Normal subgroups of the affine symmetric group

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Abstract

We classify normal subgroups of the affine symmetric group, by elementary means. As far as we can tell, these results are new.

1 The affine symmetric group

We write $\tilde{\mathfrak{S}}_n$ for the affine symmetric group of degree n. This can be defined in several ways. We will focus on $\tilde{\mathfrak{S}}_n$ as the semidirect product $T \rtimes \mathfrak{S}_n$, where T is the root lattice. To write our group multiplicatively, we'll write the root lattice as the group $\left\{ t_1^{a_1} \dots t_n^{a_n} \;\middle|\; a_1 + \dots + a_n = 0 \right\}$, where $t_1, \dots, t_n$ are commuting elements of infinite order. Then our group operation is given by $t_i \pi = \pi t_{\pi^{-1}(i)}$.

2 The main results

If $N \trianglelefteq \tilde{\mathfrak{S}}_n$, then we write

$$N^+ = \left\{ \pi \in \mathfrak{S}_n \;\middle|\; t\pi \in N \text{ for some } t \in T \right\}.$$

It is then an easy exercise to show that $N^+ \trianglelefteq \mathfrak{S}_n$.

If N^+ is trivial, then N is simply an $\mathfrak{S}_n$-invariant subgroup of T. These can be classified as follows. Take $a, b \in \mathbb{N}_0$ with $a \mid b$, and let

$$T_a^b = \left\{ t_1^{a_1} \dots t_n^{a_n} \;\middle|\; a_1 + \dots + a_n = 0,\ a_i \in a\mathbb{Z} \text{ for all } i,\ a_i - a_j \in b\mathbb{Z} \text{ for all } i, j \right\}.$$

Proposition 2.1. *Suppose U is a non-trivial $\mathfrak{S}_n$-invariant subgroup of T. Then $U = T_a^b$ for some a, b.*

Proof. For each $u \in U$ define $u_1, \dots, u_n$ by $u = t_1^{u_1} \dots t_n^{u_n}$. Now let

$$a = \operatorname{Gcd}\left\{ u_i \;\middle|\; u \in U,\ 1 \leqslant i \leqslant n \right\}, \qquad b = \operatorname{Gcd}\left\{ u_i - u_j \;\middle|\; u \in U,\ 1 \leqslant i < j \leqslant n \right\}.$$

Then we claim that $U = T_a^b$. First note that $\mathfrak{S}_n$-invariance means that

$$b = \operatorname{Gcd}\left\{ u_1 - u_2 \;\middle|\; u \in U \right\}.$$

Now we can find a finite subset $V \subset U$ such that $b = \sum_{u \in V} u_1 - u_2$. So if we let $v = \prod_{u \in V} u$, then $v_1 - v_2 = b$, with the result that $U \ni v(12)v^{-1}(12) = t_1^b t_2^{-b}$. Invariance under $\mathfrak{S}_n$ then means that $t_i^b t_j^{-b} \in U$ for all i, j, and it is clear that these elements generate T_b^b. So $T_b^b \leqslant u$.

We can also find a finite subset $W \subset U$ such that $a = \sum_{u \in W} u_1$. So if we let $w = \prod_{u \in W} u$, then $w_1 = a$, and therefore (from the definition of b) $w_i \equiv a \pmod{b}$ for all i. Now any element of T_a^b can be written as the product of a power of w and an element of T_b^b, so that $T_a^b \leqslant U$. □

In view of this, we can restrict attention to normal subgroups $N \trianglelefteq \tilde{\mathfrak{S}}_n$ for which N^+ is non-trivial. The main result that restricts possible normal subgroups is the following.

Proposition 2.2. *Suppose $N \trianglelefteq \tilde{\mathfrak{S}}_n$ and that N^+ contains a non-trivial element π with at least one fixed point. Then $T \leqslant N$, so $N = T \rtimes N^+$.*

Proof. Let k be a fixed point of π and i a non-fixed point, and let $j = \pi(i)$. Take $t \in T$ such that $t\pi \in N$. Then $N \ni t_it_k^{-1}t\pi t_i^{-1}t_k = t_it_j^{-1}t\pi$, with the result that $t_it_j^{-1} \in N$. But now normality gives $t_lt_m^{-1} \in N$ for all l, m, so that $T \leqslant N$. Hence $T\rho \subseteq N$ for every $\rho \in N^+$, so $N = T \rtimes N^+$. □

This means that our classification is complete for $n \geqslant 5$, since if $n \geqslant 5$ then every non-trivial normal subgroup of $\mathfrak{S}_n$ has a non-trivial element with a fixed point. For $n = 2, 3, 4$ we have a little more work to do.

$n = 2$

In this case $\tilde{\mathfrak{S}}_n$ is just a dihedral group, so we can classify normal subgroups by direct inspection. But let's follow the above approach. Assume $N \trianglelefteq \tilde{\mathfrak{S}}_2$ with $N^+ = \mathfrak{S}_2$ and $T \nleqslant N$. Take $t \in T$ such that $t(12) \in N$. Then $N \ni t_1t_2^{-1}t(12)t_1^{-1}t_2 = t_1^2t_2^{-2}t(12)$, so $t_1^2t_2^{-2} \in N$. So N contains the group T_2^4 generated by $t_1^2t_2^{-2}$, which has index 2 in T. So (from our assumptions) $N \cap T = T_2^4$. Now there are two possibilities: N can be $T_2^4 \rtimes \mathfrak{S}_2$, or $T_2^4 \cup t_1t_2^{-1}T_2^4(12)$. Both of these give normal subgroups. (If we let s_0, s_1 be a pair of Coxeter generators for $\tilde{\mathfrak{S}}_2$, then one of these subgroups is the normal subgroup generated by s_1, and the other is the normal subgroup generated by s_0.)

$n = 3$

If we assume that N^+ is a non-trivial normal subgroup of $\mathfrak{S}_3$ and that no non-trivial element of N has any fixed points, then $N = \mathfrak{A}_3$. So suppose $N \trianglelefteq \tilde{\mathfrak{S}}_3$ with $N^+ = \mathfrak{A}_3$ and $T \nleqslant N$. Now if we take $t \in T$ such that $t(123) \in N$ and conjugate by $t_1t_2^{-1}$, we obtain $t_1t_2^{-2}t_3 \in N$. So N contains the group T_1^3, which has index 3 in T. So our assumptions mean that $N \cap T = T_1^3$. Now there are three possibilities:

$$\begin{aligned} N &= T_1^3 \rtimes \mathfrak{A}_3, \\ N &= T_1^3 \cup t_1t_2^{-1}T_1^3(123) \cup t_1^{-1}t_2T_1^3(132), \\ N &= T_1^3 \cup t_1^{-1}t_2T_1^3(123) \cup t_1t_2^{-1}T_1^3(132). \end{aligned}$$

These are all normal subgroups of $\tilde{\mathfrak{S}}_3$; if s_0, s_1, s_2 are Coxeter generators of $\tilde{\mathfrak{S}}_3$, then these normal subgroups are the normal subgroups generated by the elements s_1s_2, s_0s_1 and s_0s_2.

$n = 4$

Finally assume $N \trianglelefteq \tilde{\mathfrak{S}}_4$ with $T \nleqslant N$ and N^+ the normal subgroup V_4 of $\mathfrak{S}_4$ generated by $(12)(34)$. Choosing $t \in T$ such that $t(12)(34) \in N$ and conjugating by $t_2t_3^{-1}$, we find that $t_1t_2t_3^{-1}t_4^{-1} \in N$. So N contains the group T_1^2, which has index 4 in T. Because $N \cap T < T$ and $N \cap T$ is $\mathfrak{S}_4$-invariant, we obtain $N \cap T = T_1^2$. This means that $\{\, t \in T \mid t(12)(34) \in N\}$ is a single coset of T_1^2 in T. The four cosets of T_1^2 in T are

$$T_1^2, \qquad T_1^2t_1t_2^{-1}, \qquad T_1^2t_1t_3^{-1}, \qquad T_1^2t_1t_4^{-1}.$$

But the last two of these don't work, because if $t_1t_k^{-1}(12)(34)$ with $k = 3$ or 4, then conjugating by (12) gives $t_2t_k^{-1}(12)(34)$ and therefore $t_1t_2^{-1} \in N$, a contradiction. So there are only two possible cosets that can appear with $(12)(34)$. This leads to two possibilities for N:

$$\begin{aligned} N &= T_1^2 \rtimes V_4, \\ N &= T_1^2 \cup t_1t_2^{-1}T_1^2(12)(34) \cup t_1t_3^{-1}T_1^2(13)(24) \cup t_1t_4^{-1}T_1^2(14)(23). \end{aligned}$$

Both of these possibilities give normal subgroups; if s_0, s_1, s_2, s_3 are Coxeter generators, then these normal subgroups are the normal subgroups of $\tilde{\mathfrak{S}}_4$ generated by s_1s_3 and s_0s_2 respectively.

We have proved the following theorem.

Theorem 2.3. *Suppose N is a non-trivial normal subgroup of $\tilde{\mathfrak{S}}_n$. Then one of the following occurs.*

1. $N = T_a^b$ *for some* $a, b \in \mathbb{N}_0$ *with* $a \mid b$.
2. $N = T \rtimes N^+$ *for some normal subgroup* $N^+ \trianglelefteq \mathfrak{S}_n$.
3. $n = 2$ *and* $N = T_2^4 \cup t_1t_2^{-1}T_2^4\pi$ *for some* $\pi \in \mathfrak{S}_2$.
4. $n = 3$ *and* $N = T_1^3 \cup t_1t_2^{-1}T_1^3\pi \cup t_1^{-1}t_2T_1^3\pi^2$ *for some* $\pi \in \mathfrak{A}_3$.
5. $n = 4$ *and* $N = T_1^2 \rtimes V_4$.
6. $n = 4$ *and* $N = T_1^2 \cup t_1t_2^{-1}T_1^2(1\,2)(3\,4) \cup t_1t_3^{-1}T_1^2(1\,3)(2\,4) \cup t_1t_4^{-1}T_1^2(1\,4)(2\,3)$.