ for arbitrary n.

2 Notation

An excellent introduction to both the classical and quantum Schur algebras can be found in the book by Martin [11]. Following Beĭlinson et al. [2], we define a basis and structure constants for $S_q(n,r)$ by considering n-step filtrations of vector spaces. So suppose q is a prime power, and let V be an n-dimensional vector space over $\mathbb{F}_q$.

Let $\mathcal{F}$ denote the set of all n-step filtrations of V; then $\mathrm{GL}_n(F)$ acts naturally on $\mathcal{F}$ and $\mathcal{F}\times\mathcal{F}$. We use $\sim$ to denote $\mathrm{GL}_n(q)$-conjugacy in both sets, and we let $O_{F,G}$ denote the orbit of $(F,G)\in\mathcal{F}\times\mathcal{F}$. These orbits are in one-to-one correspondence with the set $M(r)$ of $n\times n$ matrices with non-negative integer entries summing to r: if

$$F=(0=F_0\leqslant F_1\leqslant\ldots\leqslant F_n=V)$$

and

$$G=(0=G_0\leqslant G_1\leqslant\ldots\leqslant G_n=V),$$

then we define the matrix $A_{F,G}$ by

$$(A_{F,G})_{ij}=\dim(F_{i-1}+(F_i\cap G_j))-\dim(F_{i-1}+(F_i\cap G_{j-1}));$$

$A_{F,G}$ clearly only depends on the $\mathrm{GL}_n(q)$-orbit of (F,G).

Given $A,B,C\in M(r)$, we take $F,G,H,I,J,K\in\mathcal{F}$ such that

$$A=A_{F,G},\qquad B=A_{H,I},\qquad C=A_{J,K},$$

and define

$$\hat{g}_{A,B,C,q}=|\{L\in\mathcal{F}\mid(J,L)\sim(F,G),(L,K)\sim(H,I)\}|;$$

this does not depend on the choice of F,G,H,I,J,K.

Now let q be an indeterminate. For $A,B,C\in M(r)$, we define $g_{A,B,C,q}$ to be the unique polynomial in q such that

$$g_{A,B,C,q}=\hat{g}_{A,B,C,q}$$

whenever q is a prime power. We may now define the q-Schur algebra for arbitrary $q\in F$ to be the associative algebra with basis $M(r)$ and multiplication

$$A\circ B=\sum_{C\in M(r)}g_{A,B,C,q}C.$$

In the case $q=1$, the q-Schur algebra coincides with the classical Schur algebra $S(n,r)$.

2.1 Codeterminants

A new basis of $S(n,r)$, whose elements are called *standard codeterminants*, was introduced by J. A. Green in [6], and generalised to $S_q(n,r)$ by R. M. Green in [7]. We shall describe Schur algebra embeddings in terms of codeterminants.

In our notation, a standard codeterminant is any product $A\circ B$ in $S_q(n,r)$, where $A,B\in M(r)$ satisfy

- $a_{1k} + \cdots + a_{nk} = b_{k1} + \cdots + b_{kn}$,
- $a_{1j} + \cdots + a_{kj} \leqslant a_{1(j-1)} + \cdots + a_{(k-1)(j-1)}$,
- $b_{ji} + \cdots + b_{jk} \leqslant b_{(j-1)i} + \cdots + b_{(j-1)(k-1)}$

for any $2 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant n$, where $a_{0(j-1)}$ and $b_{(j-1)0}$ are to be treated as zero.

Theorem 2.1 (Green [6], Woodcock [12], Green [7])**.** *The standard codeterminants form a basis for $S_q(n, r)$.*

2.2 The q-Schur algebra $S_q(2, r)$

From now on, we assume $n = 2$. Since a two-step filtration of a vector space is specified simply by a subspace of that vector space, the notation for $S_q(2, r)$ is simplified considerably. $g_{A,B,C,q}$ is zero unless the row sums of A and C are the same, the column sums of B and C are the same and the column sums of A are the same as the row sums of B, so let

$$A = \begin{pmatrix} a & s-a \\ u-a & r-s-u+a \end{pmatrix}, \qquad B = \begin{pmatrix} b & u-b \\ t-b & r-t-u+b \end{pmatrix}, \qquad C = \begin{pmatrix} k & s-k \\ t-k & r-s-t+k \end{pmatrix}.$$

Now, for a prime power q, let V be an r-dimensional vector space over $\mathbb{F}_q$ and let S and T be subspaces with $\dim S = s$, $\dim T = t$, $\dim(S \cap T) = k$. Then $\hat{g}_{A,B,C,q}$ is the number of u-dimensional subspaces U of V with $\dim(U \cap S) = a$ and $\dim(U \cap T) = b$.

For $S_q(2, r)$, a standard codeterminant is simply a product

$$\begin{pmatrix} s & 0 \\ t & u \end{pmatrix} \circ \begin{pmatrix} v & w \\ 0 & u \end{pmatrix},$$

where $s, v \geqslant u$. In the next section we shall express standard codeterminants and products of standard codeterminants in terms of the standard basis elements for $S_q(2, r)$, which will enable us to prove our main results.

2.3 Quantum binomial coefficients

As usual, given q, we define the *quantum integer* $[n] = 1 + q + \cdots + q^{n-1}$ for any $n \geqslant 0$, and the *quantum factorial* $[n]! = [1][2] \ldots [n]$. We then define the *quantum binomial coefficient*

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[n-r]![r]!}$$

for non-negative integers $n \geqslant r$. If $r < 0$ or $r > n$ we define $\begin{bmatrix} n \\ r \end{bmatrix} = 0$.

Let e be the smallest positive integer such that $[e] = 0$, if such an integer exists (so e is the multiplicative order of q if $q \neq 1$, or the characteristic of F if $q = 1$). We then have the following q-analogue of Lucas's Lemma.

Lemma 2.2. *Suppose a, b, c, d are non-negative integers with $b, d < e$. Then*

$$\begin{bmatrix} ae+b \\ ce+d \end{bmatrix} = \binom{a}{c} \begin{bmatrix} b \\ d \end{bmatrix}.$$

In particular, $\begin{bmatrix} ae+b \\ ce+d \end{bmatrix} = 0$ *unless* $b \geqslant d$.

We also have the following standard results.

Lemma 2.3. *Suppose p is a prime and m, s non-negative integers.*

1. *If $r < ep^s$, then*
$$\begin{bmatrix} n+mep^s \\ r \end{bmatrix} \equiv \begin{bmatrix} n \\ r \end{bmatrix} \pmod{p}.$$
2. *If $n - r < ep^s$, then*
$$\begin{bmatrix} n+mep^s \\ r+mep^s \end{bmatrix} \equiv \begin{bmatrix} n \\ r \end{bmatrix} \pmod{p}.$$

Lemma 2.4 ([10], Theorem 3.1)**.** *Let q be a prime power and V an r-dimensional vector space over $\mathbb{F}_q$. Suppose V_1 and V_2 are subspaces of V with $\dim(V_1) = a$, $\dim(V_2) = b$ and $V_1 \cap V_2 = 0$. Then the number of m-dimensional subspaces W of V with $W \cap V_1 = 0$ and $W \geqslant V_2$ is*
$$q^{a(m-b)}\begin{bmatrix} r-a-b \\ m-b \end{bmatrix}.$$

Corollary 2.5. *Let q be a prime power and V an r-dimensional vector space over $\mathbb{F}_q$. Suppose V_1 and V_2 are subspaces of V with $\dim(V_1) = a$, $\dim(V_2) = b$ and $\dim(V_1 \cap V_2) = i$. Then the number of m-dimensional subspaces W of V with $\dim(W \cap V_1) = x$ and $W \geqslant V_2$ is*
$$q^{(a-x)(m-x-b+i)}\begin{bmatrix} a-i \\ x-i \end{bmatrix}\begin{bmatrix} r-a-b+i \\ m-x-b+i \end{bmatrix}.$$

In the next section, we shall make use of several standard properties of quantum binomial coefficients, such as the 'q-binomial Theorem'. These can be found in the book by Andrews [1].

3 Change of basis and structure constants for $S_q(2, r)$

In this section, we work out how to express a standard codeterminant or a product of two standard codeterminants for $S_q(2, r)$ in terms of the standard basis elements. We fix $s, u \leqslant r$; then the matrix
$$\beta_i = \begin{pmatrix} i & s-i \\ u-i & r-s-u+i \end{pmatrix}$$
lies in $M(r)$ provided
$$\max(0, s+u-r) \leqslant i \leqslant \min(s, u),$$
while the product
$$\gamma_x = \begin{pmatrix} s & 0 \\ x-s & r-x \end{pmatrix} \circ \begin{pmatrix} u & x-u \\ 0 & r-x \end{pmatrix}$$
is a standard codeterminant provided
$$\max(s, u, r-s, r-u) \leqslant x \leqslant r.$$
From now on, we assume that i, x lie in these ranges.

Lemma 3.1.
$$\begin{pmatrix} s & 0 \\ x-s & r-x \end{pmatrix} \circ \begin{pmatrix} u & x-u \\ 0 & r-x \end{pmatrix} = \sum_i \begin{bmatrix} r-s-u+i \\ r-x \end{bmatrix}\begin{pmatrix} i & s-i \\ u-i & r-s-u+i \end{pmatrix}.$$

Proof. Since $\left[\begin{smallmatrix} r-s-u+i \\ r-x \end{smallmatrix}\right]$ is a polynomial in q, we need only prove this in the case where q is a prime power not divisible by char(F). Suppose that V is an r-dimensional vector space over $\mathbb{F}_q$ and S and U subspaces of V with $\dim(S) = s$, $\dim(U) = u$ and $\dim(S \cap U) = i$. Then the coefficient of β_i in γ_x is the number of x-dimensional subspaces of V containing both S and U. By Lemma 2.4, this is $\left[\begin{smallmatrix} r-s-u+i \\ r-x \end{smallmatrix}\right]$. □

Now we calculate the product of two standard codeterminants. Note that the structure constants $g_{A,B,C}$ for $S_q(2,r)$ in terms of the standard basis may be easily calculated, but these are rather more unwieldy than in the case $q = 1$, essentially because the identity

$$\dim(W \cap (S + U)) = \dim(W \cap S) + \dim(W \cap U) - \dim(W \cap S \cap U)$$

does not hold for vector spaces in general. However, the standard basis elements involved in a product of standard codeterminants do not present this problem, and so we find that the formula we obtain is more manageable; in particular, it does not involve powers of $q - 1$.

Proposition 3.2. *Suppose that*

$$\begin{pmatrix} s & 0 \\ v-s & r-v \end{pmatrix} \circ \begin{pmatrix} t & v-t \\ 0 & r-v \end{pmatrix}$$

and

$$\begin{pmatrix} t & 0 \\ w-t & r-w \end{pmatrix} \circ \begin{pmatrix} u & w-u \\ 0 & r-w \end{pmatrix}$$

are standard codeterminants in $S_q(2,r)$. Then the product

$$\begin{pmatrix} s & 0 \\ v-s & r-v \end{pmatrix} \circ \begin{pmatrix} t & v-t \\ 0 & r-v \end{pmatrix} \circ \begin{pmatrix} t & 0 \\ w-t & r-w \end{pmatrix} \circ \begin{pmatrix} u & w-u \\ 0 & r-w \end{pmatrix}$$

equals

$$\sum_{i=\max(0,s+u-r)}^{\min(s,u)} \sum_{j,k} \Big(q^{(w-k+j-u)(v-k)+(v-s-j+i)(u-j)} \times \begin{bmatrix} k \\ t \end{bmatrix} \begin{bmatrix} v-j \\ k-j \end{bmatrix} \begin{bmatrix} r-v-u+j \\ w-u-k+j \end{bmatrix} \begin{bmatrix} u-i \\ j-i \end{bmatrix} \begin{bmatrix} r-s-u+i \\ v-s-j+i \end{bmatrix} \begin{pmatrix} i & s-i \\ u-i & r-s-u+i \end{pmatrix}\Big).$$

Proof. We may assume that q is a prime power not divisible by char(F), and use Corollary 2.5. We have

$$\begin{pmatrix} t & v-t \\ 0 & r-v \end{pmatrix} \circ \begin{pmatrix} t & 0 \\ w-t & r-w \end{pmatrix} = \sum_k \begin{bmatrix} k \\ t \end{bmatrix} \begin{pmatrix} k & v-k \\ w-k & r-w-v+k \end{pmatrix},$$

$$\begin{pmatrix} k & v-k \\ w-k & r-w-v+k \end{pmatrix} \circ \begin{pmatrix} u & w-u \\ 0 & r-w \end{pmatrix} = \sum_j q^{(w-k+j-u)(v-k)} \begin{bmatrix} v-j \\ k-j \end{bmatrix} \begin{bmatrix} r-v-u+j \\ w-u-k+j \end{bmatrix} \begin{pmatrix} j & v-j \\ u-j & r-v-u+j \end{pmatrix}$$

and

$$\begin{pmatrix} s & 0 \\ v-s & r-v \end{pmatrix} \circ \begin{pmatrix} j & v-j \\ u-j & r-v-u+j \end{pmatrix} = \sum_i q^{(v-s-j+i)(u-j)} \begin{bmatrix} u-i \\ j-i \end{bmatrix} \begin{bmatrix} r-s-u+i \\ v-s-j+i \end{bmatrix} \begin{pmatrix} i & s-i \\ u-i & r-s-u+i \end{pmatrix},$$

whence the result. □

If we wish, we can easily invert the change-of-basis matrix given by Lemma 3.1, and then combine this with Proposition 3.2 to obtain the structure constants for $S_q(2,r)$ in terms of the basis of standard codeterminants. However, this will not be necessary in this paper.

4 The main results

Our first result is a q-analogue of [5, Theorem 3.2], which we prove in an entirely similar way. Note that we use the word 'embedding' to mean a linear injection preserving the multiplication rule; our embeddings do not preserve the identity element.

Theorem 4.1. *Let $p = \mathrm{char}(F)$, and let s be any non-negative integer. If $r < R$ are non-negative integers with $r < 2ep^s$ and $m = R - r \equiv 0 \pmod{ep^s}$, then the linear map $\phi : S_q(2,r) \hookrightarrow S_q(2,R)$ given by*

$$\left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right) \longmapsto \begin{cases} \left(\begin{smallmatrix} e+m & f \\ g & h \end{smallmatrix}\right) & (e+f, e+g \geqslant \frac{r}{2}) \\ \left(\begin{smallmatrix} e & f+m \\ g & h \end{smallmatrix}\right) & (e+f \geqslant \frac{r}{2} > e+g) \\ \left(\begin{smallmatrix} e & f \\ g+m & h \end{smallmatrix}\right) & (e+g \geqslant \frac{r}{2} > e+f) \\ \left(\begin{smallmatrix} e & f \\ g & h+m \end{smallmatrix}\right) & (\frac{r}{2} > e+f, e+g) \end{cases}$$

is an algebra embedding.

Proof. We must show that ϕ preserves the multiplication rule in $S_q(2,r)$. We shall calculate the image of a standard codeterminant under ϕ, and then use Proposition 3.2 to show that multiplication is preserved. Suppose that

$$\left(\begin{smallmatrix} s & 0 \\ x-s & r-x \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} u & x-u \\ 0 & r-x \end{smallmatrix}\right)$$

is a standard codeterminant. Given that we want ϕ to preserve the multiplication rule, we should like

$$\phi\left(\left(\begin{smallmatrix} s & 0 \\ x-s & r-x \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} u & x-u \\ 0 & r-x \end{smallmatrix}\right)\right) = \begin{cases} \left(\begin{smallmatrix} s+m & 0 \\ x-s & r-x \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} u+m & x-u \\ 0 & r-x \end{smallmatrix}\right) & (s, u \geqslant \frac{r}{2}) \\ \left(\begin{smallmatrix} s+m & 0 \\ x-s & r-x \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} u & x-u+m \\ 0 & r-x \end{smallmatrix}\right) & (s \geqslant \frac{r}{2} > u) \\ \left(\begin{smallmatrix} s & 0 \\ x-s+m & r-x \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} u+m & x-u \\ 0 & r-x \end{smallmatrix}\right) & (u \geqslant \frac{r}{2} > s) \\ \left(\begin{smallmatrix} s & 0 \\ x-s+m & r-x \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} u & x-u+m \\ 0 & r-x \end{smallmatrix}\right) & (\frac{r}{2} > s, u). \end{cases}$$

This follows easily from Lemma 3.1; in the case where $\dfrac{r}{2} > s, u$, we have $s + u - r < 0$, and so we get

$$\begin{aligned} \phi\left(\left(\begin{smallmatrix} s & 0 \\ x-s & r-x \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} u & x-u \\ 0 & r-x \end{smallmatrix}\right)\right) &= \phi\left(\sum_{i=0}^{\min(s,u)} \left[\begin{smallmatrix} r-s-u+i \\ r-x \end{smallmatrix}\right] \left(\begin{smallmatrix} i & s-i \\ u-i & r-s-u+i \end{smallmatrix}\right)\right) \\ &= \sum_{i=0}^{\min(s,u)} \left[\begin{smallmatrix} r-s-u+i \\ r-x \end{smallmatrix}\right] \left(\begin{smallmatrix} i & s-i \\ u-i & r-s-u+i+m \end{smallmatrix}\right) \\ &= \sum_{i=0}^{\min(s,u)} \left[\begin{smallmatrix} r-s-u+i+m \\ r-x \end{smallmatrix}\right] \left(\begin{smallmatrix} i & s-i \\ u-i & r-s-u+i+m \end{smallmatrix}\right) \end{aligned}$$

(by Lemma 2.3, since $r - x < ep^s$)

$$= \phi\left(\left(\begin{smallmatrix} s & 0 \\ x-s & r-x \end{smallmatrix}\right)\right) \circ \phi\left(\left(\begin{smallmatrix} u & x-u \\ 0 & r-x \end{smallmatrix}\right)\right),$$

as required. The other cases are easier.

Given this, we may check that ϕ preserves multiplication by multiplying standard codeterminants. Suppose that

$$\begin{pmatrix} s & 0 \\ v-s & r-v \end{pmatrix} \circ \begin{pmatrix} t & v-t \\ 0 & r-v \end{pmatrix}, \qquad \begin{pmatrix} t & 0 \\ w-t & r-w \end{pmatrix} \circ \begin{pmatrix} u & w-u \\ 0 & r-w \end{pmatrix}$$

are standard codeterminants. There are several cases, depending on whether each of s, t and u is at least $\frac{r}{2}$. We treat only the case $s, t, u \geqslant \dfrac{r}{2}$; the other cases are very similar. The product

$$\phi\left(\begin{pmatrix} s & 0 \\ v-s & r-v \end{pmatrix} \circ \begin{pmatrix} t & v-t \\ 0 & r-v \end{pmatrix}\right) \circ \phi\left(\begin{pmatrix} t & 0 \\ w-t & r-w \end{pmatrix} \circ \begin{pmatrix} u & w-u \\ 0 & r-w \end{pmatrix}\right)$$

equals

$$\sum_{i=s+u-r}^{\min(s,u)} \sum_{j,k} \Big(q^{(w-k+j-u)(v+m-k)+(v-s-j+i+m)(u+m-j)} \\ \times \begin{bmatrix} k \\ t+m \end{bmatrix}\begin{bmatrix} v+m-j \\ k-j \end{bmatrix}\begin{bmatrix} r-v-u-m+j \\ w-u-k+j \end{bmatrix}\begin{bmatrix} u-i \\ j-i-m \end{bmatrix}\begin{bmatrix} r-s-u+i \\ v-s-j+i \end{bmatrix}\begin{pmatrix} i+m & s-i \\ u-i & r-s-u+i \end{pmatrix}\Big);$$

replacing j and k with $j+m$ and $k+m$ respectively, we get

$$\sum_{i=s+u-r}^{\min(s,u)} \sum_{j,k} \Big(q^{(w-k+j-u)(v-k)+(v-s-j+i)(u-j)} \\ \times \begin{bmatrix} k+m \\ t+m \end{bmatrix}\begin{bmatrix} v-j \\ k-j \end{bmatrix}\begin{bmatrix} r-v-u+j \\ w-u-k+j \end{bmatrix}\begin{bmatrix} u-i \\ j-i \end{bmatrix}\begin{bmatrix} r-s-u+i \\ v-s-j+i \end{bmatrix}\begin{pmatrix} i+m & s-i \\ u-i & r-s-u+i \end{pmatrix}\Big).$$

The summand is zero unless $k \leqslant v \leqslant r$, but this gives $k - t \leqslant \dfrac{r}{2} < ep^s$, so that $\begin{bmatrix} k+m \\ t+m \end{bmatrix} = \begin{bmatrix} k \\ t \end{bmatrix}$, and so we get

$$\phi\left(\begin{pmatrix} s & 0 \\ v-s & r-v \end{pmatrix} \circ \begin{pmatrix} t & v-t \\ 0 & r-v \end{pmatrix}\right) \circ \phi\left(\begin{pmatrix} t & 0 \\ w-t & r-w \end{pmatrix} \circ \begin{pmatrix} u & w-u \\ 0 & r-w \end{pmatrix}\right) \\ = \phi\left(\begin{pmatrix} s & 0 \\ v-s & r-v \end{pmatrix} \circ \begin{pmatrix} t & v-t \\ 0 & r-v \end{pmatrix} \circ \begin{pmatrix} t & 0 \\ w-t & r-w \end{pmatrix} \circ \begin{pmatrix} u & w-u \\ 0 & r-w \end{pmatrix}\right),$$

as required. □

Our second main result is the existence of an embedding of $S_1(2, r)$ in $S_q(2, er)$.

Theorem 4.2. *For any r, the linear map* $\psi : S_1(2, r) \hookrightarrow S_q(2, er)$ *given by*

$$\psi : \begin{pmatrix} s & 0 \\ x-s & r-x \end{pmatrix} \circ \begin{pmatrix} u & x-u \\ 0 & r-x \end{pmatrix} \longmapsto \begin{pmatrix} es & 0 \\ e(x-s) & e(r-x) \end{pmatrix} \circ \begin{pmatrix} eu & e(x-u) \\ 0 & e(r-x) \end{pmatrix}$$

for $s, u \geqslant r - x$ *is an embedding of algebras.*

Proof. We prove that ψ preserves the multiplication using Proposition 3.2. Consider the product

$$\begin{pmatrix} es & 0 \\ e(v-s) & e(r-v) \end{pmatrix} \circ \begin{pmatrix} et & e(v-t) \\ 0 & e(r-v) \end{pmatrix} \circ \begin{pmatrix} et & 0 \\ e(w-t) & e(r-w) \end{pmatrix} \circ \begin{pmatrix} eu & e(w-u) \\ 0 & e(r-w) \end{pmatrix};$$

this involves the product

$$\begin{bmatrix} ev-j \\ k-j \end{bmatrix}\begin{bmatrix} e(r-v-u)+j \\ e(w-u)-k+j \end{bmatrix},$$

which is zero unless $k - j \equiv 0$ or $k \equiv 0 \pmod{e}$, by Lemma 2.2. Similarly, the product

$$\begin{bmatrix} eu-i \\ j-i \end{bmatrix}\begin{bmatrix} e(r-s-u)+i \\ e(v-s)-j+i \end{bmatrix}$$

is zero unless either $j - i \equiv 0$ or $j \equiv 0 \pmod{e}$. So in order to reduce the product using Lemma 2.2, we have four cases to consider:

- $i \equiv j \equiv k \equiv 0 \pmod{e}$;
- $i \not\equiv j \equiv k \equiv 0 \pmod{e}$;
- $i \equiv j \not\equiv k \equiv 0 \pmod{e}$;
- $i \equiv j \equiv k \not\equiv 0 \pmod{e}$.

replacing i with $ei + \iota$ where $0 \leqslant \iota < e$, and similarly for j and k, the above product becomes

$$\sum_i \sum_{j,k} \Bigg(\binom{k}{t}\binom{v-j}{k-j}\binom{r-v-u+j}{w-u-k+j}\binom{u-i}{j-i}\binom{r-s-u+i}{r-s-j+i} \begin{pmatrix} ei & e(s-i) \\ e(u-i) & e(r-s-u+i) \end{pmatrix}$$
$$+ \sum_{\iota=1}^{e-1} \Bigg(\bigg(\binom{k}{t}\binom{v-j}{k-j}\binom{r-v-u+j}{w-u-k+j}\binom{u-i-1}{j-i-1}\binom{r-s-u+i}{r-s-j+i}$$
$$+ \binom{k}{t}\binom{v-j-1}{k-j-1}\binom{r-v-u+j}{w-u-k+j}\binom{u-i-1}{j-i}\binom{r-s-u+i}{r-s-j+i}$$
$$+ \binom{k}{t}\binom{v-j-1}{k-j}\binom{r-v-u+j}{w-u-k+j}\binom{u-i-1}{j-i}\binom{r-s-u+i}{r-s-j+i}\bigg) \begin{pmatrix} ei+\iota & e(s-i)-\iota \\ e(u-i)-\iota & e(r-s-u+i)+\iota \end{pmatrix} \Bigg) \Bigg)$$

(since the exponent of q is divisible by e in all cases). This equals

$$\sum_{i,j,k} \binom{k}{t}\binom{v-j}{k-j}\binom{r-v-u+j}{w-u-k+j}\binom{u-i}{j-i}\binom{r-s-u+i}{r-s-j+i} \begin{pmatrix} ei & e(s-i) \\ e(u-i) & e(r-s-u+i) \end{pmatrix} \sum_{\iota=0}^{e-1} \begin{pmatrix} ei+\iota & e(s-i)-\iota \\ e(u-i)-\iota & e(r-s-u+i)+\iota \end{pmatrix},$$

where we treat any matrix with a negative entry as zero.

The theorem will now follow if we show that

$$\psi\left(\begin{pmatrix} i & s-i \\ u-i & r-s-u+i \end{pmatrix}\right) = \begin{cases} \sum_{\iota=0}^{e-1} \begin{pmatrix} ei+\iota & e(s-i)-\iota \\ e(s-u)-\iota & e(r-s-u+i)+\iota \end{pmatrix} & (su > 0) \\ \begin{pmatrix} ei & e(s-i) \\ e(u-i) & e(r-s-u+i) \end{pmatrix} & (su = 0). \end{cases}$$

But note that we have

$$\begin{aligned} \psi\Bigg(\sum_{i=\max(0,s+u-r)}^{\min(s,u)} \binom{r-s-u+i}{r-x} \begin{pmatrix} i & s-i \\ u-i & r-s-u+i \end{pmatrix} \Bigg) \\ &= \psi\left(\begin{pmatrix} s & 0 \\ x-s & r-x \end{pmatrix} \circ \begin{pmatrix} u & x-u \\ 0 & r-x \end{pmatrix}\right) \\ &= \begin{pmatrix} es & 0 \\ e(x-s) & e(r-x) \end{pmatrix} \circ \begin{pmatrix} eu & e(x-u) \\ 0 & e(r-x) \end{pmatrix} \\ &= \sum_{i=\max(0,s+u-r)}^{\min(s,u)} \sum_{\iota=0}^{e-1} \begin{bmatrix} e(r-s-u+i)+\iota \\ e(r-x) \end{bmatrix} \begin{pmatrix} ei+\iota & e(s-i)-\iota \\ e(u-i)-\iota & e(r-s-u+i)+\iota \end{pmatrix} \\ &= \sum_{i=\max(0,s+u-r)}^{\min(s,u)} \binom{r-s-u+i}{r-x} \sum_{\iota=0}^{e-1} \begin{pmatrix} i & s-i \\ u-i & r-s-u+i \end{pmatrix}, \end{aligned}$$

which gives the result. □

4.1 A generalisation to $n > 2$

We conjecture that the embedding of Theorem 4.2 embedding works for all n. Given $A \in M(r)$, write eA for the matrix with $(eA)_{ij} = e(A_{ij})$. Then we propose the following.

Conjecture 4.3. *For any n, r, there is an embedding $\psi : S(n,r) \hookrightarrow S_q(n, er)$ given by mapping the standard codeterminant $A \circ B$ to $(eA) \circ (eB)$.*

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