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# 2-chains: an interesting family of posets

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**Abstract**

We introduce a new family of finite posets which we call 2-*chains*. These first arose in the study of 0-Hecke algebras, but they admit a variety of different characterisations. We give these characterisations, prove that they are equivalent and derive some numerical results concerning 2-chains.

## 1 Introduction

In this paper we introduce a family of finite partially ordered sets which we call 2-*chains*. These are remarkable in that they can be described in several quite different ways. Some of these descriptions are by direct construction, and some by restrictive properties. 2-chains originally arose in the author's paper [F] on 0-Hecke algebras, where they appear as structure posets for certain induced modules for Hecke algebras of type A. Apart from this, however, 2-chains appear not to have been considered before. The fact that they admit so many different descriptions suggests that they should be studied further, and we hope that this paper is the start of an interesting new avenue in poset theory.

The author must express his gratitude to Jeremy Rickard. Early in this work, the author posted a question about these posets on *MathOverflow*, and Jeremy's answer [R] provided some key results which underpin a lot of this paper, and without which most of this work would not have been done. We indicate later exactly which parts are due to Jeremy.

We give a brief outline of the structure of the paper. We begin in Section 2 with a definition of 2-chains. In Section 3 we consider maximal elements, giving a reduction theorem which allows all 2-chains to be constructed recursively. In Section 4 we give a direct construction of posets from binary sequences, and show that these give all 2-chains. In Section 5 we introduce a method of joining posets together which we call *splicing*, and show that this can be used to join smaller 2-chains to make larger ones. In this way, we give another construction of all 2-chains as splices of certain simple 2-chains. In Section 6 we consider a connection with graphs: writing down a definition for graphs analogous to the definition of 2-chains leads to another characterisation of 2-chains, in terms of their incomparability graphs. In Section 7 we return to the connection with Coxeter groups of type A, showing how to characterise 2-chains in terms of Coxeter elements of symmetric groups. In Section 8 we show how to count covers and linear extensions in 2-chains. In Sections 9 and 10 we consider some possible generalisations.

## 2 2-chains

In this section we will introduce 2-chains. We begin by recalling some basic definitions for posets.

As usual, a *partial order* is a reflexive asymmetric transitive binary relation. A *partially ordered set* or *poset* is a set $P$ with a given partial order $\trianglelefteqslant$. We write $p \triangleleft q$ to mean $p \trianglelefteqslant q$ and $p \neq q$. Given $p, q \in P$ with $p \trianglelefteqslant q$, we say that $q$ is *above* $p$, and that $p$ is *below* $q$. We say that $q$ *covers* $p$ if $p \triangleleft q$ and there is no $r \in P$ with $p \triangleleft r \triangleleft q$. We say that $p, q$ are *incomparable* if $p \ntrianglelefteq q \ntrianglelefteq p$. The order $\trianglelefteqslant$ is a *total order* if for all $p, q \in P$ either $p \trianglelefteqslant q$ or $q \trianglelefteqslant p$. If $Q \subseteq P$, then $\trianglelefteqslant$ induces a partial order on $Q$, which will also be denoted $\trianglelefteqslant$. $Q$ is called:

- a *chain* if this induced partial order is a total order;
- an *antichain* if $p \ntriangleleft q$ for all $p, q \in Q$;
- an *ideal* (or *down-set*) in $P$ if $p \ntrianglelefteq q$ whenever $q \in Q$ and $p \in P \setminus Q$.

The *dual* poset $P^\circ$ has the same underlying set, with $p \trianglelefteqslant q$ in $P^\circ$ if and only if $q \trianglelefteqslant p$ in $P$.

A *refinement* of $\trianglelefteqslant$ is a partial order $\trianglelefteqslant^+$ such that $p \trianglelefteqslant^+ q$ whenever $p \trianglelefteqslant q$; if there is at least one pair $p, q$ with $p \trianglelefteqslant^+ q$ but $p \ntrianglelefteq q$, then $\trianglelefteqslant^+$ is a *proper* refinement of $\trianglelefteqslant$.

An element $p \in P$ is *maximal* if there is no $q \in P$ with $p \triangleleft q$. $p$ is the *greatest* element of $P$ if $q \trianglelefteqslant p$ for all $q \in Q$. *Minimal* and *least* elements are defined similarly.

In this paper, a partial order will almost always be written as $\trianglelefteqslant$, and if we say that $P$ is a poset without specifying the partial order, we mean $(P, \trianglelefteqslant)$ is a poset. We will use terms such as *chain*, *maximal* and *above* without explicit reference to a particular partial order, and it should be understood that we are referring to $\trianglelefteqslant$. If we use such terms with respect to any other partial order, we will be explicit about which order we are referring to.

An *isomorphism* between two posets $P$ and $R$ is a bijection $\phi : P \to R$ such that for $p, q \in P$ we have $p \trianglelefteqslant q$ if and only if $\phi(p) \trianglelefteqslant \phi(q)$. $P$ and $R$ are *isomorphic* (written $P \cong R$) if there is at least one isomorphism from $P$ to $R$. An *automorphism* of $P$ is an isomorphism from $P$ to $P$.

The *Hasse diagram* of a poset $P$ is a graph drawn in the plane with vertices corresponding to the elements of $P$, with an edge drawn from $p$ to $q$ with strictly increasing $y$-coordinate whenever $q$ covers $p$.

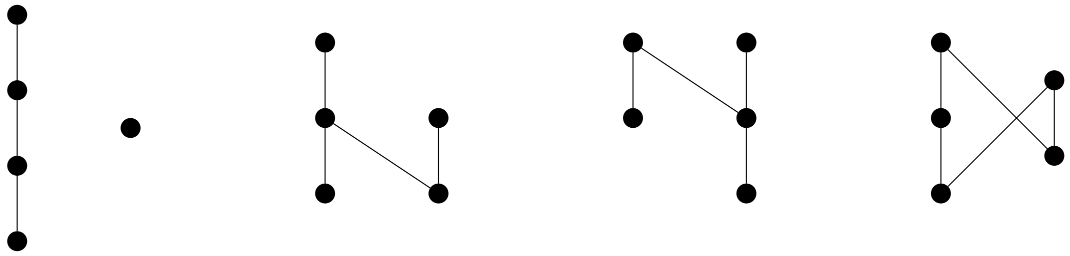
For most of this paper, we only consider *finite* posets. We make some comments on the infinite case in Section 10; until then, we take “poset” to mean “finite poset”.

Now we can give our main definition. Suppose $P$ is a poset. We say that $P$ is a 2-*chain* if the following two conditions are satisfied.

1. There is a unique way to write $P$ as the union of two chains.
2. $\trianglelefteqslant$ is maximal subject to (1), i.e. if $\trianglelefteqslant^+$ is a proper refinement of $\trianglelefteqslant$, then there is more than one way to write $P$ as the union of two $\trianglelefteqslant^+$-chains.

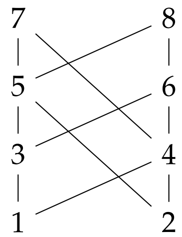
Note that when we say “union” in (1), we simply mean set union; so it is permissible to have an element of one chain lying above an element of the other.

There are four isomorphism classes of 5-element 2-chains, given by the following Hasse diagrams.



The first example can be generalised to give a 2-chain of arbitrary size: just take a poset consisting of an arbitrary chain $C$, together with an element $p$ not lying above or below any element of $C$.

For another class of 2-chains of arbitrary size, take $n$ to be a positive integer, set $\mathcal{Q}(n) = \{1, \ldots, n\}$, and define a partial order $\trianglelefteq$ on $\mathcal{Q}(n)$ by putting $i \triangleleft j$ if and only if $i \leqslant j - 2$ in the usual order on $\{1, \ldots, n\}$. For example, $\mathcal{Q}(8)$ has the following Hasse diagram.




**Proposition 2.1.** *If $n \in \mathbb{N}$, then $\mathcal{Q}(n)$ is a 2-chain.*

**Proof.** $\mathcal{Q}(n)$ can be expressed as the union of two chains, namely the set of even integers in $\mathcal{Q}(n)$ and the set of odd integers in $\mathcal{Q}(n)$. This is the unique such expression, since if $\mathcal{Q}(n)$ is expressed in any other way as the union of two subsets, then one of these subsets contains two consecutive integers and so is not a chain.

Now suppose $\trianglelefteq^+$ is a proper refinement of $\trianglelefteq$. Then there must be $i \in \{1, \ldots, n-1\}$ such that either $i \trianglelefteq^+ i+1$ or $i+1 \trianglelefteq^+ i$. Either way, we can find a new way to express $\mathcal{Q}(n)$ as the union of two $\trianglelefteq^+$-chains, namely

$$\{\ldots, i-4, i-2, i, i+1, i+3, i+5, \ldots\}$$

and

$$\{\ldots, i-3, i-1, i+2, i+4, \ldots\}.$$

This applies for any proper refinement $\trianglelefteq^+$, so $\mathcal{Q}(n)$ is a 2-chain. $\square$

We now proceed to give several other characterisations of 2-chains.

## 3 Maximal elements

In this section we consider maximal elements in 2-chains; this leads to some numerical results and a recursive construction for all 2-chains. Several of the results in this section are due to Jeremy Rickard.

First we introduce some terminology. Say that a maximal element in a poset is *supermaximal* if it lies above all the non-maximal elements.

**Lemma 3.1 (Rickard).** *Suppose $P$ is a 2-chain with $|P| \geqslant 3$. Then $P$ contains exactly two maximal elements, exactly one of which is supermaximal.*

For example, looking at the four 5-element 2-chains depicted in Section 2, we can see that each has exactly two maximal elements; in each diagram, the left-hand maximal element is supermaximal, but the right-hand one is not.

**Proof.** By definition $P$ is the union of two chains. No two maximal elements can lie in the same chain, so there can be at most two maximal elements in $P$. If there is only one maximal element $p$, then $p$ is the greatest element of $P$. But then there cannot be a unique way to express $P$ as the union of two chains, since if $B$ and $C$ are chains with $B \cup C = P$ and $p \in B$, then $B \setminus \{p\}$ and $C \cup \{p\}$ are also chains with union $P$.

So $P$ has exactly two maximal elements $p$ and $q$. We show by contradiction that exactly one of $p$ and $q$ is supermaximal.

If both $p$ and $q$ are supermaximal, then $P$ cannot be uniquely expressed as the union of two chains: indeed, if $P = B \cup C$ with $B, C$ chains, then without loss of generality we may assume that $p \in B$ and $q \in C$. But $p$ and $q$ lie above all other elements of $P$, so $B \setminus \{p\} \cup \{q\}$ and $C \setminus \{q\} \cup \{p\}$ are also chains whose union is $P$, so there is more than one way to write $P$ as the union of two chains, a contradiction.

So suppose instead that neither $p$ nor $q$ is supermaximal. This means that there are $r, s \in P \setminus \{p, q\}$ such that $r \ntrianglelefteq q$ and $s \ntrianglelefteq p$. Now let $P = B \cup C$ be the unique decomposition of $P$ as the union of two chains. Then $p$ and $r$ lie in one of these chains, say $B$, and $q$ and $s$ lie in the other. Let $\trianglelefteq^+$ be the refinement of $\trianglelefteq$ obtained by setting $r \trianglelefteq^+ q$ and extending transitively; that is,

$$a \trianglelefteq^+ b \quad \Longleftrightarrow \quad a \trianglelefteq b \text{ or } (a \trianglelefteq r \text{ and } b = q).$$

Since $\trianglelefteq^+$ is a proper refinement of $\trianglelefteq$, the definition of a 2-chain means there are at least two ways to write $P$ as the union of two $\trianglelefteq^+$-chains. One of these is $P = B \cup C$; let $P = D \cup E$ be another. Note that $p$ and $q$ are $\trianglelefteq^+$-incomparable, and so are $p$ and $s$ and also $r$ and $s$. So $p$ and $r$ lie in one of the two chains, say $D$, while $q$ and $s$ lie in the other. But now we claim that $D$ and $E$ are also $\trianglelefteq$-chains, which contradicts the assumption that $P$ can be uniquely written as the union of two $\trianglelefteq$-chains. Given two elements of $D$, we can write them as $a \trianglelefteq^+ b$, and since $b \neq q$, the definition of $\trianglelefteq^+$ means that $a \trianglelefteq b$. So $D$ is a $\trianglelefteq$-chain. Given two elements of $Q$, write them as $a \trianglelefteq^+ b$. Then the definition of $\trianglelefteq^+$ gives $a \trianglelefteq b$ except possibly in the case where $a \trianglelefteq r$ and $b = q$; so assume this is the case. Since $E$ is a $\trianglelefteq^+$-chain, $a$ must be $\trianglelefteq^+$-comparable with $s$, and hence $\trianglelefteq$-comparable with $s$. We cannot have $s \trianglelefteq a$, since then we would get $s \trianglelefteq a \trianglelefteq r \trianglelefteq p$, contrary to assumptions. So instead $a \trianglelefteq s$, and then $a \trianglelefteq q = b$, as required. So in any case $a \trianglelefteq b$, and $E$ is a $\trianglelefteq$-chain. $\square$

Now we give a reduction result which leads to a recursive construction of all 2-chains.

**Proposition 3.2. (Rickard).** *Suppose $P$ is a poset with $|P| \geqslant 3$. Suppose $P$ has exactly two maximal elements $p$ and $q$, and that $p$ is supermaximal. Then $P$ is a 2-chain if and only if $P \setminus \{p\}$ is a 2-chain.*

**Proof.** For this proof we write $P^- = P \setminus \{p\}$. First consider the case where $q$ is also supermaximal in $P$. Then neither $P$ nor $P^-$ is a 2-chain: $P^-$ fails to be a 2-chain because it has a greatest element $q$, so (as in the proof of Lemma 3.1) is not uniquely expressible as the union of two chains; $P$ fails to be a 2-chain by Lemma 3.1 because it has two supermaximal elements.

So we can assume that $q$ is not supermaximal in $P$, and we consider decompositions of $P$ and $P^-$ as unions of two chains. Suppose we can decompose $P$ as the union of two chains $B, C$, with $p \in B$ and $q \in C$. Then we can decompose $P^-$ as the union of two chains $B \setminus \{p\}$ and $C$. Conversely, if we can decompose $P^-$ as the union of two chains $D, E$ with $q \in E$, then $q \notin D$ (because $q$ is not supermaximal in $P$, and is therefore incomparable with some element of $P \setminus \{p\}$) so $P$ is the union of the two chains $D \cup \{p\}$ and $E$. Hence $P$ has a unique expression as the union of two chains if and only if $P^-$ does. So we can assume for the rest of the proof that $P$ and $P^-$ each have a unique decomposition as the union of two chains.

Suppose $P^-$ is not a 2-chain. Then there is a proper refinement $\trianglelefteq^+$ of $\trianglelefteq$ on $P^-$ such that $P^-$ has a unique expression as the union of two $\trianglelefteq^+$-chains. If we extend $\trianglelefteq^+$ to all of $P$ by setting $r \trianglelefteq^+ p$ for all $r \neq q$ and keeping $p, q$ incomparable, then $\trianglelefteq^+$ is a proper refinement of $\trianglelefteq$ on $P$. Moreover, $P$ is uniquely expressible as the union of two $\trianglelefteq^+$-chains (since if there were more than one such expression, there would be more than one such expression for $P^-$). So $P$ is not a 2-chain.

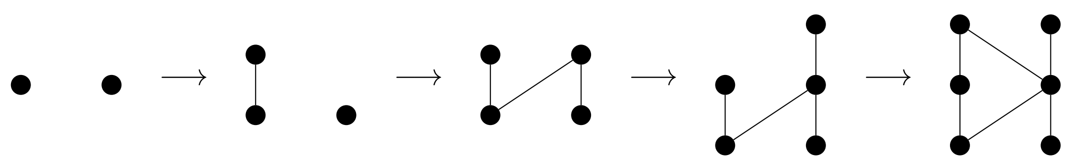
Conversely, suppose $P$ is not a 2-chain. Then there is a proper refinement $\trianglelefteq^+$ of $\trianglelefteq$ such that $P$ is uniquely expressible as the union of two $\trianglelefteq^+$-chains. This means in particular that $P$ must have two $\trianglelefteq^+$-maximal elements (namely $p$ and $q$), so the restriction of $\trianglelefteq^+$ to $P^-$ is a proper refinement of the restriction of $\trianglelefteq$ to $P^-$. Now $P^-$ is uniquely expressible as the union of two $\trianglelefteq^+$-chains because $P$ is, so $P^-$ is not a 2-chain. □

An immediate consequence is that we can determine all automorphisms of 2-chains.

**Proposition 3.3.** *Suppose $P$ is a 2-chain with $|P| \geqslant 3$. Then $P$ has no automorphisms except the identity.*

**Proof.** We use induction on $|P|$. The case where $|P| = 3$ is trivial to check, so assume $|P| \geqslant 4$. By Lemma 3.1 $P$ has a unique supermaximal element $p$. Clearly $p$ must be fixed by any automorphism of $P$. Hence any automorphism of $P$ restricts to an automorphism of $P \setminus \{p\}$. By Proposition 3.2 $P \setminus \{p\}$ is a 2-chain, so by induction $P \setminus \{p\}$ has no non-trivial automorphisms. Hence neither does $P$. □

Rickard has pointed out that Proposition 3.2 gives a way to construct all 2-chains recursively: starting from the unique 2-element 2-chain, we repeatedly add new supermaximal elements; at each step, we simply have to choose which of the two existing maximal elements should remain maximal. For example, we may construct a 6-element 2-chain by adding elements to form a sequence of 2-chains as follows.



We can make this precise in the following proposition, in which we determine the number of $n$-element 2-chains up to isomorphism.

**Proposition 3.4.** *Suppose $n \geqslant 3$. Then there are exactly $2^{n-3}$ 2-chains of size $n$ up to isomorphism.*

**Proof.** We use induction on $n$, with the case $n = 3$ being easy. Assuming $n \geqslant 4$, Proposition 3.2 shows that we have a function $P \mapsto P^-$ from the set of $n$-element 2-chains to the set of $(n-1)$-element 2-chains, defined by removing the unique supermaximal element from a 2-chain. By the induction hypothesis it suffices to show that given an $(n-1)$-element 2-chain $Q$, there are exactly two 2-chains $P$ up to isomorphism with $P^- = Q$. To reconstruct $P$ from $Q$, we just have to add a new element $p$ which is supermaximal in $P$, with $P$ having exactly one maximal element other than $p$. Since $Q$ has two maximal elements, we just choose which of these two elements will not lie below $p$ in $P$. So there are two possibilities for $P$; these are non-isomorphic, since any isomorphism from one to the other would restrict to a non-trivial automorphism of $Q$; but by Proposition 3.3 $Q$ has no non-trivial automorphisms. □

Next we show that the number of pairs of comparable elements in an $n$-element 2-chain depends only on $n$; this will be very useful in later sections.

**Proposition 3.5.** *Suppose $P$ is a 2-chain with $|P| = n$. Then there are exactly $\binom{n}{2} + 1$ pairs $(p, q)$ in $P$ with $p \trianglelefteq q$, and hence exactly $n-1$ pairs $(p, q)$ with $p \not\trianglelefteq q \not\trianglelefteq p$.*

**Proof.** Suppose $n \geqslant 3$. By Lemma 3.1 $P$ has a supermaximal element $p$ and one other maximal element $q$, and by Proposition 3.2 $P \setminus \{p\}$ is a 2-chain. By induction there are exactly $\binom{n-1}{2} + 1$ comparable pairs in $P \setminus \{p\}$; adding the supermaximal element $p$ adds $n-1$ comparable pairs (i.e. $r \trianglelefteq p$ for every $r \neq q$), giving $\binom{n}{2} + 1$ comparable pairs in $P$. □

We can also count ideals in 2-chains.

**Proposition 3.6.** *Suppose $P$ is a 2-chain with $n$ elements, and $1 \leqslant m < n$. Then $P$ has exactly two $m$-element ideals, and if $m \geqslant 2$ then exactly one of these two ideals is a 2-chain.*

**Proof.** We use induction on $n$, with the case $n = 2$ being trivial. Assuming $n \geqslant 3$, let $p$ be the unique supermaximal element of $P$. Then by Proposition 3.2 $P^- = P \setminus \{p\}$ is a 2-chain. Since $p$ is supermaximal and there is only one other maximal element $q$, there are $n-2$ elements $r \in P$ with $r \triangleleft p$. Hence $p$ cannot be contained in any ideal with fewer than $n-1$ elements. So if $m \leqslant n-2$, then any $m$-element ideal of $P$ is contained in $P^-$, and the result follows by the inductive hypothesis. So it remains to consider the case $m = n-1$. There are clearly two ideals with $n-1$ elements, namely $P^-$ and $P \setminus \{q\}$. Of these, $P^-$ is a 2-chain but $P \setminus \{q\}$ is not, because it contains only one maximal element, namely $p$. □

**Example.** Consider the 2-chain $\mathcal{Q}(n)$ introduced in Section 2. For $m \in \{1, \ldots, n-1\}$, the two $m$-element ideals of $\mathcal{Q}(n)$ are $\{1, \ldots, m\}$ and $\{1, \ldots, m-1\} \cup \{m+1\}$. Of these, only the first is a 2-chain.

In fact, there is a converse to Proposition 3.6, which provides another characterisation of 2-chains.

**Proposition 3.7.** *Suppose $P$ is an $n$-element poset, and that $P$ has exactly two $m$-element ideals, for each $m \in \{1, \ldots, n-1\}$. Then $P$ is a 2-chain.*

**Proof.** We use induction on $n$, and for the inductive step we assume $n \geqslant 3$. Since $P$ has exactly two $(n-1)$-element ideals, it has exactly two maximal elements $p, q$. We claim that at least one of them must be supermaximal. If not, then there are elements $r, s \in P \setminus \{p, q\}$ with $r \ntrianglelefteq q$ and $s \ntrianglelefteq p$. If we take $r, s$ to be maximal with these properties, then $P$ has three $(n-2)$-element ideals

$$P \setminus \{p, q\}, \qquad P \setminus \{p, r\}, \qquad P \setminus \{q, s\},$$

a contradiction. So at least one of $p, q$, say $p$, is supermaximal. As observed in the proof of Proposition 3.6, this means that every ideal with fewer than $n-1$ elements is contained in $P^- = P \setminus \{p\}$. So $P^-$ has exactly two $m$-element ideals, for each $m \in \{1, \ldots, n-2\}$, so by induction $P^-$ is a 2-chain. Hence by Proposition 3.2 $P$ is a 2-chain too. □

# 4 Two concrete constructions of 2-chains

Here we give a concrete construction of posets, which will turn out to be 2-chains and in fact to give all 2-chains up to isomorphism. This construction comes originally from the author's paper [F].

First we set out some notation for binary sequences. A *binary sequence* of length $n$ means a word $a = a_1 \ldots a_n$ in the alphabet $\{0, 1\}$. Given such a sequence $a$, we define $\bar{a} = \bar{a}_1 \ldots \bar{a}_n$, where $\bar{0} = 1$ and $\bar{1} = 0$. We say that two binary sequences $b = b_1 \ldots b_r$ and $c = c_1 \ldots c_r$ are *opposed* if for some $i, j$ we have $b_i > c_i$ and $b_j < c_j$, i.e. if the subsets of $\{1, \ldots, r\}$ corresponding to $b$ and $c$ are incomparable in the containment order.

Now we can give our constructions. We fix a binary sequence $a = a_1 \ldots a_n$ of length $n \geqslant 0$, and define $n+2$ sequences $a(0), \ldots, a(n+1)$ of length $n+1$ as follows:

$$\begin{aligned}
a(n+1) &= a_1\, a_2\, a_3 \ldots a_{n-1}\;\; a_n \;\; 0 \\
a(n) &= a_1\, a_2\, a_3 \ldots a_{n-1}\;\; 0 \;\; 1 \\
a(n-1) &= a_1\, a_2\, a_3 \ldots \;\; 0 \;\; 1 \;\; a_n \\
&\;\;\vdots \\
a(2) &= a_1\; 0\; 1 \ldots a_{n-2}\, a_{n-1}\, a_n \\
a(1) &= 0\; 1\; a_2 \ldots a_{n-2}\, a_{n-1}\, a_n \\
a(0) &= 1\; a_1\, a_2 \ldots a_{n-2}\, a_{n-1}\, a_n.
\end{aligned}$$

**Lemma 4.1 [F, Proposition 6.1].** *The sequences $a(0), \ldots, a(n+1)$ are distinct.*

**Proof.** If the sequences $a(i)$ and $a(j)$ are equal for $i < j$, then

$$1 = a_{i+1} = a_{i+2} = \cdots = a_{j-1} = 0,$$

a contradiction. □

In fact, there is another description of the set $\{a(0), \ldots, a(n+1)\}$.

**Lemma 4.2.** *$a(0), \dots, a(n+1)$ are precisely the sequences that can be obtained by inserting 0 or 1 at some point in $a$.*

**Proof.** First we observe that each $a(i)$ is obtained by inserting a 0 or a 1 in $a$. For $i = 0, n+1$ this is trivial, so suppose $1 \leqslant i \leqslant n$. If $a_i = 0$, then $a(i)$ is obtained from $a$ by inserting a 1 immediately after $a_i$, while if $a_i = 1$, then $a(i)$ is obtained from $a$ by inserting a 0 immediately before $a_i$.

Conversely, suppose $b$ is the sequence obtained by inserting $x \in \{0, 1\}$ immediately after position $i$. We assume $x = 0$, as the case $x = 1$ is similar. Let $j > i$ be maximal such that $a_{i+1} = \dots = a_{j-1} = 0$. Then $b = a(j)$; indeed, both sequences equal

$$a_1 \dots a_i 00 \dots 01 a_{j+1} \dots a_n$$

where there are $j - i$ 0s, and the 1 should be omitted if $j = n+1$. □

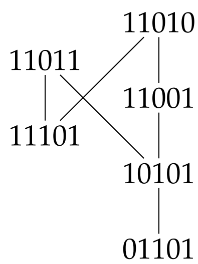
The description of the set $\{a(0), \dots, a(n+1)\}$ given by Lemma 4.2 is simpler than the original definition. However, the ordering $a(0), \dots, a(n+1)$ is significant for defining our partial order, which we do next.

Set $\mathcal{P}_a = \{a(0), \dots, a(n+1)\}$ and define a binary relation $\trianglelefteqslant$ on $\mathcal{P}_a$ by setting $a(i) \lhd a(j)$ whenever $i < j$ and $a(i)$ and $a(j)$ are opposed.

**Example.** Take $a = 1101$. Then we have

$$\begin{aligned}
a(5) &= 11010 \\
a(4) &= 11001 \\
a(3) &= 11011 \\
a(2) &= 10101 \\
a(1) &= 01101 \\
a(0) &= 11101
\end{aligned}$$

and the relation $\trianglelefteqslant$ is a partial order, with the following Hasse diagram.




It is not obvious from the definition that $\trianglelefteqslant$ is a partial order in general – in particular, that it is transitive. In fact this was shown in [F], where $\mathcal{P}_a$ is shown to be the "structure poset" of a multiplicity-free module for a 0-Hecke algebra. We will show it in a different (and purely combinatorial) way by giving a different definition of $\trianglelefteqslant$.

Keeping the binary sequence $a$ fixed, define a second list of binary sequences $a[0], \ldots, a[n+1]$ by

$$\begin{aligned}
a[n+1] &= a_1\, a_2\, a_3 \ldots a_{n-1}\;\; a_n \;\; 1 \\
a[n] &= a_1\, a_2\, a_3 \ldots a_{n-1}\;\; 1 \;\; 0 \\
a[n-1] &= a_1\, a_2\, a_3 \ldots\; 1 \;\; 0 \;\; a_n \\
&\vdots \\
a[2] &= a_1\; 1\; 0 \ldots a_{n-2}\, a_{n-1}\, a_n \\
a[1] &= 1\; 0\; a_2 \ldots a_{n-2}\, a_{n-1}\, a_n \\
a[0] &= 0\; a_1\, a_2 \ldots a_{n-2}\, a_{n-1}\, a_n.
\end{aligned}$$

**Lemma 4.3.** $\mathcal{P}_a = \{a[0], \ldots, a[n+1]\}$.

**Proof.** Lemma 4.2 says that $\mathcal{P}_a = \{a(0), \ldots, a(n+1)\}$ is the set of all sequences that can be obtained by inserting a symbol in $a$. But the latter description is symmetric in 0 and 1, and therefore the same applies to the set $\{a[0], \ldots, a[n+1]\}$. □

As a consequence of Lemma 4.3, we have a permutation $w_a$ of $\{0, \ldots, n+1\}$ defined by $a(i) = a[w_a(i)]$. This allows us to give a second description of the relation $\trianglelefteq$. First we give a simple description of the permutation $w_a$; we leave the proof as an exercise.

**Lemma 4.4.** *Suppose $i \in \{0, \ldots, n+1\}$.*

- *If $i = 0$ or $a_i = 0$, then $w_a(i)$ is the smallest $k > i$ such that $k = n+1$ or $a_k = 0$.*
- *If $i = n+1$ or $a_i = 1$, then $w_a(i)$ is the largest $k < i$ such that $k = 0$ or $a_k = 1$.*

**Proposition 4.5.** *Given $i, j \in \{0, \ldots, n+1\}$ we have $a(i) \trianglelefteq a(j)$ if and only if $i \leqslant j$ and $w_a(i) \leqslant w_a(j)$. Hence $\trianglelefteq$ is a partial order on $\mathcal{P}_a$.*

**Proof.** We need to show that for $0 \leqslant i < j \leqslant n+1$ the sequences $a(i)$ and $a(j)$ are opposed if and only if $w_a(i) < w_a(j)$. We consider several cases; in the remainder of this proof we read $a_0$ as 0 and $a_{n+1}$ as 1.

1. Suppose $a_i = a_j$. Then $a(i)$ and $a(j)$ are certainly opposed because they are different but have the same sum. Moreover, $w_a(i) < w_a(j)$, by Lemma 4.4.

2. Suppose $a_i = 1$ and $a_j = 0$. Then $a(i)$ and $a(j)$ are opposed, since

   $$a(i)_i = a(j)_k = 0, \qquad a(i)_k = a(j)_i = 1,$$

   where $k > i$ is minimal such that $a_k = 0$. Furthermore, $w_a(i) < i < j < w_a(j)$ by Lemma 4.4.

3. Finally suppose $a_i = 0$ and $a_j = 1$.

   - If $w_a(j) < i$, then certainly $w_a(i) > w_a(j)$. And in this case $a_{i+1} = \cdots = a_{j-1} = 0$ by Lemma 4.4, so that $a(i)$ and $a(j)$ differ only in the $(i+1)$th position and are not opposed.

- Similarly if $w_a(i) > j$, then $w_a(i) > w_a(j)$ and $a(i), a(j)$ are not opposed.
- If $w_a(i) = w_a(j) + 1$, then the sequence $a_i a_{i+1} \dots a_j$ has the form $011 \dots 1100 \dots 001$. So $a(i)$ and $a(j)$ are not opposed: they differ only in the $w_a(i)$th position.
- The remaining possibility is that $i < w_a(i) < w_a(j) < j$. In this case let $k = w_a(i)$, and let $l > k$ be minimal such that $a_l = 1$. Then by assumption $l < j$, so

$$a(i)_k = a(j)_l = 1, \qquad a(i)_l = a(j)_k = 0,$$

and $a(i)$ and $a(j)$ are opposed. □

Proposition 4.5 shows that $\trianglelefteqslant$ is the intersection of the two total orders

$$a(0) < \cdots < a(n+1) \qquad \text{and} \qquad a[0] < \cdots < a[n+1],$$

so the poset $\mathcal{P}_a$ has dimension 2 (in general, the *dimension* of a poset $P$ is the smallest $d$ such that the partial order on $P$ can be expressed as the intersection of $d$ total orders). In fact, this follows from a more general result of Pretzel [P, Theorem 1], which says that the dimension of a poset is no more than its width, i.e. the size of its largest antichain; clearly in a 2-chain the largest antichain has size 2.

In Section 7 we will examine the permutations $w_a$ in more detail.

**Corollary 4.6.** *There is an isomorphism*

$$\begin{aligned} \phi : \mathcal{P}_a &\longrightarrow \mathcal{P}_{\bar{a}} \\ a(i) &\longmapsto \overline{a(i)}. \end{aligned}$$

**Proof.** Notice that $\overline{a(i)} = \bar{a}[i]$ and $\overline{a[i]} = \bar{a}(i)$ for each $i$, so $\phi$ is certainly a bijection from $\mathcal{P}_a$ to $\mathcal{P}_{\bar{a}}$, and Proposition 4.5 guarantees that $\phi(a(i)) \trianglelefteqslant \phi(a(j))$ if and only if $a(i) \trianglelefteqslant a(j)$. □

We note in passing that this corollary yields a third way to describe the partial order on $\mathcal{P}_a$: we have $a[i] \triangleleft a[j]$ if and only if $i < j$ and $a[i]$ and $a[j]$ are opposed.

Now we prove the main result of this section.

**Theorem 4.7.** *Suppose $n \geqslant 0$.*

1. *If $a$ is a binary sequence of length $n$, then $\mathcal{P}_a$ is a 2-chain.*
2. *If $P$ is an $(n+2)$-element 2-chain, then $P \cong \mathcal{P}_a$ for some binary sequence $a$.*
3. *If $a, b$ are binary sequences with $\mathcal{P}_a \cong \mathcal{P}_b$, then $b = a$ or $b = \bar{a}$.*

**Proof.**

1. We use induction on $n$, and Proposition 3.2. The case $n = 0$ is trivial, so take $n \geqslant 1$. Using Corollary 4.6 we can assume that $a_n = 1$. First we claim that $\mathcal{P}_a$ has exactly two maximal elements $a(n+1)$ and $a[n+1]$, with $a(n+1)$ being supermaximal. Proposition 4.5 certainly shows that $a(n+1)$ and $a[n+1]$ are maximal. Furthermore, the assumption $a_n = 1$ means that $a(n+1) = a[n]$. So (again using Proposition 4.5) $a(n+1)$ lies above every

element of $\mathcal{P}_a$ except $a[n+1]$. So $a(n+1)$ is supermaximal, and there are no maximal elements other than $a(n+1)$ and $a[n+1]$.

Now define $a^-$ to be the binary sequence $a_1 \ldots a_{n-1}$. Let $\mathcal{P}_a^- = \{a(0), \ldots, a(n)\}$, with the partial order $\trianglelefteq$ induced from $\mathcal{P}_a$. Then we observe that $\mathcal{P}_a^-$ is isomorphic to $\mathcal{P}_{a^-}$: each of the sequences $a(0), \ldots, a(n)$ ends in a 1, and deleting this 1 from each sequence yields the sequences $a^-(0), \ldots, a^-(n)$, preserving opposedness. So by induction $\mathcal{P}_a^-$ is a 2-chain, and hence by Proposition 3.2 so is $\mathcal{P}_a$.

2. Again we use induction on $n$, and the case $n = 0$ is trivial. By Lemma 3.1 $P$ has one supermaximal element $p$, and one other maximal element $q$. Let $P^- = P \setminus \{p\}$. Then by Proposition 3.2 $P^-$ is a 2-chain, so by induction there is an isomorphism $P^- \cong \mathcal{P}_{a^-}$ for some binary sequence $a^-$ of length $n-1$. Under this isomorphism, the two maximal elements of $P^-$ (one of which is $q$) map to $a^-(n)$ and $a^-[n]$. We suppose $q$ maps to $a^-[n]$ (the other case is similar). Define a sequence $a$ by adding a 1 at the end of $a^-$; then we claim that $P \cong \mathcal{P}_a$. As explained in the proof of (1), $a(n+1)$ is a supermaximal element of $\mathcal{P}_a$, and $\mathcal{P}_a \setminus \{a(n+1)\} \cong \mathcal{P}_{a^-}$, and so there is an isomorphism $\theta : \mathcal{P}_a \setminus \{a(n+1)\} \to P^-$, with $\theta(a[n+1]) = q$. Since $a(n+1)$ is incomparable with $a[n+1]$ in $\mathcal{P}_a$ and lies above every other element, and $p$ is incomparable with $q$ in $P$ but above every other element, we can extend $\theta$ by setting $\theta(a(n+1)) = p$ to give an isomorphism from $\mathcal{P}_a$ to $P$.

3. Again, we use induction on $n$, and assume here that $n \geqslant 2$. Let $\phi : \mathcal{P}_a \to \mathcal{P}_b$ be an isomorphism; in fact, by Proposition 3.3 $\phi$ is the unique isomorphism from $\mathcal{P}_a$ to $\mathcal{P}_b$. Then $\phi$ must map the supermaximal element of $\mathcal{P}_a$ to the supermaximal element of $\mathcal{P}_b$, and the other maximal element of $\mathcal{P}_a$ to the other maximal element of $\mathcal{P}_b$. Arguing as in the proof of (1) above, we see that the supermaximal element of $\mathcal{P}_a$ is $a_1 \ldots a_n \overline{a_n}$, and the other maximal element is $a_1 \ldots a_n a_n$. A similar statement holds for $\mathcal{P}_b$, so $\phi$ maps $a_1 \ldots a_n a_n$ to $b_1 \ldots b_n b_n$.

Let $\mathcal{P}_a^-$ be the 2-chain obtained by deleting the supermaximal element of $\mathcal{P}_a$, and define $\mathcal{P}_b^-$ similarly. Then (as in the proof of (1), interchanging 0 and 1 if necessary) $\mathcal{P}_a^- \cong \mathcal{P}_{a^-}$, with the unique isomorphism $\theta : \mathcal{P}_a^- \to \mathcal{P}_{a^-}$ defined by deleting the last digit $a_n$ from the end of each element of $\mathcal{P}_a^-$. Similarly, there is a unique isomorphism $\kappa : \mathcal{P}_b^- \to \mathcal{P}_{b^-}$, given by deleting the digit $b_n$ at the end of each element of $\mathcal{P}_b^-$.

Now we have an isomorphism $\kappa \circ \phi \circ \theta^{-1} : \mathcal{P}_{a^-} \to \mathcal{P}_{b^-}$ which maps $a$ to $b$. By induction we see that $b^-$ equals either $a^-$ or $\overline{a^-}$. But the unique isomorphism from $\mathcal{P}_{a^-}$ to $\mathcal{P}_{a^-}$ maps $a$ to $a$, while the unique isomorphism from $\mathcal{P}_{a^-}$ to $\mathcal{P}_{\overline{a^-}}$ maps $a$ to $\bar{a}$ by Corollary 4.6. So $b$ must equal either $a$ or $\bar{a}$. □

Theorem 4.7 allows us to label 2-chains in a canonical way: each 2-chain can be written as $\mathcal{P}_a$ for some binary sequence $a$ which is unique up to replacing $a$ with $\bar{a}$. Note that given Proposition 3.4 and Corollary 4.6, any two parts of Theorem 4.7 imply the other part. But proving all three parts directly yields a new proof of Proposition 3.4.

Now we consider duality. Let us write $P^\circ$ for the poset dual to $P$. It is obvious from the definition in Section 2 that the dual of a 2-chain is again a 2-chain. Using the labelling for 2-chains provided by Theorem 4.7, we can be more specific.

**Proposition 4.8.** *Suppose $a$ is a binary sequence of length $n$, and let $\operatorname{rev} a$ be the binary sequence $a_n \dots a_1$. Then $\mathcal{P}_a^\circ \cong \mathcal{P}_{\operatorname{rev} a}$.*

**Proof.** This follows from Proposition 4.5, which says that $\trianglelefteqslant$ is the intersection of the total orders

$$a(0) < \cdots < a(n+1) \qquad \text{and} \qquad a[0] < \cdots < a[n+1]$$

on $\mathcal{P}_a$. Hence the partial order on $\mathcal{P}_a^\circ$ is the intersection of the total orders

$$a(n+1) < \cdots < a(0) \qquad \text{and} \qquad a[n+1] < \cdots < a[0].$$

But now observe that $(\operatorname{rev} a)(i) = \operatorname{rev}(a[n+1-i])$ and $(\operatorname{rev} a)[i] = \operatorname{rev}(a(n+1-i))$ for each $i$. So we have an isomorphism from $\mathcal{P}_a^\circ$ to $\mathcal{P}_{\operatorname{rev} a}$ given by mapping $a(i) \mapsto \operatorname{rev}(a(i))$ for each $i$. $\square$

As a consequence of Proposition 4.8, we see that a 2-chain $P = \mathcal{P}_a$ is self-dual if and only if $\operatorname{rev} a$ equals either $a$ or $\bar{a}$. Clearly a self-duality preserves the unique decomposition of $P$ as a union of two chains; one can show that the self-duality preserves the two chains if $\operatorname{rev} a = a$, and interchanges them if $\operatorname{rev} a = \bar{a}$.

## 5 Splicing 2-chains

In this section we give a simple way of joining two 2-chains together to create a larger one. This yields a further construction of all 2-chains, starting from a family of "indecomposable" 2-chains.

We start with a more general definition. Suppose $P$ and $Q$ are posets, and that $P$ has exactly two maximal elements $p_0, p_1$, with only $p_0$ being supermaximal, and suppose $Q$ has exactly two minimal elements $q_0, q_1$ with only $q_0$ being superminimal (i.e. lying below every non-minimal element). We define a new poset $P \boxslash Q$ called the *splice* of $P$ and $Q$. Informally, this is defined by placing $Q$ above $P$, and identifying $p_0$ with $q_0$ and $p_1$ with $q_1$. Formally, we start by defining a partial order on the disjoint union $P \sqcup Q$ via the following rules:

- if $a, b \in P$, then $a \trianglelefteqslant b$ in $P \sqcup Q$ if and only if $a \trianglelefteqslant b$ in $P$;
- if $a, b \in Q$, then $a \trianglelefteqslant b$ in $P \sqcup Q$ if and only if $a \trianglelefteqslant b$ in $Q$;
- if $a \in P$ and $b \in Q$, then $a \trianglelefteqslant b$ if and only if for some $i \in \{0, 1\}$ we have $a \trianglelefteqslant p_i$ in $P$ and $q_i \trianglelefteqslant b$ in $Q$;
- if $a \in Q$ and $b \in P$, then $a \ntrianglelefteq b$.

Now define $P \boxslash Q$ to be the quotient poset obtained by identifying $p_0$ with $q_0$ and $p_1$ with $q_1$. Note that $P$ and $Q$ are then naturally subposets of $P \boxslash Q$.

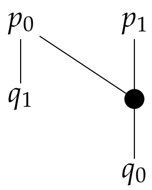
Now we consider the special case of 2-chains. Recall that a 2-chain of size at least 3 has exactly two maximal elements, with exactly one being supermaximal. Dually, a 2-chain of size at least 3 has exactly two minimal elements, with exactly one being superminimal. Hence the splice of two 2-chains each with at least three elements is well-defined. In fact we can easily extend this definition to include the case where either or both of the posets is the unique 2-element 2-chain $\mathcal{P}_\varnothing$, even though both elements of $\mathcal{P}_\varnothing$ are both supermaximal and superminimal. We then get $P \boxbslash \mathcal{P}_\varnothing = P$ and $\mathcal{P}_\varnothing \boxbslash Q = Q$.

**Proposition 5.1.** *If $P$ and $Q$ are 2-chains, then so is $P \boxbslash Q$.*

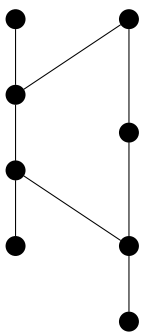
**Proof.** Since $P$ and $Q$ can each be expressed as the union of two chains, so can $P \boxbslash Q$: we take the union of the chain in $P$ containing $p_0$ and the chain in $Q$ containing $q_0$ to give one chain, and do the same for $p_1, q_1$ to give the other. Moreover, this is the unique way to express $P \boxbslash Q$ as the union of two chains, since any other such expression would restrict to give a new expression for either $P$ or $Q$ as the union of two chains.

Now suppose $\trianglelefteq^+$ is a proper refinement of $\trianglelefteq$ on $P \boxbslash Q$. Note that for every $p \in (P \boxbslash Q) \setminus Q$ and $q \in (P \boxbslash Q) \setminus P$ we already have $p \trianglelefteq q$, since $p \trianglelefteq p_0$ in $P$ and $q_0 \trianglelefteq q$ in $Q$. So in order for $\trianglelefteq^+$ to be a proper refinement of $\trianglelefteq$ on $P \boxbslash Q$, we must have $a \trianglelefteq^+ b$ but $a \ntrianglelefteq b$ either for some $a, b \in P$ or for some $a, b \in Q$. We assume the former case; then the restriction of $\trianglelefteq^+$ to $P$ is a proper refinement of $\trianglelefteq$ on $P$. Since $P$ is a 2-chain, this means that there are at least two different ways to express $P$ as the union of two $\trianglelefteq^+$-chains. But now (via the construction in the first paragraph of the proof) there are at least two ways to express $P \boxbslash Q$ as the union of two $\trianglelefteq^+$-chains. $\square$

**Example.** Take $P = Q = \mathcal{P}_{001}$. Then the Hasse diagram of $P$ with the elements $p_0, p_1, q_0, q_1$ marked is as follows.




Hence the splice $P \boxbslash Q$ is given by the following diagram.



We can make Proposition 5.1 more explicit using the labelling for 2-chains introduced in Section 4. Given binary sequences $a, b$, write $a|b$ for their concatenation.

**Proposition 5.2.** *Suppose $a = a_1 \dots a_r$ and $b = b_1 \dots b_s$ are binary sequences, and define*

$$c = \begin{cases} a|b & \text{if } a_r = b_1 \\ a|\bar{b} & \text{if } a_r \neq b_1. \end{cases}$$

*Then* $\mathcal{P}_a \boxslash \mathcal{P}_b \cong \mathcal{P}_c$.

(Note that we include the case where $r$ or $s$ equals 0; in this case the condition $a_r = b_1$ doesn't make sense, but the conclusion is trivially true regardless.)

**Proof.** By replacing $b$ with $\bar{b}$ if necessary and using the fact that $\mathcal{P}_b \cong \mathcal{P}_{\bar{b}}$, we can assume that $a_r = b_1$. In fact, by Corollary 4.6 we can assume that $a_r = 0 = b_1$. Then the supermaximal element of $\mathcal{P}_a$ is $a|1 = a[r+1]$, with the other maximal element being $a|0 = a(r+1)$. Similarly, the superminimal element of $\mathcal{P}_b$ is $1|b = b(0)$, and the other minimal element is $0|b = b[0]$.

We define an injective function

$$\begin{aligned} \phi_b : \mathcal{P}_a &\longrightarrow \mathcal{P}_{a|b} \\ a[i] &\longmapsto (a|b)[i]. \end{aligned}$$

This function can be more simply described as mapping $p \mapsto p|b$ for each $p \in \mathcal{P}_a$, which shows that it is order-preserving.

We also define an injective function

$$\begin{aligned} \phi^a : \mathcal{P}_b &\longrightarrow \mathcal{P}_{a|b} \\ b(i) &\longmapsto (a|b)(r+i). \end{aligned}$$

This function can be more simply described as mapping $q \longmapsto a|q$, which shows that it too is order-preserving.

The images of $\phi_b$ and $\phi^a$ intersect in the two points

$$\phi_b(a|0) = a|0|b = \phi^a(0|b)$$

and

$$\phi_b(a|1) = a|1|b = \phi^a(1|b).$$

Since $a|0$ is supermaximal in $\mathcal{P}_a$ and $0|b$ is superminimal in $\mathcal{P}_b$, we get

$$p \trianglelefteqslant a|0|b \trianglelefteqslant q$$

for all $p \in \operatorname{im}\phi_b \setminus \{a|1|b\}$ and $q \in \operatorname{im}\phi^a \setminus \{a|1|b\}$. Putting this together with the partial orders on $\operatorname{im}\phi_b$ and $\operatorname{im}\phi^a$, we see that $\mathcal{P}_{a|b} \cong \mathcal{P}_a \boxslash \mathcal{P}_b$. □

As a consequence of Proposition 5.2, we can see that whenever a binary sequence $a$ contains two consecutive equal entries, $\mathcal{P}_a$ can be expressed as a splice of two smaller 2-chains. This means that we can construct all 2-chains from the 2-chains of the form $\mathcal{P}_{1010\dots}$. In fact, we have already seen these 2-chains constructed in a very simple way: they are the posets $\mathcal{Q}(n)$ defined in Section 2, as we now show.

**Proposition 5.3.** *Suppose $a$ is the alternating binary sequence* $1010\dots$ *of length $n$. Then* $\mathcal{P}_a \cong \mathcal{Q}(n+2)$.

**Proof.** Suppose $0 \leqslant i < j \leqslant n+1$. We can check that $a(i)$ and $a(j)$ are opposed unless (and only unless) one of the following happens:

- $i$ is even and $j = i+1$;
- $i$ is even and $j = i+3$;
- $n$ is odd, $i = n-1$ and $j = n+1$.

Hence we can define an isomorphism from $\mathcal{P}_a$ to $\mathcal{Q}(n+2)$ by mapping

$$a(i) \longmapsto \begin{cases} i & \text{if } i \text{ is odd} \\ i+2 & \text{if } i \text{ is even and } i \neq n+1 \\ i+1 & \text{if } i \text{ is even and } i = n+1. \end{cases}$$ □

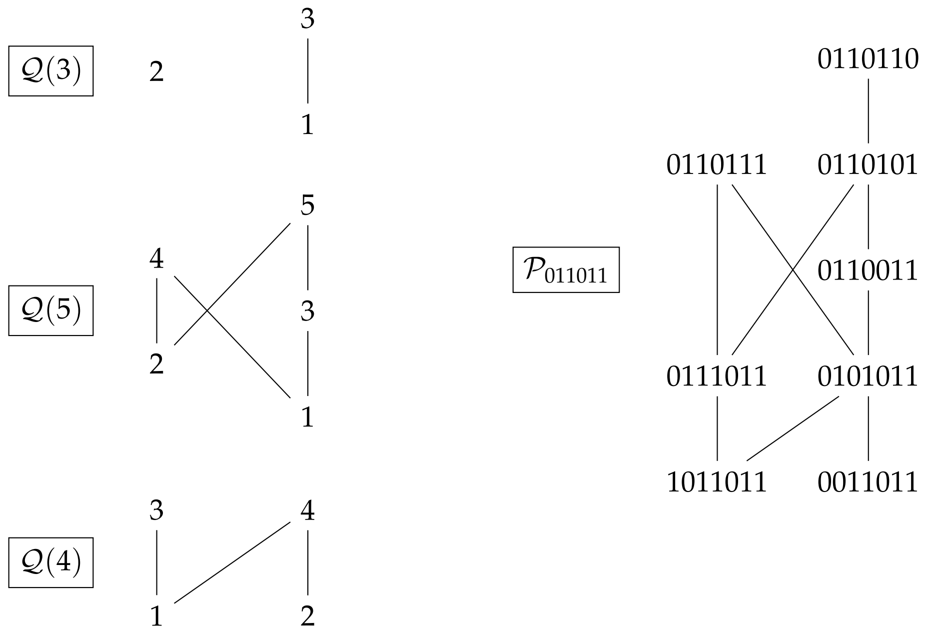
As a consequence, we deduce the following.

**Proposition 5.4.** *Suppose $P$ is a 2-chain with $n$ elements, where $n \geqslant 3$. Then $P$ is isomorphic to $\mathcal{Q}(n_1) \boxbslash \cdots \boxbslash \mathcal{Q}(n_s)$ for a unique choice of $n_1, \ldots, n_s \geqslant 3$ with $n_1 + \cdots + n_s = n + 2s - 2$.*

**Proof.** The existence of such a decomposition comes from Propositions 5.2 and 5.3: writing $P \cong \mathcal{P}_a$ for a binary sequence $a$, we break $a$ into subsequences $a^{(1)}, \ldots, a^{(s)}$ where each $a^{(i)}$ is an alternating sequence $\ldots 0101 \ldots$, and the last term of $a^{(i)}$ equals the first term of $a^{(i+1)}$ for each $i$. Then $n_1, \ldots, n_s$ are just the lengths of the sequences $a^{(1)}, \ldots, a^{(s)}$.

For the uniqueness, it suffices to use Proposition 3.4 and to count the number of possible expressions $n_1 + \cdots + n_s = n + 2s - 2$, which is a simple exercise. □

**Example.** The 2-chain $\mathcal{P}_{011011}$ is isomorphic to $\mathcal{Q}(4) \boxbslash \mathcal{Q}(5) \boxbslash \mathcal{Q}(3)$, as we see from the following Hasse diagrams.



# 6 Graphs

In this section we consider an analogue of 2-chains in the context of graphs. We consider only finite graphs. Let's say that a graph $G = (V, E)$ is a 2-*clique* if $V$ can be uniquely expressed as the union of two cliques (i.e. complete subgraphs) and $G$ is maximal with this property: adding any edge to $E$ breaks the uniqueness.

In fact, 2-cliques are easy to understand, by looking at complements. A graph is a 2-clique if and only if its complement is bipartite with a unique bipartition, and is minimal with this property (i.e. removing any edge breaks the uniqueness). Being bipartite with a unique bipartition is the same as being connected and bipartite, and a minimal connected (bipartite) graph is a tree, so a 2-clique is simply the complement of a tree.

However, 2-cliques do have a direct relationship with 2-chains. The *comparability graph* of a poset $P$ has vertex set $P$, with an edge from $p$ to $q$ if and only if either $p \lhd q$ or $q \lhd p$. Our main result here is the following, which gives yet another characterisation of 2-chains. The author is grateful to an anonymous referee for suggesting the proof of this result, which is simpler than the proof in the original version of this paper.

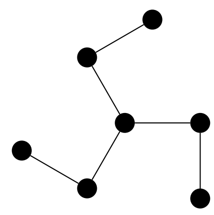
**Proposition 6.1.** *Suppose $P$ is a poset. Then $P$ is a* 2*-chain if and only if its comparability graph is a* 2*-clique.*

**Proof.** Let $G$ be the *in*comparability graph of $P$, in which there is an edge from $p$ to $q$ if and only if $p \ntrianglelefteq q \ntrianglelefteq p$. From the above discussion, we need to show that $P$ is a 2-chain if and only if $G$ is a tree. Let $n = |P|$, and proceed by induction on $n$, with the case $n = 2$ being trivial.

First consider the number of maximal elements of $P$. If $P$ has only one maximal element $p$, then by Lemma 3.1 $P$ is not a 2-chain and $G$ is not a tree, because $p$ is an isolated vertex of $G$. If $P$ has three (or more) maximal elements, then by Lemma 3.1 $P$ is not a 2-chain, and $G$ is not a tree, since it contains a triangle.

So we can assume that $P$ has exactly two maximal elements. Furthermore, we can assume that one of these maximal elements is supermaximal: if not, then by Lemma 3.1 $P$ is not a 2-chain, and $G$ contains a 4-cycle, so is not a tree. So let $p$ be a supermaximal element of $P$, let $P^- = P \setminus \{p\}$, and let $G^-$ be the incomparability graph of $P^-$. Since $p$ is supermaximal, there is only one element of $P$ incomparable with $p$ in $P$, so $G$ is obtained from $G^-$ by adding a leaf. So $G$ is a tree if and only if $G^-$ is a tree, and by Proposition 3.2 $P$ is a 2-chain if and only if $P^-$ is a 2-chain, so the result follows by induction. $\square$

One can ask which trees can occur as the incomparability graphs of posets (and hence of 2-chains). The general question of which graphs are incomparability graphs of posets was answered completely by Gallai [G]. His theorem gives an explicit list of graphs, and says that a graph is the incomparability graph of a poset if and only if it has no induced subgraph in the given list. We are interested in the special case of trees; for this, we just check that the only forest in Gallai's list is the *triskelion*



which leads to the following result.

**Theorem 6.2 (Gallai).** *Suppose G is a tree. Then G is the incomparability graph of a poset if and only if G is a* caterpillar*, i.e. if the non-leaves in G form a path.*

Proposition 6.1 and Theorem 6.2 show that the incomparability graph of a 2-chain is a caterpillar, and that every caterpillar arises as the incomparability graph of a 2-chain. We now show that this 2-chain is unique up to duality.

**Proposition 6.3.** *Suppose G is a caterpillar. Then G is the incomparability graph of a* 2*-chain P, which is unique up to duality.*

**Proof.** We use a counting argument. Sending a poset to its incomparability graph defines a function from (isomorphism classes of) 2-chains to caterpillars, and Proposition 6.1 and Theorem 6.2 show that this function is surjective. Moreover, dual 2-chains obviously map to the same caterpillar. So it suffices to show that the number of dual pairs of $n$-element 2-chains equals the number of isomorphism types of caterpillars with $n$ vertices. Proposition 4.8 and the discussion preceding Proposition 3.4 show that the former number is the number of binary sequences of length $n-2$ modulo reversal and modulo interchanging 0s and 1s; an easy exercise shows that this number is $2^{n-4} + 2^{\lfloor (n-4)/2 \rfloor}$. [HS, Theorem 2.1] shows that this is also the number of isomorphism types of caterpillars on $n$ vertices. $\square$

By way of example, we consider two special cases of caterpillars. The first example is the *star*, where one vertex is attached to all the others. This is the incomparability graph of the 2-chain consisting of a chain and an isolated point. The second example is the *path* with $n$ vertices: this is the incomparability graph of the 2-chain $\mathcal{Q}(n)$ introduced in Section 2.

# 7 Coxeter elements in the symmetric group

In this section we give yet another construction of 2-chains, relating to permutations. Recall from Section 4 the second construction of the poset $\mathcal{P}_a$: we define a permutation $w_a$ of $\{0, \dots, n+1\}$ by $a(i) = a[w_a(i)]$; then we have $a(i) \trianglelefteq a(j)$ in $\mathcal{P}_a$ if $i \leqslant j$ and $w_a(i) \leqslant w_a(j)$. Our focus in this section is on the permutations $w_a$: we consider the analogous definition for an arbitrary permutation, and determine exactly which permutations yield 2-chains.

We recall some basic Coxeter theory of the symmetric group (for more details see the book by Humphreys [H]). Let $\mathfrak{S}_n$ denote the group of all permutations of $\{1, \dots, n\}$. We write $s_1, \dots, s_{n-1}$ for the *Coxeter generators* of $\mathfrak{S}_n$; here $s_i$ is the transposition $(i, i+1)$. Any permutation $w \in \mathfrak{S}_n$ can be written in the form $s_{i_1} \dots s_{i_l}$ for some $i_1, \dots, i_l$; the smallest $l$ for which we can do this is called the *length* $l(w)$. The length of $w$ can be also be defined as the number of *inversions* of $w$, i.e. pairs $i < j$ such that $w(i) > w(j)$.

A *Coxeter element* of $\mathfrak{S}_n$ is an element which can be written as a product of the Coxeter generators, each appearing exactly once, in some order. Alternatively, a Coxeter element may be described as an element $w \in \mathfrak{S}_n$ such that for every $r \neq 1, n$, one of $w(r)$, $w^{-1}(r)$ is greater than $r$ and the other is less than $r$.

**Example.** The Coxeter elements in $\mathfrak{S}_4$ are the 4-cycles

$$(1,2,3,4), \qquad (1,3,4,2), \qquad (1,2,4,3), \qquad (1,4,3,2).$$

Coxeter elements are defined for all finite Coxeter groups, and play an important role in Coxeter theory. For the symmetric group, we will need the following result.

**Proposition 7.1.** *Suppose $n \geqslant 2$. Then there are exactly $2^{n-2}$ Coxeter elements in $\mathfrak{S}_n$, and they all have length $n-1$ and order $n$. In particular, if $n \geqslant 3$, then no Coxeter element of $\mathfrak{S}_n$ is self-inverse.*

The number of Coxeter elements in $\mathfrak{S}_n$ is a special case of a theorem of Shi [S, Theorem 1.5]. The order of the Coxeter element $s_1 \dots s_{n-1}$ is easily seen to be $n$; since (as proved by Coxeter himself) the Coxeter elements are all conjugate, they all have the same order.

Now we consider posets defined by permutations. Given $w \in \mathfrak{S}_n$, define $\mathcal{R}_w$ to be the set $\{1, \dots, n\}$, with $i \trianglelefteq j$ if and only if $i \leqslant j$ and $w(i) \leqslant w(j)$. Note that if $w \in \mathfrak{S}_n$ then $\mathcal{R}_w$ and $\mathcal{R}_{w^{-1}}$ are isomorphic: in fact $w$ is itself an isomorphism from $\mathcal{R}_w$ to $\mathcal{R}_{w^{-1}}$.

Our main result in this section is the following.

**Theorem 7.2.**

1. *Suppose $w \in \mathfrak{S}_n$. Then $\mathcal{R}_w$ is a 2-chain if and only if $w$ is a Coxeter element of $\mathfrak{S}_n$.*

2. *Every $n$-element 2-chain is isomorphic to $\mathcal{R}_w$ for some Coxeter element $w \in \mathfrak{S}_n$.*

3. *Suppose $v$ and $w$ are Coxeter elements of $\mathfrak{S}_n$. Then $\mathcal{R}_v \cong \mathcal{R}_w$ if and only if $v \in \{w, w^{-1}\}$.*

**Proof.** We assume $n \geqslant 3$. We begin with the "only if" part of (1). The construction of $\mathcal{R}_w$ means that the number of pairs of incomparable elements in $\mathcal{R}_w$ is $l(w)$. So by Proposition 3.5 $\mathcal{R}_w$ can only be a 2-chain if $l(w) = n-1$. If $l(w) = n-1$ but $w$ is not a Coxeter element, then $w$ can be written as a product of Coxeter generators with one Coxeter generator, say $s_m$, not occurring. This means that $w$ is contained in the *Young subgroup* $\mathfrak{S}_{(m,n-m)}$: this is the subgroup of $\mathfrak{S}_n$ consisting of all permutations $v \in \mathfrak{S}_n$ such that $v(\{1, \dots, m\}) = \{1, \dots, m\}$. So in $\mathcal{R}_w$ we have $i \trianglelefteq j$ whenever $i \leqslant m < j$. But now the incomparability graph of $\mathcal{R}_w$ is disconnected, so by Proposition 6.1 $\mathcal{R}_w$ is not a 2-chain.

So $\mathcal{R}_w$ is a 2-chain only if $w$ is a Coxeter element. Proposition 4.5 and Theorem 4.7 show that every 2-chain is isomorphic to $\mathcal{R}_w$ for some permutation $w$, which must therefore be a Coxeter element, so (2) is proved. Now consider the function $\psi : w \mapsto \mathcal{R}_w$ from the set of Coxeter elements of $\mathfrak{S}_n$ to the set of isomorphism classes of $n$-element posets. We have just seen that that the image of $\psi$ contains all $n$-element 2-chains, and therefore this image has size at least $2^{n-3}$, by Proposition 3.4. On the other hand, since $w \neq w^{-1}$ but $\psi(w) = \psi(w^{-1})$ for each Coxeter element $w$, the image of $\psi$ has size at most $2^{n-3}$, by Proposition 7.1. So equality holds everywhere; hence the image of $\psi$ is precisely the set of $n$-element 2-chains (which is the "if" part of (1)), and different inverse pairs of Coxeter elements map to different 2-chains (which gives (3)). $\square$

## 8 Covers and linear extensions

In this section we answer a question posed by one of the referees for this paper, showing how to count covers and linear extensions of a 2-chain.

First we count covers in a 2-chain $P$, i.e. pairs $a, b \in P$ such that $b$ covers $a$. Our main result is the following.

**Proposition 8.1.** *Suppose $P$ is a 2-chain of size $n$, and write $P = \mathcal{Q}(n_1) \boxslash \cdots \boxslash \mathcal{Q}(n_s)$ for $n_1, \ldots, n_s \geqslant 3$ with $n_1 + \cdots + n_s = n + 2s - 2$. Then the number of covers in $P$ is $2n - s - 4$.*

**Proof.** We use induction on $s$: in the case $s = 1$, we have $P = \mathcal{Q}(n)$, and the covers are the pairs $(a, a+2)$ and $(a, a+3)$, of which there are $2n - 5$. For the inductive step it suffices to observe that for two 2-chains $P$ and $Q$ the number of covers in $P \boxslash Q$ equals the number of covers in $P$ plus the number of covers in $Q$; this is easy to see from the construction of splices. $\square$

Now we consider the number of linear extensions. For any 2-chain $P$, let $l(P)$ denote the number of linear extensions of $P$, i.e. total orders that refine the partial order on $P$. Our main result here is a recurrence relation for $l(P)$.

**Proposition 8.2.** *Suppose $P$ is a 2-chain with $n$ elements, and write $P = \mathcal{Q}(n_1) \boxslash \cdots \boxslash \mathcal{Q}(n_s)$ with $n_1, \ldots, n_s \geqslant 3$ and $n_1 + \cdots + n_s = n + 2s - 2$. If $n_1 = \cdots = n_s = 3$ (so that $s = n - 2$), then $l(P) = n$. Otherwise, let $r \leqslant s$ be maximal such that $n_r \geqslant 4$. Then*

$$l(P) = l\big(\mathcal{Q}(n_1) \boxslash \cdots \boxslash \mathcal{Q}(n_{s-1}) \boxslash \mathcal{Q}(n_s - 1)\big) + l\big(\mathcal{Q}(n_1) \boxslash \cdots \boxslash \mathcal{Q}(n_{r-1}) \boxslash \mathcal{Q}(n_r - 2)\big).$$

To prove this, we need to consider the poset obtained by removing a maximal element from a 2-chain. Recall from Lemma 3.1 that a 2-chain with at least three elements has exactly two maximal elements, of which exactly one is supermaximal.

We also recall some standard poset notation: we write $\underline{r}$ for a totally-ordered set with $r$ elements, and given posets $P$ and $Q$, we write $P \oplus Q$ for the poset obtained by taking the union of disjoint copies of $P$ and $Q$, and setting $p \trianglelefteq q$ for every $p \in P$ and $q \in Q$.

**Lemma 8.3.** *Suppose $P$ is a 2-chain with at least three elements, and write $P = \mathcal{Q}(n_1) \boxslash \cdots \boxslash \mathcal{Q}(n_s)$ with $n_1, \ldots, n_s \geqslant 3$.*

1. *The poset obtained by removing the supermaximal element from $P$ is isomorphic to*

$$\mathcal{Q}(n_1) \boxslash \cdots \boxslash \mathcal{Q}(n_{s-1}) \boxslash \mathcal{Q}(n_s - 1).$$

2. *Suppose $1 \leqslant r \leqslant n$ with $n_r \geqslant 4$ and $n_{r+1} = \cdots = n_s = 3$, and let $Q$ be the poset obtained by removing the non-supermaximal maximal element from $P$. Then $Q$ is isomorphic to*

$$\mathcal{Q}(n_1) \boxslash \cdots \boxslash \mathcal{Q}(n_{r-1}) \boxslash \mathcal{Q}(n_r - 2) \ \oplus\ \underline{s - r + 1}.$$

**Proof.**

1. First consider the case $s = 1$, so that $P = \mathcal{Q}(n_1)$. The supermaximal element of $\mathcal{Q}(n_1)$ is $n_1$, and removing this leaves the poset $\mathcal{Q}(n_1 - 1)$, as claimed. The general case follows by applying the splicing operation.

2. First consider the case $r = s = 1$, so that $P = \mathcal{Q}(n_1)$. The non-supermaximal maximal element of $\mathcal{Q}(n_1)$ is $n_1 - 1$; removing this leaves a poset with a greatest element $n_1$, with the other elements forming the poset $\mathcal{Q}(n_1 - 2)$. So $\mathcal{Q}(n_1) \setminus \{n_1 - 1\} \cong \mathcal{Q}(n_1 - 2) \oplus \underline{1}$, as claimed.

   To generalise to the case $r = 1 \leqslant s$, observe that splicing $\mathcal{Q}(n_1)$ with $\mathcal{Q}(3) \boxslash \cdots \boxslash \mathcal{Q}(3)$ involves adding a chain of length $s - r$ lying above all elements of $\mathcal{Q}(n_1)$ except $n_1 - 1$. So the non-supermaximal maximal element is still $n_1 - 1$, and removing this leaves $(\mathcal{Q}(n_1 - 2) \oplus \underline{1}) \oplus \underline{s - r} \cong \mathcal{Q}(n_1 - 2) \oplus \underline{s - r + 1}$.

   Now the case $r > 1$ follows by applying the splicing operation. □

**Proof of Proposition 8.2.** The case where $n_1 + \cdots + n_s = 3$ is precisely the case where $P$ consists of a chain of size $n - 1$ and an isolated element; clearly in this case $P$ has $n$ linear extensions. So assume that $n_r \geqslant 4$ for some $r$, and let $r$ be maximal with this property.

In any linear extension of $P$, the greatest element must be one of the maximal elements of $P$. Moreover, if $p$ is a maximal element of $P$, then the number of linear extensions of $P$ having $p$ as greatest element equals the number of linear extensions of $P \setminus \{p\}$. Now the result follows from Lemma 3.1 and Lemma 8.3. □

Now we can deduce bounds for the number of covers and the number of linear extensions of a 2-chain. Let $F_n$ denote the $n$th Fibonacci number, with the convention that $F_0 = 0$. The following result follows from Propositions 8.1 and 8.2 by an easy induction.

**Corollary 8.4.** *Suppose $P$ is a 2-chain of size $n \geqslant 3$, and let $c$ denote the number of covers in $P$ and $l$ the number of linear extensions of $P$. Then:*

1. *$c \geqslant n - 2$ and $l \geqslant n$, with equality in each case if and only if $P \cong \mathcal{Q}(3) \boxslash \cdots \boxslash \mathcal{Q}(3)$;*

2. *$c \leqslant 2n - 5$ and $l \leqslant F_{n+1}$, with equality in each case if and only if $P \cong \mathcal{Q}(n)$.*

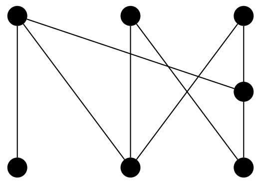
## 9 3-chains and beyond

In this section we briefly discuss a natural generalisation of 2-chains. Given $r \in \mathbb{N}$, define an *$r$-chain* to be a poset $P$ such that $P$ is uniquely expressible as the union of $r$ chains, and the partial order on $P$ is maximal with this property.

For a first family of examples, we can generalise the posets $\mathcal{Q}(n)$ defined in Section 2. Given $n \in \mathbb{N}$, define $\mathcal{Q}(n)^r = \{1, \dots, n\}$, with $i \trianglelefteqslant j$ if and only if $i \leqslant j - r$ in the usual order on $\{1, \dots, n\}$.

To construct all $r$-chains, it seems to be possible to generalise the recursive construction suggested after Proposition 3.3: starting from the unique $r$-element $r$-chain, repeatedly add supermaximal elements, at each stage choosing which $r-1$ of the existing $r$ maximal elements should remain maximal. From this it should follow that the number of isomorphism classes of *labelled* $r$-chains $P$ with $n \geqslant r$ elements (i.e. with the $r$ chains labelled $1, \dots, r$, and with isomorphisms required to preserve labelling) is $r^{n-r}$.

However, the other results in this paper remain to be extended. The author cannot see an obvious way to extend the constructions in Section 4 to the cases $r \geqslant 3$. For the results in Section 5, it should be possible to splice $r$-chains to make larger ones, but a direct analogue of Proposition 5.4 fails: there are $r$-chains other than $\mathcal{Q}(1)^r, \mathcal{Q}(2)^r, \dots$ which cannot be written as splices of smaller $r$-chains. The first example with $r = 3$ is as follows.



The results of Section 7 also have no obvious extension beyond $r = 2$; indeed, $r$-chains need not have dimension 2 when $r > 2$, so not all $r$-chains have the form $\mathcal{R}_w$ for $w$ a permutation. For example, $\mathcal{Q}(7)^3$ has dimension 3.

We hope to say more about $r$-chains in future work.

## 10 Infinite 2-chains

In this section we briefly consider infinite 2-chains. The definition of a 2-chain still makes sense for infinite posets, and many of the results in this paper appear to have analogues, provided we restrict attention to *locally finite* posets, i.e. posets $P$ such that for any two elements $p, q$, there are only finitely many elements $r$ with $p \leqslant q \leqslant r$. So for this section we assume all posets are locally finite, and we explain how to generalise some of the results in this paper. Some of the statements made here are conjectural; we expect they should not be hard to prove, though of course the inductive approach used in this paper for finite 2-chains will not work.

The simple examples of 2-chains in Section 2 generalise naturally: we can take any locally finite chain (which may have a greatest or a least element, or neither) together with an isolated element to give an infinite 2-chain. We can also construct analogues of the finite 2-chains $\mathcal{Q}(n)$, by taking the underlying set to be $\mathbb{N}$, $-\mathbb{N}$, or $\mathbb{Z}$, with $i \lhd j$ when $i \leqslant j-2$. We call the resulting 2-chains $\mathcal{Q}(+\infty)$, $\mathcal{Q}(-\infty)$, $\mathcal{Q}(\infty)$ respectively.

Now we consider maximal elements. An infinite 2-chain has at most two maximal elements; it has a supermaximal element (which is unique) if and only if it has two maximal elements, and this in turn happens if and only if every element lies below a maximal element. A corresponding statement holds for minimal elements. As a consequence, an infinite 2-chain cannot have two maximal and two minimal elements. We refer to a 2-chain with at most one maximal element and at most one minimal element as a *doubly-infinite* 2-chain.

The construction using binary sequences in Section 4 can be generalised to the infinite case, though with slight complications. First we consider infinite 2-chains with two minimal elements. To give a construction of these, take an infinite binary sequence $a = a_1 a_2 \dots$. Define an infinite family of sequences $a(0), a(1), \dots$ by

$$a(n) = a_1 \, \dots \, a_{n-1} \, 0 \, 1 \, a_{n+1}, a_{n+2}, \, \dots$$

(with the 0 omitted in the case $n = 0$). Additionally, if $a_r = 0$ for all $r \gg 0$, define the sequence $a(\infty) = a$. The sequences defined in this way are all the sequences that can be obtained by inserting a 0 or a 1 into $a$, and the poset $\mathcal{P}_a$ can be defined as in Section 4, and is a 2-chain with two minimal elements (namely, the sequences obtained by adding a 0 or a 1 at the start of $a$). $\mathcal{P}_a$ has a maximal element (namely, $a$ itself) if and only if $a_r$ is constant for $r \gg 0$.

A dual construction (using backwards-infinite sequences $\dots a_{-2}a_{-1}$) works for infinite 2-chains with two maximal elements. For doubly-infinite 2-chains, we need to use doubly-infinite binary sequences $a = \dots a_{-1}a_0a_1a_2\dots$. Given such a sequence, we define sequences $a(n)$ for $n \in \mathbb{Z}$ by

$$a(n)_r = \begin{cases} a_r & (r < n) \\ 0 & (r = n) \\ 1 & (r = n+1) \\ a_{r-1} & (r > n+1). \end{cases}$$

In addition:

- if $a_r = 0$ for all $r \gg 0$, define the sequence $a(\infty) = a$;
- if $a_r = 1$ for all $r \ll 0$, define the sequence $a(-\infty)$ to be the sequence $\vec{a}$ obtained from $a$ by shifting entries by 1, i.e. $\vec{a}_r = a_{r-1}$ for all $r$.

The sequences constructed are then all the sequences that can be obtained from $a$ by inserting a symbol in $a$ and shifting by 1 every symbol to the right of the inserted symbol. Now we can follow the construction of Section 4, yielding an infinite 2-chain $\mathcal{P}_a$. This 2-chain has a maximal element (namely $a$) if and only if $a_r$ is constant for $r \gg 0$, and has a minimal element (namely $\vec{a}$) if and only if $a_r$ is constant for $a \ll 0$. It seems likely that these constructions give all locally finite 2-chains.

Locally finite 2-chains can also be constructed by splicing, using the finite 2-chains $\mathcal{Q}(n)$ as well as the 2-chains $\mathcal{Q}(+\infty)$, $\mathcal{Q}(-\infty)$ and $\mathcal{Q}(\infty)$ introduced above. We suspect that every infinite 2-chain is obtained by splicing 2-chains $\mathcal{Q}(n)$. More specifically, every infinite 2-chain with two minimal elements is either of the form

$$\mathcal{Q}(n_1) \boxbslash \mathcal{Q}(n_2) \boxbslash \cdots$$

for some infinite sequence $n_1, n_2, \dots$ of integers greater than 2, or of the form

$$\mathcal{Q}(n_1) \boxbslash \cdots \boxbslash \mathcal{Q}(n_s) \boxbslash \mathcal{Q}(+\infty)$$

for some finite sequence $n_1, \dots, n_s$. Similarly every doubly-infinite 2-chain should have one of the forms

$$\cdots \boxbslash \mathcal{Q}(n_{-1}) \boxbslash \mathcal{Q}(n_0) \boxbslash \mathcal{Q}(n_1) \boxbslash \cdots,$$
$$\cdots \boxbslash \mathcal{Q}(n_{-2}) \boxbslash \mathcal{Q}(n_{-1}) \boxbslash \mathcal{Q}(+\infty),$$
$$\mathcal{Q}(-\infty) \boxbslash \mathcal{Q}(n_1) \boxbslash \mathcal{Q}(n_2) \boxbslash \cdots,$$
$$\mathcal{Q}(\infty)$$

for an appropriate (finite or infinite) sequence of integers $n_r \geqslant 3$. One way to prove this would be as follows: define a *super* element in a 2-chain $P$ to be an element $p$ which is comparable with every element of $P$ except one (say $p^*$). Then the ideal generated by $p$ and $p^*$ should be a 2-chain, and the coideal generated by $p$ and $p^*$ (that is, the set $\{a \in P \mid p \leqslant a \text{ or } p^* \leqslant a\}$) should be a 2-chain, and $P$ is the splice of these two 2-chains. The main task then is to prove that a 2-chain with no super elements is isomorphic to $\mathcal{Q}(\infty)$ (and similarly that a 2-chain whose only super element is minimal is isomorphic to $\mathcal{Q}(+\infty)$).

We expect the results on graphs in Section 6 to generalise to infinite 2-chains; the incomparability graphs of 2-chains should then be caterpillars in which all vertices have finite degree except possibly the head and the tail (i.e. the endpoints (if there are any) of the path formed by the non-leaves).

Finally, we expect that there should be analogues of the results in Section 7, for suitable infinite analogues of Coxeter elements. We begin with the case of 2-chains having two minimal elements. Let $\mathfrak{S}_{\mathbb{N}}$ denote the group of permutations of $\mathbb{N}$, and define a *Coxeter element* in $\mathfrak{S}_{\mathbb{N}}$ to be a permutation $w$ with the property that for every $r > 1$, one of $w(r)$, $w^{-1}(r)$ is greater than $r$ and the other is less than $r$. Now the poset $\mathcal{R}_w$ defined as in Section 7 should be a 2-chain with two minimal elements and no maximal elements. For 2-chains with two minimal elements and a maximal element, a more complicated notion is needed: rather than permutations, one should take bijections from $\mathbb{N}$ to $\mathbb{N} \cup \{\infty\}$. For doubly-infinite 2-chains, we make a similar definition: define a Coxeter element in $\mathfrak{S}_{\mathbb{Z}}$ to be a permutation of $\mathbb{Z}$ such that for every $r$ one of $w(r)$ and $w^{-1}(r)$ is greater than $r$ and the other is less than $r$. Then the poset $\mathcal{R}_w$ should be a 2-chain with no maximal or minimal elements. To obtain doubly-infinite 2-chains with a maximal or minimal element, one needs to consider elements $\pm\infty$; we leave the reader to work out the details.

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