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A non-recursive criterion for weights of a highest-weight module for an affine Lie algebra

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Abstract

Let Λ be a dominant integral weight of level k for the affine Lie algebra $\mathfrak{g}$ and let α be a non-negative integral combination of simple roots. We address the question of whether the weight $\eta = \Lambda - \alpha$ lies in the set $P(\Lambda)$ of weights in the irreducible highest-weight module with highest weight Λ. We give a non-recursive criterion in terms of the coefficients of α modulo an integral lattice kM, where M is the lattice parameterizing the abelian normal subgroup T of the Weyl group. The criterion requires the preliminary computation of a set no larger than the fundamental region for kM, and we show how this set can be efficiently calculated.

1 Introduction

To an affine Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ corresponds a vector space $\mathfrak{h}$ containing the Cartan subalgebra and a dual vector space $\mathfrak{h}^*$. An important class of modules are the *integrable highest-weight modules* $L(\Lambda)$ for $\Lambda \in \mathfrak{h}^*$. We will study the set $P(\Lambda)$ of weights of the homogeneous elements of $L(\Lambda)$.

For each affine Lie algebra there is a certain integral lattice M of weights of rank l defined in [Ka, 6.5.8], parameterizing an abelian subgroup T of the Weyl group. We determine a set $\bar{N}$ of maximal weights which is in bijection with the image of $P(\Lambda)$ in a fundamental region for kM. We give a criterion for a weight η to lie in $P(\Lambda)$, which is a generalization of [Ka, 12.6.3] to levels $k > 1$. In the process we also prove that every weight in $\Lambda - Q$ can be δ-shifted to an element of $P(\Lambda)$ (where Q denotes the root lattice and δ the null root). Finally, using a case-by-case study of the affine families and the affine exceptional algebras, we show how to separate the weights into Q-classes by congruences and how to find the maximal dominant weights in each class.

The original motivation for this research was an investigation of the existence of the block H^{Λ}_{α} of the cyclotomic Hecke algebra $H^{\Lambda}_{d}(\mathbb{F}, \xi)$ [AK], where $\xi \in \mathbb{F}^{\times}$ is a primitive $(l{+}1)$-th root of unity. This question is typically settled by a recursive construction of the weights of blocks up to rank d or by the construction of a multipartition with content α. By the categorification result in [AM], such a block exists if and only if the corresponding weight η is in $P(\Lambda)$ for the affine Lie algebra $\mathfrak{g}(A^{(1)}_{l})$, so our non-recursive criterion gives a criterion in terms of the residues of the coefficients of α modulo k. In this case the set to be computed is of order k^l. A similar procedure involving $A^{(2)}_{2l}$ corresponds to cyclotomic Hecke algebras related to spin representations of the symmetric groups [Kl].

1.1 Affine Lie algebras

In this paper we work with the affine Lie algebra $\mathfrak{g}$ defined by an $(l{+}1) \times (l{+}1)$ Cartan matrix $A = [a_{ij}]$. There are two families ($A_l^{(1)}$ and $D_l^{(1)}$) with symmetric Cartan matrices, several other families ($B_l^{(1)}$, $C_l^{(1)}$, $D_l^{(2)}$, $A_{2l-1}^{(2)}$ and $A_{2l}^{(2)}$) with non-symmetric matrices, and a number of exceptional algebras [Ka, Chap. 4]. The algebras with exponent (1) will be called untwisted, and the algebras with exponent (2) or (3) will be called twisted.

The algebra $\mathfrak{g}$ has simple roots α_i and simple coroots h_i for $i \in \{0, \dots, l\}$, with a pairing given by the Cartan matrix entries

$$\langle h_i, \alpha_j \rangle = a_{ij}.$$

We choose a set of fundamental weights $\{\Lambda_0, \dots, \Lambda_l\}$ in $\mathfrak{h}^*$ which satisfy $\langle h_i, \Lambda_j \rangle = \delta_{ij}$. In the affine case, we have a *null root*

$$\delta = a_0\alpha_0 + \cdots + a_l\alpha_l$$

with integer coefficients, which generates the kernel when A acts from the left, and, dually, a canonical central element

$$c = a_0^\vee h_0 + \cdots + a_l^\vee h_l,$$

where the integral coefficients of c are chosen so that that $\langle c, \alpha_i \rangle = 0$ for any simple root α_i.

If we let D be the matrix $\mathrm{diag}(a_0^{-1}a_0^\vee, \dots, a_l^{-1}a_l^\vee)$, then the matrix $B = DA$ is symmetric. This matrix determines an invariant symmetric bilinear form $(\cdot|\cdot)$, for which we have

$$(\alpha_i|\Lambda_j) = \frac{a_i^\vee}{a_i}\delta_{ij}, \qquad (\alpha_i|\alpha_j) = \frac{a_i^\vee}{a_i}a_{ij}, \qquad (\alpha_i|\delta) = 0, \qquad (\delta|\delta) = 0.$$

Following Kac, we define

$$\begin{aligned} P &= \{\eta \in \mathfrak{h}^* \mid \langle h_i, \eta \rangle \in \mathbb{Z} \text{ for } i = 0, \dots, l\}, \\ P_+ &= \{\eta \in P \mid \langle h_i, \eta \rangle \geqslant 0 \text{ for } i = 0, \dots, l\}, \\ Q &= \textstyle\sum_{i=0}^l \mathbb{Z}\alpha_i, \\ Q_+ &= \textstyle\sum_{i=0}^l \mathbb{Z}_{\geqslant 0}\alpha_i. \end{aligned}$$

The weights in P are called *integral weights*, and those in P_+ are *dominant integral weights*.

Definition 1.1. The *level* of a dominant integral weight Λ is the integer $k = \langle c, \Lambda \rangle$, or equivalently $(\Lambda|\delta)$.

1.2 The weights of an integrable highest-weight module

In this paper we fix a dominant integral weight $\Lambda \in P_+$. We let $P(\Lambda)$ denote the set of weights labeling non-zero weight spaces in the integrable highest-weight module $L(\Lambda)$. Then $P(\Lambda)$ is a subset of $\Lambda - Q_+$.

Although the set $P(\Lambda)$ is discussed in detail in [Ka], it is not easy to determine whether a given weight lies in $P(\Lambda)$. The purpose of this paper is to give a non-recursive way to do this.

A weight $\eta \in P(\Lambda)$ is called *maximal* if $\eta + \delta$ is not a weight for Λ, and the set of maximal weights is denoted $\max(\Lambda)$. By [Ka, (12.6.1)] the set of weights for $L(\Lambda)$ is the union of the negative shifts by the null root δ of the maximal weights:

$$P(\Lambda) = \{\lambda - s\delta \mid \lambda \in \max(\Lambda), s \in \mathbb{Z}_{\geqslant 0}\}\,. \tag{1.2.1}$$

1.3 The Weyl group

Let W denote the Weyl group of $\mathfrak{g}$; considered as a group acting on the weight space $\mathfrak{h}^*$, this is generated by the reflections $s_0, \dots, s_l$ defined by

$$s_i : \eta \longmapsto \eta - \langle h_i, \eta\rangle\alpha_i.$$

The Weyl group is of critical importance to this paper, since it fixes the set $P(\Lambda)$ of weights for $L(\Lambda)$. It also fixes the null root δ, and therefore fixes the set $\max(\Lambda)$.

The criterion we shall prove in the next section depends on the decomposition of W as a semi-direct product $T \rtimes \mathring{W}$, where $\mathring{W}$ is the (finite) subgroup generated by $s_1, \dots, s_l$, and T is a torsion-free abelian group. The elements of T correspond to a certain lattice M in $\mathfrak{h}^*$, which has generators

$$\frac{d_1\alpha_1}{a_0}, \dots, \frac{d_l\alpha_l}{a_0}$$

for certain positive integers $d_1, \dots, d_l$, which are given in [Ka, (6.5.8)]. When the Cartan matrix is symmetric (types $A_l^{(1)}$ and $D_l^{(1)}$) or of twisted type other than $A_{2l}^{(2)}$, all the d_i are equal to 1. For $\alpha \in M$, the corresponding element t_α of T is given by [Ka, (6,5,2)]:

$$t_\alpha : \eta \mapsto \eta + k\alpha - \Big((\eta|\alpha) + \tfrac{1}{2}(\alpha|\alpha)k\Big)\,\delta,$$

where $k = \langle c, \eta\rangle$ is the level of η.

2 The criterion for membership in $P(\Lambda)$

In this section, we give a theorem which yields an algorithm for determining whether a given weight $\eta \in \Lambda - Q_+$ lies in $P(\Lambda)$. Writing $\eta = \Lambda - \sum_i b_i\alpha_i$, we may refer to η by its *content* $b = (b_0, \dots, b_l)$, if Λ is understood. We will give another representation of η in terms of the decomposition $W = T \rtimes \mathring{W}$.

By [Ka, Corollary 10.1], a weight $\eta \in P(\Lambda)$ is equivalent under the action of the affine Weyl group W to a unique element of $P_+ \cap P(\Lambda)$. From this we can obtain a description of the set $P(\Lambda)$ modulo δ. First we need to know that every weight of positive level is W-equivalent to a dominant weight.

Proposition 2.1. *Suppose $\eta \in P$ with $\langle c, \eta\rangle > 0$. Then there is $w \in W$ such that $w\eta \in P_+$.*

Proof. This is essentially the result of [Ka, Proposition 5.8(b)], though one must interchange $\mathfrak{g}$ with its dual. What Kac proves in [*loc. cit.*] is that the Tits cone

$$W \cdot \{h \in \mathfrak{h}_{\mathbb{R}} \mid \langle h, \alpha_i\rangle \geqslant 0 \text{ for all } i\}$$

includes all elements $h \in \mathfrak{h}_{\mathbb{R}}$ for which $\langle h, \delta\rangle > 0$. Applying this result to the algebra ${}^t\mathfrak{g}$ dual to $\mathfrak{g}$ (that is, the Kac–Moody algebra whose Cartan matrix is the transpose of the Cartan matrix of $\mathfrak{g}$), and then interchanging h_i and α_i for each i, one obtains that for $\mathfrak{g}$ the cone

$$W \cdot \left\{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \,\middle|\, \langle h_i, \lambda\rangle \geqslant 0 \text{ for all } i\right\}$$

includes all elements $\eta \in \mathfrak{h}_{\mathbb{R}}^*$ for which $\langle c, \eta\rangle > 0$. □

Now we can show that, modulo δ, $P(\Lambda)$ coincides with $\Lambda + Q$.

Proposition 2.2. *Suppose $\eta \in \Lambda + Q$. Then there is some $s \in \mathbb{Z}$ such that $\eta + s\delta \in P(\Lambda)$.*

Proof. In the case where η is a dominant weight, this follows from [Ka, Proposition 11.2], since then every weight $\eta + s\delta$ is dominant, and for sufficiently small s we must have $\eta + s\delta \leqslant \Lambda$.

In general, we have $\langle c, \eta\rangle = \langle c, \Lambda\rangle = k > 0$, and so by Proposition 2.1, η is the image of some dominant weight ξ under the action of the Weyl group. Since the Weyl group action involves adding elements of Q, ξ also lies in $\Lambda - Q$, and so the present proposition holds for ξ. Since δ and $P(\Lambda)$ are fixed by the Weyl group action, the result holds for η too. □

Lemma 2.3. *Suppose $\eta \in \Lambda + Q$. Then there is a maximal integer $s(\eta)$ such that $\eta + s(\eta)\delta \in P(\Lambda)$. Moreover, $\eta \in P(\Lambda)$ if and only if $s(\eta) \geqslant 0$.*

Proof. This is immediate from (1.2.1) and Proposition 2.2. □

Definition 2.4. Suppose Λ is a dominant integral weight, and $\eta \in \Lambda + Q$. Define the *delta-shift* $s(\eta)$ to be the largest integer s such that $\eta + s\delta \in P(\Lambda)$.

By Lemma 2.3, in order to test whether $\eta \in P(\Lambda)$, it suffices to calculate the delta-shift $s(\eta)$. This calculation is our main result (Theorem 2.7).

Now recall from §1.3 the decomposition $W = T \rtimes \mathring{W}$, and the integers $d_1, \dots, d_l$ defining the lattice M.

Definition 2.5. Suppose Λ is of level k.

- For any $b = (b_0, \dots, b_l) \in \mathbb{Z}^{l+1}$, not necessarily non-negative, let $\eta(b)$ be the weight $\Lambda - (b_0\alpha_0 + \cdots + b_l\alpha_l)$.
- For any $\eta = \eta(b)$, let

$$\tilde{\eta} = \Big((a_0b_1 - a_1b_0) \bmod kd_1, \dots, (a_0b_l - a_lb_0) \bmod kd_l\Big) \in \prod_{i=1}^{l}(\mathbb{Z}/kd_i\mathbb{Z}).$$

Note that if $\eta, \zeta \in \Lambda - Q$, with $\zeta - \eta = \sum_{i=0}^{l} c_i\alpha_i$, then we have $(\tilde{\zeta} - \tilde{\eta})_i \equiv a_ic_0 - a_0c_i \pmod{kd_i}$.

Let N be the set of maximal dominant weights, and let $\tilde{N} = \mathring{W} \cdot N$ be the union of the orbits under the finite Weyl group $\mathring{W}$. Say that two elements of $\tilde{N}$ lie in the same *T-class* if one can be moved to the other by the action of an element of T, and let $\bar{N}$ be a set of representatives of the T-classes of $\tilde{N}$.

Proposition 2.6. *Suppose Λ is a dominant integral weight of level k.*

1. *If $\eta, \zeta \in \Lambda - Q$, then we have $\eta = t_\alpha(\zeta) - s\delta$ for some $\alpha \in M$ and some $s \in \mathbb{Z}$ if and only if $\tilde{\eta} = \tilde{\zeta}$.*
2. *If $f : \bar{N} \to \prod_{i=1}^{l}(\mathbb{Z}/kd_i\mathbb{Z})$ denotes the restriction of $\tilde{\cdot}$, then f is injective.*
3. *If $\mathfrak{g}$ is of any type other than $A_{2l}^{(2)}$, then f is surjective, and thus $\bar{N}$ has $k^l \prod_{i=1}^{l} d_i$ elements.*
4. *If $\mathfrak{g} = \mathfrak{g}(A_{2l}^{(2)})$, then the image of f is $(2\mathbb{Z}/2k\mathbb{Z})^{l-1} \times \mathbb{Z}/k\mathbb{Z}$, and thus $\bar{N}$ has k^l elements.*

Proof.

1. First assume that $\eta = t_\alpha(\zeta) - s\delta$ for some $\alpha \in M$, and write

$$\alpha = \sum_{i=1}^{l} \frac{n_i d_i}{a_0} \alpha_i$$

for integers $n_1, \dots, n_l$. From above, we have

$$\eta = t_\alpha(\zeta) - s\delta = \zeta + k\alpha - \left((\zeta|\alpha) + \tfrac{1}{2}(\alpha|\alpha)k + s\right)\delta.$$

Writing the coefficient $(\zeta|\alpha) + \frac{1}{2}(\alpha|\alpha)k + s$ as N, we get

$$\begin{aligned}\zeta - \eta &= N\delta - k\alpha \\ &= N\sum_{i=0}^{l} a_i\alpha_i - k\sum_{i=1}^{l}\frac{n_i d_i}{a_0}\alpha_i;\end{aligned}$$

hence the ith component of $\tilde{\zeta} - \tilde{\eta}$ is

$$\left(a_i N a_0 - a_0\left(N a_i - k\frac{n_i d_i}{a_0}\right)\right) \bmod kd_i,$$

which is zero.

Conversely, suppose that $\tilde{\eta} = \tilde{\zeta}$ with $\eta = \eta(b)$ and $\zeta = \eta(b')$; then we can write

$$(a_0b_1' - a_1b_0', \dots, a_0b_l' - a_lb_0') - (a_0b_1 - a_1b_0, \dots, a_0b_l - a_lb_0) = k(n_1d_1, \dots, n_ld_l)$$

for some $n_1, \dots, n_l \in \mathbb{Z}$. Thus, if we set $\alpha = \frac{n_1d_1}{a_0}\alpha_1 + \cdots + \frac{n_ld_l}{a_0}\alpha_l$, then modulo $\mathbb{Q}\delta$ we have

$$\begin{aligned}t_\alpha(\zeta) &\equiv \zeta + k\alpha \\ &\equiv \Lambda - \sum_{i=0}^{l} b_i'\alpha_i + k\sum_{i=1}^{l}\frac{n_i d_i}{a_0}\alpha_i \\ &\equiv \Lambda - (b_0 + (b_0' - b_0))\alpha_0 - \sum_{i=1}^{l}\left(b_i' - \frac{a_0b_i' - a_ib_0' - a_0b_i + a_ib_0}{a_0}\right)\alpha_i \\ &\equiv \Lambda - \sum_{i=0}^{l} b_i\alpha_i - \frac{b_0' - b_0}{a_0}\sum_{i=0}^{l} a_i\alpha_i \\ &\equiv \eta - \frac{b_0' - b_0}{a_0}\delta \\ &\equiv \eta.\end{aligned}$$

So $\eta = t_\alpha(\zeta) + s\delta$ for some rational number s; but since $\eta - t_\alpha(\zeta) \in Q$ and $Q \cap \mathbb{Q}\delta = \mathbb{Z}\delta$, s is in fact an integer.

2. If η and ζ are in $\bar{N}$ with $f(\eta) = f(\zeta)$, then from (1) we have $\eta = t_\alpha(\zeta) + s\delta$ for some s. But since η, ζ lie in $\bar{N}$, they are both maximal weights and thus $s = 0$, so they are both in the same T-class. Since $\bar{N}$ contains a unique representative of each T-class, $\eta = \zeta$ and thus f is injective.

3. Since $\mathfrak{g}$ is not of type $A^{(2)}_{2l}$, we have $a_0 = 1$. Given an element $(b_1, \dots, b_l) \in \prod_{i=1}^{l}(\mathbb{Z}/kd_i\mathbb{Z})$, consider the content $b = (0, b_1, \dots, b_l)$ (regarding each b_i as an integer in the range $\{0, \dots, kd_i - 1\}$). By Proposition 2.1 and Proposition 2.2, the corresponding weight $\eta(b)$ can be written as $t'w_0(\zeta') - s\delta$ for some $\zeta' \in N$. If ζ is the representative of the T-class of $w_0\zeta'$ in $\bar{N}$, then we replace t' by another element $t \in T$ such that $t'w_0(\zeta') = t(\zeta)$, giving $\eta = t(\zeta) - s\delta$. By part (1) of this proposition, we have $f(\zeta) = \tilde{\zeta} = \widetilde{\eta(b)} = (b_1, \dots, b_l)$, so f is indeed surjective.

   The cardinality of $\bar{N}$ is now immediate, since the number of elements in $\prod_{i=1}^{l}(\mathbb{Z}/kd_i\mathbb{Z})$ is $k^l \prod_{i=1}^{l} d_i$.

4. In this case, we have $a_i = d_i = 2$ for $i = 1, \dots, l{-}1$, while $a_l = d_l = 1$. If $\eta = \eta(b)$, then for $i < l$ we have $\tilde{\eta}_i = 2b_i - 2b_0$, which is even; so the image of f is certainly contained in $(2\mathbb{Z}/2k\mathbb{Z})^{l-1} \times \mathbb{Z}/k\mathbb{Z}$. Conversely, suppose we are given integers $c_1, \dots, c_l$ with $c_1, \dots, c_{l-1}$ even. If c_l is also even, then let b be the content $\left(0, \frac{c_1}{2}, \dots, \frac{c_l}{2}\right)$ and let $\eta = \eta(b)$. Repeating the argument from above, there is $\zeta \in \bar{N}$ with $f(\zeta) = \tilde{\eta} = (c_1, \dots, c_l)$. If instead c_l is odd, let $b = \left(1, \frac{c_1}{2} + 1, \dots, \frac{c_{l-1}}{2} + 1, \frac{c_l + 1}{2}\right)$ and repeat the argument. □

Now we can give our main result, which is the calculation of the delta-shift $s(\eta)$, for $\eta \in \Lambda + Q$. Since by Lemma 2.3 $\eta \in P(\Lambda)$ if and only if $s(\eta) \geqslant 0$, this result gives a criterion for testing whether $\eta \in P(\Lambda)$.

Theorem 2.7. *Suppose $\mathfrak{g}$ is an affine Lie algebra, with Weyl group $W = T \rtimes \mathring{W}$. Let Λ be a dominant integral weight, N the set of maximal dominant weights in $P(\Lambda)$, and $\bar{N}$ a set of T-class representatives of the set $\mathring{W} \cdot N$. For $\eta = \eta(b)$, let ζ be the unique element from $\bar{N}$ such that $\tilde{\eta} = \tilde{\zeta}$. Let $\alpha \in M$ and $s(\eta) \in \mathbb{Z}$ be determined by the formula $\eta = t_\alpha(\zeta) - s(\eta)\delta$. Writing $\eta - \zeta = c_0\alpha_0 + \dots + c_l\alpha_l$, we have*

$$\alpha = \frac{1}{ka_0}\Big((a_0c_1 - a_1c_0)\alpha_1 + \dots + (a_0c_l - a_lc_0)\alpha_l\Big),$$

and hence

$$s(\eta) = -\frac{c_0}{a_0} - \Big((\zeta|\alpha) + \tfrac{1}{2}(\alpha|\alpha)k\Big).$$

Proof. By Proposition 2.6, there is a unique $\zeta \in \bar{N}$ such that $\tilde{\eta} = \tilde{\zeta}$. There is an $\alpha \in M$ and an $s \in \mathbb{Z}$ such that $\eta = t_\alpha(\zeta) - s\delta$. However, since $\zeta \in \max(\Lambda)$, we get $s = s(\eta)$ by Definition 2.4.

Now we have

$$\begin{aligned}\sum_{i=0}^{l} c_i\alpha_i &= \eta - \zeta \\ &= t_\alpha(\zeta) - s(\eta)\delta - \zeta \\ &= k\alpha - \Big((\zeta|\alpha) + \tfrac{1}{2}(\alpha|\alpha)k + s(\eta)\Big)\delta.\end{aligned}$$

Since α lies in the span of $\alpha_1, \dots, \alpha_l$, we can compare coefficients of α_0 to get

$$-\Big((\zeta|\alpha) + \tfrac{1}{2}(\alpha|\alpha)k + s(\eta)\Big)a_0 = c_0,$$

from which we get

$$\begin{aligned}\alpha &= \frac{1}{k}\sum_{i=0}^{l} c_i\alpha_i - \frac{c_0}{a_0}\delta \\ &= \frac{1}{ka_0}\sum_{i=1}^{l}(a_0c_i - a_ic_0)\alpha_i\end{aligned}$$

and also

$$s(\eta) = -\frac{c_0}{a_0} - \Big((\zeta|\alpha) + \tfrac{1}{2}(\alpha|\alpha)k\Big). \qquad \square$$

We provide an example below, but first we make another definition.

Definition 2.8. [BK, 3.11] The *defect* of the weight $\eta = \Lambda - \alpha$ is given by

$$\mathrm{def}_\Lambda(\eta) = (\Lambda|\alpha) - \tfrac{1}{2}(\alpha|\alpha).$$

If $\eta \in P(\Lambda)$, then $\mathrm{def}_\Lambda(\eta)$ is non-negative: to see this, note that if $\eta \in P(\Lambda)$, then by Proposition 2.1 we have $\eta = w \cdot \xi$ for some $w \in W$ and some dominant weight $\xi \in P(\Lambda)$. Writing $\xi = \Lambda - \alpha$ for $\alpha \in Q_+$, we have $(\Lambda|\alpha), (\xi|\alpha) \geqslant 0$ since Λ, ξ are both dominant, and this yields $(\xi|\xi) \leqslant (\Lambda|\Lambda)$. Since $(\cdot|\cdot)$ is W-invariant, we get $(\eta|\eta) \leqslant (\Lambda|\Lambda)$, which gives $\mathrm{def}_\Lambda(\eta) \geqslant 0$.

If $s \in \mathbb{Z}$, then

$$\mathrm{def}_\Lambda(\eta + s\delta) = \mathrm{def}_\Lambda(\eta) + s(\Lambda|\delta) = \mathrm{def}_\Lambda(\eta) + sk,$$

since

$$(\Lambda - \alpha - \delta|\Lambda - \alpha - \delta) = (\Lambda - \alpha|\Lambda - \alpha) - 2(\Lambda - \alpha|\delta) + (\delta|\delta)$$

and

$$(\delta|\delta) = (\alpha|\delta) = (\delta|\alpha) = 0, \quad (\Lambda|\delta) = k.$$

From these results we see that the defect of a weight η of level k is at least k times its δ-shift s. Since by [Ka, Lemma 12.6], every maximal weight in level 1 is in $W \cdot \Lambda$ and thus of defect 0, this shows that for level 1, the defect is equal to the δ-shift s.

There is a unique dominant weight of defect zero, namely Λ. Indeed, the defect of η is zero only if $(\Lambda|\Lambda) = (\eta|\eta)$, and by [Ka, Prop. 11.4] this implies that η lies in the W-orbit of Λ. Since any W-orbit contains a unique dominant weight, we see that if η has defect zero and is dominant, then $\eta = \Lambda$.

Remark 2.9. In the level one case, as just described, one can determine the maximal weights because they all have defect zero. For higher levels, the defect is not sufficient to determine whether or not a weight is a maximal weight. Although any weight of defect less that k must be a maximal weight, there may be weights of defect greater than k which are maximal weights. If, for example, $\mathfrak{g} = \mathfrak{g}(A_l^{(1)})$, $l \geqslant 2$ and $\Lambda = 2m\Lambda_0$, then there is a weight $\eta = \Lambda - m\alpha_0$ for which we have $\langle \eta, h_i \rangle = m(\delta_{i1} + \delta_{il})$ (so η is dominant), and

$$\mathrm{def}_\Lambda(\eta) = (2m\Lambda_0|m\alpha_0) - \tfrac{1}{2}(m\alpha_0|m\alpha_0) = 2m^2 - m^2 = m^2.$$

For $m > 2$, this is larger than $k = 2m$.

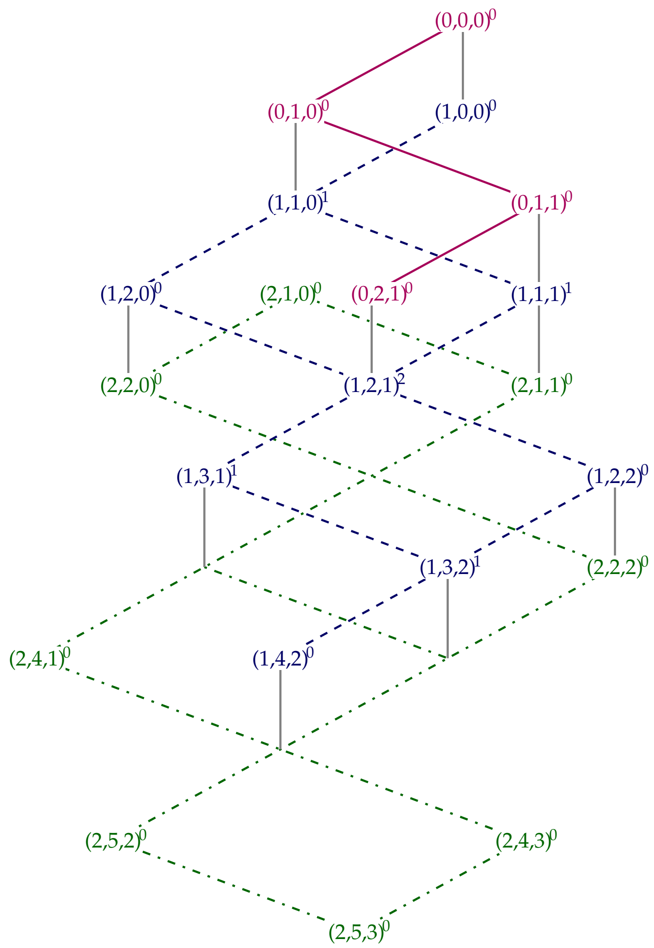
Example 2.10. The weight $\Lambda = \Lambda_0 + \Lambda_1$ for the twisted affine algebra $\mathfrak{g}(A_4^{(2)})$ is of level 3. We represent the elements of $P(\Lambda)$ as nodes in a graph, with coordinates corresponding to content. We wish to find the weights $\zeta = \eta(d) \in \bar{N}$.

Using the results of the next section, we can check that there are three maximal dominant weights in $P(\Lambda)$, which we label by their contents as follows:

$$a^0 = (0,0,0), \qquad a^1 = (1,1,0), \qquad a^2 = (1,2,1).$$

To find $\tilde{N}$, we apply the generators s_1, s_2 of $\mathring{W}$ to these three weights. Each of a_0, a_1, a_2 yields a different orbit; these orbits have sizes $4, 4, 1$ respectively. Taking the nine elements $\zeta \in \tilde{N}$ and computing the vectors $\tilde{\zeta}$, we obtain a complete set of representatives for $(\mathbb{Z}/3\mathbb{Z})^2$. The weights $\eta(b) \in L(\Lambda)$ with $b_0 \leqslant 2$ are depicted in Figure 2.10, which is drawn as the projection of a three-dimensional model; the diagonal lines indicate subtraction of α_1 and α_2, while the vertical lines indicate subtraction of α_0. For clarity, the only vertical lines drawn are those which would be visible in an opaque model, and only the maximal weights are labeled. The superscript on the content of any weight η is its defect. Note that the defects are preserved by the action of the Weyl group, which acts by reflection on lines.

In Section 4, we treat another example (in type $A_2^{(1)}$) in more detail.



3 Finding the dominant weights in $P(\Lambda)$ for affine Lie algebras

In this section we describe how to find the dominant weights in the weight space $L(\Lambda)$, for a fixed dominant integral weight Λ.

Definition 3.1. We say that integral weights Λ and Λ' for $\mathfrak{g}$ are *equivalent*, written $\Lambda \equiv \Lambda'$, if $\Lambda - \Lambda' \in Q$.

Since each element of Q has level 0, two equivalent weights must have the same level. Moreover, since the set of weights of level 0 is precisely the $\mathbb{C}$-span of Q, it is clear that two weights Λ, Λ' of the same level differ by a linear combination of simple roots. However, we do not have $\Lambda' - \Lambda \in Q$ unless all the coefficients are integers. We shall study this equivalence relation among integral weights of the same level below, giving a necessary and sufficient condition for Λ and Λ' to be equivalent.

3.1 A criterion for maximality of dominant weights in $P(\Lambda)$

If we can find the set of all dominant weights Λ' equivalent to Λ, it is straightforward to find the maximal dominant weight for each one, using the following result.

Proposition 3.2. *Suppose η is a dominant weight in $P(\Lambda)$, and write $\eta = \Lambda - \sum_i \gamma_i\alpha_i$. If η is maximal, then we have $\gamma_i < a_i$ for some i.*

Proof. For affine Kac–Moody algebras, [Ka, Proposition 11.2] tells us that the dominant weights in $P(\Lambda)$ are precisely the dominant weights η such that $\eta \leqslant \Lambda$. If η is a dominant weight in $\max(\Lambda)$ with $\gamma_i \geqslant a_i$ for each i, then we have $\eta + \delta \leqslant \Lambda$. $\eta + \delta$ is a dominant weight, and therefore must lie in $P(\Lambda)$, contradicting the maximality of η. □

Using this proposition, we can find the maximal dominant weights in $P(\Lambda)$ simply by finding which dominant weights lie in $\Lambda - Q + \mathbb{C}\delta$: given such a weight μ, there is a weight $\Lambda' \in P(\Lambda)$ with $\Lambda' \in \mu + \mathbb{C}\delta$. The unique maximal such weight is obtained by adding a multiple of δ to μ, and Proposition 3.2 tells us what this multiple of δ must be, since there is a unique multiple of δ which will make each γ_i a non-negative integer with $\gamma_i < a_i$ for some i.

Example 3.3. Suppose we are in type $D_5^{(1)}$, and $\Lambda = \Lambda_2$, which is of level 2. Using the results of the next section for this type, a dominant weight $\mu = \sum_{i=0}^{5} \mu_i\Lambda_i$ lies in $\Lambda - Q + \mathbb{C}\delta$ if and only if

$$\mu_0 + \mu_1 + 2\mu_2 + 2\mu_3 + \mu_4 + \mu_5 = 2$$

and

$$\mu_0 - \mu_1 + 2\mu_2 + 2\mu_4 \in 2 + 4\mathbb{Z}.$$

It is easy to check that the dominant weights satisfying these criteria are those with

$$(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5) = (0,0,1,0,0,0),\ (2,0,0,0,0,0),\ (0,2,0,0,0,0) \text{ or } (0,0,0,0,1,1).$$

The corresponding maximal weights in $P(\Lambda)$ are

$$\begin{aligned}&\Lambda_2,\\&\Lambda_2-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4-\alpha_5,\\&\Lambda_2-\alpha_0-2\alpha_2-2\alpha_3-\alpha_4-\alpha_5,\\&\Lambda_2-\alpha_0-\alpha_1-2\alpha_2-\alpha_3.\end{aligned}$$

3.2 Finding the maximal dominant weights in $P(\Lambda)$

In order to apply the membership criterion in Theorem 2.7, we must find all the maximal dominant elements of $P(\Lambda)$, and we have shown above that it suffices to determine which $\mu \in P_+$, with the same level as Λ, lie in $\Lambda - Q$. The difference $\psi = \Lambda - \mu$ is a weight of level zero, and so we seek a necessary and sufficient condition for a weight $\psi = \psi_0\Lambda_0 + \dots + \psi_l\Lambda_l$ of level 0 to lie in $Q + \mathbb{C}\delta$.

We will do this by considering the finite-dimensional Kac–Moody algebra $\mathring{\mathfrak{g}}$ associated to $\mathfrak{g}$ in [Ka, (6.3)]. The weight space $\mathring{\mathfrak{h}}^*$ of $\mathring{\mathfrak{g}}$ is the $\mathbb{C}$-span of $\alpha_1,\dots,\alpha_l$, while the $\mathbb{Z}$-span of these roots is the root lattice $\mathring{Q}$ of $\mathring{\mathfrak{g}}$.

Suppose $i \geqslant 1$. By expressing Λ_i in terms of the basis $\{\Lambda_0,\delta,\alpha_1,\dots,\alpha_l\}$ for $\mathfrak{h}^*$, we can write

$$\Lambda_i = x_i\Lambda_0 + y_i\delta + \mathring{\Lambda}_i$$

where $\mathring{\Lambda}_i \in \mathring{\mathfrak{h}}^*$. In fact, we have

$$x_i = x_i\langle\Lambda_0,c\rangle = \langle x_i\Lambda_0 + y_i\delta + \mathring{\Lambda}_i, c\rangle = \langle\Lambda_i,c\rangle = a_i^\vee.$$

Moreover, for $j \geqslant 1$ we have

$$\langle\mathring{\Lambda}_i,h_j\rangle = \langle\Lambda_i,h_j\rangle - a_i^\vee\langle\Lambda_0,h_j\rangle - y_i\langle\delta,h_j\rangle = \delta_{ij};$$

since $\mathring{\Lambda}_1,\dots,\mathring{\Lambda}_l$ lie in $\mathring{\mathfrak{h}}^*$, they therefore coincide with the fundamental weights for $\mathring{\mathfrak{g}}$. So the $\mathbb{Z}$-span $\mathring{P}$ of $\mathring{\Lambda}_1,\dots,\mathring{\Lambda}_l$ is the lattice of integral weights for $\mathring{\mathfrak{g}}$.

Now if $\psi = \sum_{i=0}^{l}\psi_i\Lambda_i$, then

$$\psi = \left(\sum_{i=0}^{l} a_i^\vee\psi_i\right)\Lambda_0 + \left(\sum_{i=1}^{l} y_i\psi_i\right)\delta + \sum_{i=1}^{l}\psi_i\mathring{\Lambda}_i \qquad \text{(since } a_0^\vee = 1\text{)}.$$

If in addition ψ has level zero, then $\sum_{i=0}^{l} a_i^\vee\psi_i = 0$, so we get

$$\psi \equiv \sum_{i-1}^{l}\psi_i\mathring{\Lambda}_i \pmod{\mathbb{C}\delta}.$$

So we have the following.

Lemma 3.4. *Suppose $\psi = \sum_{i=0}^{l}\psi_i\Lambda_i$ is an integral weight of level 0. Then*

$$\psi \in Q + \mathbb{C}\delta \qquad \textit{if and only if} \qquad \sum_{i=1}^{l}\psi_i\mathring{\Lambda}_i \in Q + \mathbb{C}\delta.$$

Using this lemma, we can find the criterion for ψ to lie in $Q + \mathbb{C}\delta$.

Proposition 3.5. *Suppose $\mathfrak{g}$ is an affine Lie algebra, and $\psi = \psi_0\Lambda_0 + \cdots + \psi_l\Lambda_l$ is an integral weight of level 0. Then $\psi \in Q + \mathbb{C}\delta$ if and only if the coordinates of ψ satisfy the congruences given in the following table.*

Type	Congruences
$A_l^{(1)} (l \geqslant 1)$	$\psi_1 + 2\psi_2 + \cdots + l\psi_l \equiv 0 \pmod{l+1}$
$A_{2l}^{(2)}$ $(l \geqslant 1)$	None
$A_{2l-1}^{(2)}$ $(l \geqslant 3)$	$\psi_1 + \psi_3 + \psi_5 + \cdots \equiv 0 \pmod 2$
$B_l^{(1)}$ $(l \geqslant 3)$	$\psi_l \equiv 0 \pmod 2$
$C_l^{(1)}$ $(l \geqslant 2)$	$\psi_1 + \psi_3 + \psi_5 + \cdots \equiv 0 \pmod 2$
$D_l^{(1)}$ $(l \geqslant 4, l \text{ even})$	$\psi_{l-1} + \psi_l \equiv 0 \pmod 2$ and $\psi_1 + \psi_3 + \psi_5 + \cdots + \psi_{l-1} \equiv 0 \pmod 2$
$D_l^{(1)}$ $(l \geqslant 5, l \text{ odd})$	$2(\psi_1 + \psi_3 + \psi_5 + \cdots + \psi_{l-2}) + \psi_{l-1} + \psi_l \equiv 0 \pmod 4$
$D_{l+1}^{(2)}$ $(l \geqslant 2)$	$\psi_l \equiv 0 \pmod 2$
$D_4^{(3)}$	None
$E_6^{(1)}$	$\psi_1 + 2\psi_2 \equiv \psi_5 + 2\psi_4 \pmod 3$
$E_7^{(1)}$	$\psi_4 + \psi_6 + \psi_7 \equiv 0 \pmod 2$
$E_8^{(1)}$	None
$E_6^{(2)}$	None
$F_4^{(1)}$	None
$G_2^{(1)}$	None

Proof. Using Lemma 3.4, we want to find when the weight $\mathring{\psi} := \sum_{i=1}^{l} \psi_i\mathring{\Lambda}_i$ lies in $Q + \mathbb{C}\delta$.

Suppose first that $\mathfrak{g}$ is not of type $A_{2l}^{(2)}$. then $a_0 = 1$, which means that $\alpha_0 \in \delta + \mathring{Q}$, and hence $Q + \mathbb{C}\delta = \mathring{Q} + \mathbb{C}\delta$. Since $\mathring{\psi} \in \mathbb{C}\mathring{Q} \not\ni \delta$, this means that $\mathring{\psi} \in Q + \mathbb{C}\delta$ if and only if $\mathring{\psi} \in Q$. Since $a_0^\vee = 1$ in all cases, the coefficients $\psi_1, \dots \psi_l$ can take any integer values (and still yield an integral weight ψ of level 0), so we are simply trying to find a condition for an arbitrary element of $\mathring{P}$ to lie in $\mathring{Q}$ in terms of the coefficients of $\mathring{\Lambda}_1, \dots, \mathring{\Lambda}_l$.

The Cartan matrix $\mathring{A}$ for $\mathring{\mathfrak{g}}$ is obtained from A by deleting the 0th row and column of A, and is non-singular with positive determinant. The (i, j)-entry of $\mathring{A}$ gives the coefficient of $\mathring{\Lambda}_i$ in α_j, and hence $\mathring{\psi} \in \mathring{Q}$ if and only if $(\psi_1, \dots, \psi_l) \in \mathring{A}\mathbb{Z}^l$. In particular, the index $|\mathring{P} : \mathring{Q}|$ equals $\det(\mathring{A})$.

Now let $\hat{Q}$ be the lattice of weights in $\mathring{P}$ satisfying the congruences given in the table. Since $\psi_1, \dots, \psi_l$ can take any integer values, it is easy to determine the index of $\hat{Q}$ in $\mathring{P}$: for example, if (as in most cases) $\hat{Q}$ is defined by a congruence modulo m on $\psi_1, \dots, \psi_l$, then $|\mathring{P} : \hat{Q}| = m$.

So to show that $\hat{Q} = \mathring{Q}$, it suffices to show that $|\mathring{P} : \hat{Q}| = |\mathring{P} : \mathring{Q}|$ (which just involves inspection of the congruences and computation of $\det(\mathring{A})$), and to show that $\mathring{Q} \subseteq \hat{Q}$ (which just involves inspecting $\mathring{A}$ to verify that the column satisfy the given congruences). We can now consider the individual cases.

Type $A_l^{(1)}$ In this case, $\mathring{A}$ is the Cartan matrix of type A_l, and it is easy to verify by induction that $\det(\mathring{A}) = l{+}1 = |\mathring{P} : \hat{Q}|$. For $i, j \geqslant 1$, the matrix entry a_{ij} equals $2\delta_{ij} - \delta_{i(j+1)} - \delta_{i(j-1)}$; so for a given $j \geqslant 1$,

$$a_{1j} + 2a_{2j} + \cdots + la_{lj} = (l{+}1)\delta_{lj} \equiv 0 \pmod{l{+}1},$$

so $\mathring{Q} \subseteq \hat{Q}$, as required.

Types $B_l^{(1)}$ and $D_{l+1}^{(2)}$ In these cases, $\mathring{A}$ is the Cartan matrix of type B_l, and it is easy to check by induction that $\det(\mathring{A}) = 2$. The lattice $\hat{Q}$ is defined by the condition that ψ_l is even, so $|\mathring{P} : \hat{Q}| = 2$. Since the bottom row of $\mathring{A}$ has only even entries, we have $\mathring{Q} \subseteq \hat{Q}$, and we are done.

Types $C_l^{(1)}$ and $A_{2l-1}^{(2)}$ In these cases $\mathring{A}$ is of type C_l, and so (since this is the transpose of the matrix in the previous case) $\det(\mathring{A}) = 2 = |\mathring{P} : \hat{Q}|$. It is simple to check that for each j

$$a_{1j} + a_{3j} + a_{5j} + \cdots \equiv 0 \pmod{2},$$

so $\alpha_j \in \hat{Q}$, so $\mathring{Q} \subseteq \hat{Q}$.

Type $D_l^{(1)}$ The Cartan matrix in this case is of type D_l, of determinant 4:

$$\begin{pmatrix}
2 & -1 & 0 & \cdots & & 0 & 0 \\
-1 & 2 & -1 & 0 & & \vdots & \vdots \\
0 & -1 & 2 & \ddots & \ddots & & \\
 & 0 & \ddots & \ddots & -1 & 0 & 0 \\
\vdots & & & \ddots & 2 & -1 & -1 \\
0 & \cdots & & 0 & -1 & 2 & 0 \\
0 & \cdots & & 0 & -1 & 0 & 2
\end{pmatrix}.$$

Whether l is even or odd, the congruences imply that $|\mathring{P} : \hat{Q}| = 4$. We observe that in each column the sum of the last two entries is even, so if $\psi \in \mathring{Q}$ then certainly $\psi_{l-1} + \psi_l$ is even. Furthermore, if l is even, then the sum of the odd-numbered entries in each column is even, so ψ satisfies the required congruences in this case. A similar check guarantees the required congruence in the case where l is odd. So $\mathring{Q} \subseteq \hat{Q}$.

Type $E_6^{(1)}$ Here $\mathring{A}$ is the Cartan matrix of type E_6, of determinant 3:

$$\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & -1 \\
0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix}.$$

$\hat{Q}$ is defined by the congruence $\psi_1 + 2\psi_2 \equiv \psi_5 + 2\psi_4 \equiv 0 \pmod 3$, so $|\mathring{P} : \hat{Q}| = 3$. One can easily verify that for any j we have $a_{1j} + 2a_{2j} \equiv a_{5j} + 2a_{4j} \pmod 3$, so $\mathring{Q} \subseteq \hat{Q}$.

Type $E_7^{(1)}$ In this case $\mathring{A}$ is the Cartan matrix of type E_7, of determinant 2:

$$\begin{pmatrix} 2 & {}^-1 & 0 & 0 & 0 & 0 & 0 \\ {}^-1 & 2 & {}^-1 & 0 & 0 & 0 & 0 \\ 0 & {}^-1 & 2 & {}^-1 & 0 & 0 & {}^-1 \\ 0 & 0 & {}^-1 & 2 & {}^-1 & 0 & 0 \\ 0 & 0 & 0 & {}^-1 & 2 & {}^-1 & 0 \\ 0 & 0 & 0 & 0 & {}^-1 & 2 & 0 \\ 0 & 0 & {}^-1 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

One can easily check that in any column, the sum of the 4th, 6th and 7th entries is even, so $\mathring{Q} \subseteq \hat{Q}$.

Types $E_8^{(1)}$, $E_6^{(2)}$, $F_4^{(1)}$, $G_2^{(1)}$ and $D_4^{(3)}$ In each of these cases, $\mathring{A}$ (which is of type E_8, F_4 or G_2) has determinant 1, so we have $\mathring{Q} = \mathring{P} = \hat{Q}$.

It remains to consider the case where $\mathfrak{g}$ is of type $A_{2l}^{(2)}$. In this case we have

$$\delta = 2(\alpha_0 + \cdots + \alpha_{l-1}) + \alpha_l,$$

so $Q + \mathbb{C}\delta = \mathring{Q} + \frac{1}{2}\mathbb{Z}\alpha_l + \mathbb{C}\delta$. Arguing as in the other cases, we therefore need to determine when the weight $\mathring{\psi}$ lies in $\mathring{Q} + \frac{1}{2}\mathbb{Z}\alpha_l$.

The matrix $\mathring{A}$ is of type C_l, so has determinant 2, so $|\mathring{P} : \mathring{Q}| = 2$. Since $\frac{1}{2}\alpha_l \notin \mathring{Q}$, we have $|\mathring{Q} + \frac{1}{2}\mathbb{Z}\alpha_l : \mathring{Q}| = 2$ also, so $\mathring{P} = \mathring{Q} + \frac{1}{2}\mathbb{Z}\alpha_l$, and $\psi_1, \ldots, \psi_l$ do not need to satisfy any additional congruence. □

4 Finding the weights in $P(\Lambda)$ – an example

The results in this paper give a way to determine whether a given integral weight η lies in the weight space $P(\Lambda)$. To do this, we begin by finding the set N of all maximal dominant weights in $P(\Lambda)$. There are only a finite number of maximal dominant weights of level k, and the congruences in Proposition 3.5 allow us to find exactly which of them lie in N.

Our next task is to compute $\tilde{N} = \mathring{W} \cdot N$. Recall that the Weyl group W may be regarded as a group of isometries of the weight space of $\mathfrak{g}$; the generating reflections $s_0, \ldots, s_l$ act via

$$s_i : \eta \longmapsto \eta - \langle h_i, \eta \rangle \alpha_i.$$

The group $\mathring{W}$ is generated by $s_1, \ldots, s_l$ and leaves the coefficient of α_0 fixed. We define the jth *floor* of $P(\Lambda)$ to be the set of all weights with 0-content j; then each floor is a union of $\mathring{W}$-orbits. The reflection $s_i = r_{\alpha_i}$ reflects all i-strings for $i = 1, \ldots, l$. If $j \geqslant a_0$, the floor j contains a δ-shifted copy of floor $j - a_0$ and whatever new maximal weights appear on that floor.Then $\tilde{N}$ is contained in the union of all the floors with a representative in N. This set can be quickly and efficiently generated once we can determine the lengths of the i-strings, by the method we now describe.

4.1 Positive hubs

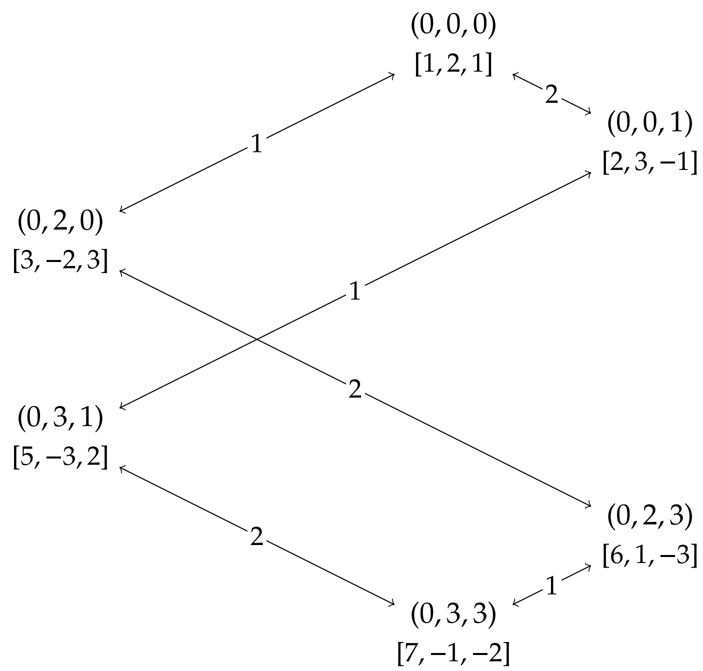
Definition 4.1. Suppose $\eta \in \mathfrak{h}^*$. The *hub* of η is the $(l+1)$-tuple $\theta(\eta) = (\theta_0, \ldots, \theta_l)$ defined by $\theta_i = \langle h_i, \eta \rangle$ for each i. We say that the hub of η is *positive* if each θ_i is non-negative, i.e. η is dominant.

Remark 4.2.

1. Note that $\theta(\eta)$ is just the projection of η onto the first $l+1$ components in its representation with respect to the basis
$$B_H = \{\Lambda_0, \Lambda_1, \ldots, \Lambda_l, \delta\}$$
for $\mathfrak{h}^*$. As such, it determines η up to addition of a multiple of δ; hence if θ is the hub of a weight in $P(\Lambda)$, there will be a unique maximal weight in $P(\Lambda)$ with hub θ.

2. If $\eta = \Lambda - \alpha$ is the first weight in an i-string $\eta, \eta - \alpha_i, \eta - 2\alpha_i, \ldots, \eta - n\alpha_i$, then the i-component of $\theta(\eta)$ will be n. This information allows us to quickly generate the entire floor from a few weights lying on 0-strings which started on floors j' for $j' < j$.

3. In an earlier work on cyclotomic Hecke algebras by the second author [Fa], the term "hub" was used for the negative of the hub defined here, and then the hubs of interest were the "negative hubs". We have reversed the sign here to make our work compatible with the conventional notation in affine Lie algebras.



4.2 An example

In this section we study the example where $\mathfrak{g} = \widehat{\mathfrak{sl}_3}$ (that is, of type $A_2^{(1)}$) and $\Lambda = \Lambda_0 + 2\Lambda_1 + \Lambda_2$ (so $l = 2$ and $k = 4$). We represent a weight $\Lambda - \gamma_0\alpha_0 - \gamma_1\alpha_1 - \gamma_2\alpha_2$ by its content $(\gamma_0, \gamma_1, \gamma_2)$, and we write the hub as $[\theta_0, \theta_1, \theta_2]$.

To begin with, we find the dominant weights in $P(\Lambda)$, using Proposition 3.5. Λ is of level 4, so the weights $\theta = \theta_0\Lambda_0 + \theta_1\Lambda_1 + \theta_2\Lambda_2$ in $P(\Lambda) + \mathbb{C}\delta$ are those for which $\theta_0 + \theta_1 + \theta_2 = 4$ and $\theta_1 + 2\theta_2 \equiv 1 \pmod 3$. It is easy to check that the only hubs $[\theta_0, \theta_1, \theta_2]$ satisfying these conditions are

$$[1,2,1], \quad [2,0,2], \quad [3,1,0], \quad [0,4,0], \quad [0,1,3].$$

Using Proposition 3.2, we can find the corresponding maximal weights, which we label a^0, a^1, a^2, A^1, A^2, respectively. The contents of these weights are as follows:

$$\gamma(a^0) = (0,0,0), \quad \gamma(a^1) = (0,1,0), \quad \gamma(a^2) = (0,1,1),$$
$$\gamma(A^1) = (1,0,1), \quad \gamma(A^2) = (1,1,0).$$

The superscript in the notation for each weight indicates the defect.

Next we compute $\tilde{N}$ by applying the reflections $r_{\alpha_1}, r_{\alpha_2}$ to these five weights. Applying r_{α_i} to a weight μ means adding $\langle h_i, \mu\rangle$ copies of $-\alpha_i$. For example, for the weight a^0, we get the picture in Figure 4 (where we write both the content and hub of each weight, and arrows labeled i represent the reflections $s_i = r_{\alpha_i}$).

It turns out that $\tilde{N}$ contains 21 weights, comprising four $\mathring{W}$-orbits. We describe these, together with their images under the reflections $r_{\alpha_1}, r_{\alpha_2}$, in the following table.

η	content	hub	$r_{\alpha_1}(\eta)$	$r_{\alpha_2}(\eta)$	$\tilde{\eta}$
a^0	$(0,0,0)$	$[1,2,1]$	b^0	d^0	$(0,0)$
b^0	$(0,2,0)$	$[3,-2,3]$	a^0	c^0	$(2,0)$
c^0	$(0,2,3)$	$[6,1,-3]$	f^0	b^0	$(0,3)$
d^0	$(0,0,1)$	$[2,3,-1]$	e^0	a^0	$(0,1)$
e^0	$(0,3,1)$	$[5,-3,2]$	d^0	f^0	$(3,1)$
f^0	$(0,3,3)$	$[7,-1,-2]$	c^0	e^0	$(3,3)$
a^1	$(0,1,0)$	$[2,0,2]$	a^1	b^1	$(1,0)$
b^1	$(0,1,2)$	$[4,2,-2]$	c^1	a^1	$(1,2)$
c^1	$(0,3,2)$	$[6,-2,0]$	b^1	c^1	$(3,2)$
a^2	$(0,1,1)$	$[3,1,0]$	b^2	a^2	$(1,1)$
b^2	$(0,2,1)$	$[4,-1,1]$	a^2	c^2	$(2,1)$
c^2	$(0,2,2)$	$[5,0,-1]$	c^2	b^2	$(2,2)$
A^1	$(1,0,1)$	$[0,4,0]$	B^1	A^1	$(3,0)^*$
B^1	$(1,4,1)$	$[4,-4,4]$	A^1	C^1	$(3,0)$
C^1	$(1,4,5)$	$[8,0,-4]$	C^1	B^1	$(3,0)$
A^2	$(1,1,0)$	$[0,1,3]$	B^2	D^2	$(0,3)^*$
B^2	$(1,2,0)$	$[1,-1,4]$	A^2	C^2	$(1,3)^*$
C^2	$(1,2,4)$	$[5,3,-4]$	F^2	B^2	$(1,3)$
D^2	$(1,1,3)$	$[3,4,-3]$	E^2	A^2	$(0,2)^*$

η	content	hub	$r_{\alpha_1}(\eta)$	$r_{\alpha_2}(\eta)$	$\tilde{\eta}$
E^2	$(1,5,3)$	$[7,-4,1]$	D^2	F^2	$(0,2)$
F^2	$(1,5,4)$	$[8,-3,-1]$	C^2	E^2	$(0,3)$

In the cases where the T-class has more than one element, we have chosen the first occurrence in the table as representative and marked it with an asterisk. Using this table, one can then test a given weight $\eta \in \Lambda - Q$ to see whether it lies in $P(\Lambda)$, by calculating $\tilde{\eta}$. For example, if $\eta = (2,7,3)$, we compute

$$\tilde{\eta} = ((7-2) \bmod 4, (3-2) \bmod 4) = (1,3) = \tilde{a}^2.$$

Letting $\zeta = a^2 = \Lambda - \alpha_1 - \alpha_2$, we have

$$\eta - \zeta = -2\alpha_0 - 6\alpha_1 - 2\alpha_2.$$

Hence $\eta = t_\alpha(\zeta)$, where $\alpha = -\alpha_1$. To determine whether or not ξ is in $P(\Lambda)$, we now calculate (from Theorem 2.7)

$$s(\eta) = 2 - \Big((\zeta|-\alpha_1) - \tfrac{1}{2}(-\alpha_1|-\alpha_1)4\Big).$$

Since $(\zeta|-\alpha_1) = -2 + 2 - 1 = -1$, and $(-\alpha_1|-\alpha_1) = 2$, we get $s(\eta) = -1 < 0$, showing that η is not in $P(\Lambda)$ by Lemma 2.3.

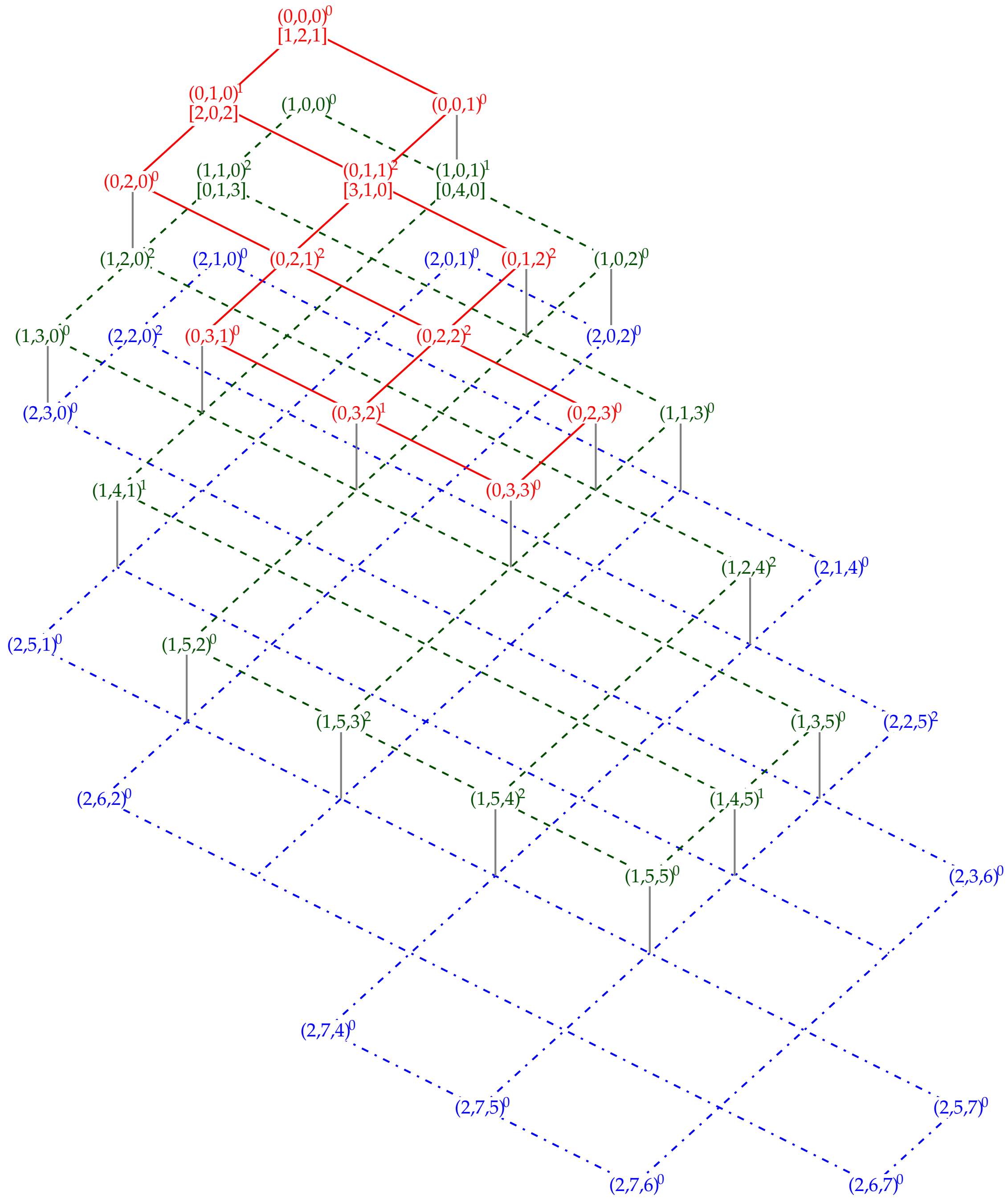
In Figure 4.2 we have given a three-dimensional representation of $P(\Lambda)$. We have recorded the contents of the maximal weights, with an exponent giving the defect of each maximal weight. The contents and defects of the other weights can be deduced by shifting by δ. Since all the dominant weights have defect less than k, one can get the δ-shifts by subtracting the residue of the defect modulo k and dividing by k. For the five maximal dominant weights we have also indicated the hubs in square brackets. The reflection r_{α_1} inverts strings going diagonally down to the left, while the reflection r_{α_2} inverts strings going diagonally down to the right. All the weights along a given horizontal line in the two-dimensional representation have the same height, so in the corresponding cyclotomic Hecke algebra they correspond to the blocks of H^Λ_d for a fixed rank d.

The weights on the 0th floor are all maximal weights, and the remaining maximal weights appear on 0-strings or at the ends of strings on other floors. We shall see in Lemma 5.2 below that there cannot be more than $\max a_i$ maximal weights at either end of a string, unless all the weights in the string are maximal.

5 δ-shifts

In this section we examine further the δ-shift of a weight in $P(\Lambda)$; in particular, we examine how δ-shift varies along a string in $P(\Lambda)$.

In Lemma 2.3 we showed that if Λ is a dominant integral weight, and η is any other equivalent weight, i.e. $\eta \in \Lambda - Q$, then there exists an integer s such that $\eta + s\delta \in P(\Lambda)$. The δ-shift $s(\eta)$ of $\eta \in P(\Lambda)$ was defined as the largest s such that $\eta + s\delta \in P(\Lambda)$. So $s(\eta) = 0$ if and only if η is maximal.



(0,0,0)0
[1,2,1]
(0,1,0)1
[2,0,2]
(1,0,0)0
(0,0,1)0
(0,2,0)0
(1,1,0)2
[0,1,3]
(0,1,1)2
[3,1,0]
(1,0,1)1
[0,4,0]
(1,2,0)2
(2,1,0)0
(0,2,1)2
(2,0,1)0
(0,1,2)2
(1,0,2)0
(1,3,0)0
(2,2,0)2
(0,3,1)0
(0,2,2)2
(2,0,2)0
(2,3,0)0
(0,3,2)1
(0,2,3)0
(1,1,3)0
(1,4,1)1
(0,3,3)0
(1,2,4)2
(2,1,4)0
(2,5,1)0
(1,5,2)0
(1,5,3)2
(1,3,5)0
(2,2,5)2
(2,6,2)0
(1,5,4)2
(1,4,5)1
(1,5,5)0
(2,3,6)0
(2,7,4)0
(2,7,5)0
(2,5,7)0
(2,7,6)0
(2,6,7)0

Recall that Δ^{re} denotes the set of *real roots*, i.e. the images of the simple roots under the action of the Weyl group. For $\alpha \in \Delta^{\mathrm{re}}$, an *$\alpha$-string* is a set of weights

$$\lambda, \lambda + \alpha, \lambda + 2\alpha, \dots, \lambda + t\alpha$$

all lying in $P(\Lambda)$, with $\lambda - \alpha, \lambda + (t+1)\alpha \notin P(\Lambda)$. If α is the simple root α_i, then we call an α-string an *i-string*.

We would like to study the behavior of the δ-shifts along an α-string. First we prove a simple statement which holds in any type.

Proposition 5.1. *Suppose $\alpha \in \Delta^{\mathrm{re}}$ and $\{\lambda, \lambda + \alpha, \dots, \lambda + t\alpha\}$ is an α-string. The delta-shifts along this string are symmetric and unimodal, i.e.*

$$s(\lambda + i\alpha) = s(\lambda + (s - i)\alpha) \qquad \text{for every } 0 \leqslant i \leqslant t$$

and

$$s(\lambda) \leqslant s(\lambda + \alpha) \leqslant \dots \leqslant s(\lambda + \lfloor \tfrac{t}{2} \rfloor \alpha).$$

Proof. [Ka, Proposition 11.1(a)] gives

$$\langle \lambda, \alpha^\vee \rangle = -\langle \lambda + t\alpha, \alpha^\vee \rangle,$$

so that the reflection $r_\alpha \in W$ reverses the alpha-string. Now the symmetry follows, since $P(\Lambda)$ and δ (and hence delta-shifts) are W-invariant.

For the unimodality, suppose we have $s = s(\lambda + i\alpha) > s(\lambda + (i+1)\alpha)$ for some $0 \leqslant i < \frac{t}{2}$. Then we have

$$\lambda + i\alpha + s\delta \in P(\Lambda), \qquad \lambda + (i+1)\alpha + s\delta \notin P(\Lambda), \qquad \lambda + (s - i)\alpha + s\delta \in P(\Lambda),$$

but this contradicts [Ka, Proposition 11.1(a)], which says that for any η, the set $(\eta + \mathbb{Z}\alpha) \cap P(\Lambda)$ must be an α-string. □

Now we provide some more precise information in types $A^{(1)}$ and $D^{(2)}$. First we need a lemma.

Lemma 5.2. *Suppose $\alpha \in \Delta^{\mathrm{re}}$ and $\eta \in P(\Lambda)$. Let $a = \max\{a_0, \dots, a_l\}$. Suppose that $s(\eta) = 0$, and that $\eta + a\alpha, \eta - a\alpha$ both lie in $P(\Lambda)$. Then every weight in the α-string containing η has δ-shift 0.*

Proof. By Proposition 2.1, we can find $w \in W$ such that $w\eta$ has a positive hub. Since the action of W preserves $P(\Lambda)$, the α-string S containing η is mapped by w to the $(w\alpha)$-string wS containing $w\eta$ as well as $w\eta \pm aw\alpha$. Furthermore, since δ is fixed by the action of W, the k-values along wS will be the same as the k-values along S.

In particular, $w\eta$ has δ-shift zero, so it is a maximal weight with positive hub. Hence by Proposition 3.2, some component of the content of $w\eta$, say the ith component γ_i, is less than $a_i \leqslant a$. If we write $w\alpha = \sum_j t_j \alpha_j$, then the ith component of the content of $w\eta \pm aw\alpha$ is $\gamma_i \pm at_i$; but $w\eta \pm aw\alpha$ lies in $P(\Lambda) \subseteq \Lambda - Q_+$, so has content in which every component is non-negative. Hence we must have $t_i = 0$. This means that the ith component of the content of every weight in the string wS equals $\gamma_i < a_i$, so every weight in the string wS is maximal. So the δ-shifts of all the weights in wS are zero, and hence the δ-shifts of all the weights in S are zero. □

Corollary 5.3. *Suppose $\mathfrak{g}$ is of type $A_l^{(1)}$ or $D_{l+1}^{(2)}$. Then along any α-string, the δ-shifts are either constant or strictly increasing to a symmetric central portion on which the δ-shifts are fixed, after which they are strictly decreasing.*

Proof. Write the α-string as $\{\lambda, \lambda + \alpha, \dots, \lambda + t\alpha\}$. Using Proposition 5.1, we just need to show that if there is some $0 \leqslant i < \frac{t-1}{2}$ for which $s(\lambda + i\alpha) = s(\lambda + (i{+}1)\alpha)$, then we have $s(\lambda + i\alpha) = s(\lambda + (i{+}1)\alpha) = \cdots = s(\lambda + (t - i)\alpha)$.

Let $s = s(\lambda + i\alpha)$, and consider the α-string containing $\lambda + i\alpha + s\delta$. This α-string contains $\lambda + i\alpha + s\delta$, $\lambda + (i{+}1)\alpha + s\delta$ and $\lambda + (t - i)\alpha + s\delta$, and $s(\lambda + (i{+}1)\alpha + s\delta) = 0$. So by Lemma 5.2 we have

$$s(\lambda + i\alpha + s\delta) = s(\lambda + (i{+}1)\alpha + s\delta) = \cdots = s(\lambda + (t - i)\alpha + s\delta) = 0,$$

and so

$$s(\lambda + i\alpha) = s(\lambda + (i{+}1)\alpha) = \cdots = s(\lambda + (t - i)\alpha).$$ □

We remark that this result is certainly not true in other types. In general, it is difficult to describe precisely the behaviour of δ-shifts along a string, but they seem to vary approximately quadratically along the string.

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References

[AK] S. Ariki & K. Koike, *A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$ and construction of its irreducible representations*, Adv. Math. **106** (1994), 216–243.

[AM] S. Ariki & A. Mathas, *The number of simple modules of the Hecke algebras of type $G(r, 1, n)$*, Math. Z. **233** (2000), 601–623.

[BK] J. Brundan and A. Kleshchev, *Graded decomposition numbers for cyclotomic Hecke algebras*, Adv. Math. **222** (2009), 1883–1942.

[Fa] M. Fayers, *Weights of multipartitions and representations of Ariki–Koike algebras*, Adv. Math. **206** (2006), 112–144.

[Ka] V. Kac, *Infinite Dimensional Lie Algebras* 3rd ed., Cambridge University Press (1990).

[Kl] A. Kleshchev, *Representation theory of symmetric groups and related Hecke algebras*, Bull. Amer. Math. Soc. **47** (2010), 419–481.