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James's Conjecture holds for weight four blocks of Iwahori–Hecke algebras

Matthew Fayers
Queen Mary, University of London, Mile End Road, London E1 4NS, U.K.*
Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge MA 02139-4307, U.S.A.†
m.fayers@qmul.ac.uk

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Abstract

James's Conjecture suggests that in certain cases, the decomposition numbers for the Iwahori–Hecke algebra of the symmetric group over a field of prime characteristic (and in particular, the decomposition numbers for the symmetric group itself) coincide with the decomposition numbers for a corresponding Iwahori–Hecke algebra defined over $\mathbb{C}$, and hence can be computed using the LLT algorithm. We prove this conjecture for blocks of weight 4.

1 Introduction

Let $\mathbb{F}$ be a field and n a non-negative integer. Let $\mathfrak{S}_n$ denote the symmetric group on n letters. The representation theory of $\mathfrak{S}_n$ over $\mathbb{F}$ has been extensively studied, often as a special case of the representation theory of the Iwahori–Hecke algebra $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$. The most important outstanding problem in the representation theory of these algebras is to determine the decomposition numbers, i.e. the composition multiplicities of the simple modules D^μ in the Specht modules S^λ, where λ and μ range over the set of partitions of n. In the case where $\mathbb{F} = \mathbb{C}$, this problem has been solved, in that there is an algorithm for computing the decomposition numbers. This result is an important step towards the general case, since it is known that the decomposition matrix in prime characteristic may be obtained from the corresponding decomposition matrix in infinite characteristic by post-multiplying by an 'adjustment matrix'. Very few adjustment matrices are known, but James's Conjecture suggests that for certain blocks of the Iwahori–Hecke algebra, the adjustment matrix should be the identity matrix. This conjecture is closely related to the celebrated Lusztig Conjecture, and both conjectures seem to be very hard.

Some progress has been made with small cases, where 'small' applies not to n but to the *weight* of the block in question. James's Conjecture concerns the case where the weight of the block is less than the characteristic of the underlying field, and a great deal of work has been done on blocks of small weight. Blocks of weight at most 1 have been understood for some time; in fact, they are of finite type, with blocks of weight 0 being simple. Blocks of weight 2 were systematically addressed by Scopes [19]

*Correspondence address.

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and Richards [16]; the latter gave an explicit description of the decomposition numbers for weight 2 blocks in odd or infinite characteristic, from which it follows that James's Conjecture holds for weight 2 blocks. Blocks of weight 3 presented difficulties for some years, until the present author was finally able to prove that over fields of characteristic at least 5, the decomposition numbers for weight 3 blocks are bounded above by 1. In the course of proving this, the author verified James's Conjecture for blocks of weight 3.

The task undertaken in this paper is to prove James's Conjecture for blocks of weight 4. The techniques used are similar to those used for blocks of weight 3, though an important difference is that we do not have a guiding conjecture as to what the decomposition numbers for weight 4 blocks look like; certainly the decomposition numbers can be bigger than 1. Accordingly, our proof involves few calculations of decomposition numbers; instead, we work directly with the entries of the adjustment matrix.

Of course, this leaves us a long way from a proof of James's Conjecture in general, and we do not expect that it could be proved using only the techniques in this paper. Our hope is that some of the techniques and cases could be generalised to arbitrary weight, in order to prove particular cases of James's Conjecture and thereby shed light on what the 'difficult' cases look like.

In the next section, we introduce the background theory that we shall require. There is a great deal of this, and accordingly some of it is treated very briefly. Then we outline the proof of the main theorem. The remaining sections of the paper give the details of the proof.

2 Background and method of proof

2.1 Representations of Iwahori–Hecke algebras

An excellent introduction to the representation theory of $\mathcal{H}_n$ is to be found in the book [14] by Mathas; we summarise the relevant points here. Suppose q is a non-zero element of a field $\mathbb{F}$, and write $\mathcal{H}_n$ for the Iwahori–Hecke algebra $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ of $\mathfrak{S}_n$ over $\mathbb{F}$ with parameter q. Let e denote the least integer such that $1 + q + \cdots + q^{e-1} = 0$ in $\mathbb{F}$, assuming throughout the paper that such an integer exists. Thus, e is an integer greater than 1; if $q = 1$ then e is the characteristic of $\mathbb{F}$, and otherwise e is the multiplicative order of q.

We assume that the reader is familiar with the combinatorics of partitions, e-regular partitions, Young diagrams and addable and removable nodes and their residues. To each partition λ of n is associated a *Specht module* S^λ for $\mathcal{H}_n$. (Note that we use the Dipper–James version [3] of Specht modules, rather than that of Mathas.) If λ is e-regular, then S^λ has an irreducible cosocle D^λ, and the D^λ give all the irreducible $\mathcal{H}_n$-modules as λ ranges over the set of e-regular partitions of n. The point of this paper is to compare different Iwahori–Hecke algebras, and we may write $S^\lambda_{\mathbb{F},q}$ and $D^\lambda_{\mathbb{F},q}$ if there is a danger of ambiguity.

The central problem in the representation theory of $\mathcal{H}_n$ is to determine the *decomposition numbers* $[S^\lambda : D^\mu]$ for all pairs (λ, μ) of partitions of n with μ e-regular. These are conventionally recorded in the *decomposition matrix*, which has rows indexed by partitions, and columns indexed by e-regular partitions, with the (λ, μ)-entry being $[S^\lambda : D^\mu]$.

2.2 Canonical bases and v-decomposition numbers

When $\mathbb{F}$ has infinite characteristic (we adopt the convention in this paper that the characteristic of a field is the order of its prime subfield) and q is a primitive eth root of unity in $\mathbb{F}$, there is an algorithm for computing the decomposition numbers for $\mathcal{H}_n$. The Fock space representation of the quantum group $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ contains a submodule isomorphic to the irreducible integrable highest weight module $L(\Lambda_0)$, and this submodule possesses a *canonical basis* $\{G(\mu) \mid \mu \text{ an } e\text{-regular partition}\}$. Expanding each $G(\mu)$ with respect to the natural basis $\{\lambda \mid \lambda \text{ a partition}\}$ for the Fock space, one obtains

$$G(\mu) = \sum_{\lambda} d^{(e)}_{\lambda\mu}(v)\lambda,$$

where each $d^{(e)}_{\lambda\mu}(v)$ is a polynomial with non-negative integer coefficients, which is zero unless $|\lambda| = |\mu|$. This is known as a 'v-decomposition number', in view of the following theorem, due to Ariki.

Theorem 2.1. [1, Theorem 4.4] *Suppose* $\mathrm{char}(\mathbb{F}) = \infty$, *and* λ *and* μ *are partitions of* n *with* μ *e-regular. Then*

$$[S^{\lambda}_{\mathbb{F},q} : D^{\mu}_{\mathbb{F},q}] = d^{(e)}_{\lambda\mu}(1).$$

More details of the Fock space, and the important properties of the canonical basis, can be found in the paper by Lascoux, Leclerc and Thibon [13], in which the 'LLT algorithm' for computing the canonical basis is also given.

2.3 Blocks of $\mathcal{H}_n$ and the abacus

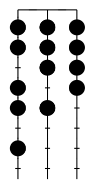
The abacus is a combinatorial device frequently used in the representation theory of symmetric groups and Iwahori–Hecke algebras. Since the choice of notation can vary, we make explicit the version we use here.

With e as above, take an abacus with e vertical runners, numbered $0, \dots, e-1$ from left to right, and mark positions $0, 1, \dots$ on the runners, increasing from left to right along successive 'rows' (so that on runner i, the non-negative integers congruent to i modulo e appear in increasing order from the top down). Given a partition λ, take an integer $r \geqslant \lambda'_1$, and define $\beta_i = \lambda_i + r - i$ for $i = 1, \dots, r$. Now place a bead on the abacus at position β_i, for each i. The resulting configuration is the *abacus display for* λ *with* r *beads*.

From an abacus display for λ, we can read of the *e-quotient* of λ as follows: for each i, we regard runner i on its own as an abacus with only one runner, and read off the corresponding partition, which we denote $\lambda(i)$; thus $\lambda(i)_j$ is the number of empty spaces above the jth lowest bead on runner i. The (e-)quotient of λ is then simply the e-tuple $(\lambda(0), \dots, \lambda(e-1))$. Of course, taking an abacus display with a different number of beads changes the quotient; specifically, if we increase the number of beads by 1, then the components of the quotient are permuted cyclically, so that $(\lambda(0), \dots, \lambda(e-1))$ becomes $(\lambda(e-1), \lambda(0), \dots, \lambda(e-2))$.

It is easy to see that a partition λ is determined by its quotient and the number of beads on each runner of the abacus. Accordingly, we may write λ as $\langle 0_{\lambda(0)}, \dots, e{-}1_{\lambda(e-1)} \mid b_0, \dots, b_{e-1} \rangle$, where b_i is the number of beads on runner i, for each i. We omit $i_{\lambda(i)}$ if $\lambda(i) = \varnothing$, and write $i_{\lambda(i)}$ simply as i if $\lambda(i) = (1)$. We may also group together equal b_is, or omit the b_is altogether if they are understood.

For example, if $e = 3$ and $\lambda = (6, 2^3, 1^3)$, then (taking $r = 13$) λ has an abacus display



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and we may write λ as $\langle 0_{2,1^2}, 1 \mid 5, 4^2\rangle$.

If λ has e-quotient $(\lambda(0), \dots, \lambda(e-1))$, then the *(e-)weight* of λ is defined as $|\lambda(0)|+\dots+|\lambda(e-1)|$. The *(e-)core* of λ is the partition whose abacus display is obtained from an abacus display for λ by moving all the beads as high as possible on their runners; if λ is a partition of n, then its core is a partition of $n - ew$, where w is the weight of λ. The definition of the e-core of a partition is vital to the following theorem, which is still called the 'Nakayama Conjecture'.

Theorem 2.2. [14, Corollary 5.38] *Suppose λ and μ are partitions of n. Then the Specht modules S^λ and S^μ lie in the same block of $\mathcal{H}_n$ if and only if λ and μ have the same e-core.*

Given Theorem 2.2, we abuse notation by saying that two partitions λ and μ of n lie in the same block of $\mathcal{H}_n$ to mean that λ and μ have the same e-core. Theorem 2.2 allows us to define the weight and core of any block of $\mathcal{H}_n$, meaning the e-weight and e-core of any partition lying in that block. Note that if we have an abacus display for a partition λ, then the e-core of λ is determined by the numbers $b_0, \dots, b_{e-1}$ of beads on the runners, and so we may define an abacus for a block by specifying the number of beads on each runner, without specifying their positions. A block is then determined by its weight and its abacus, so we may speak of the weight w block with the $\langle b_0, \dots, b_{e-1}\rangle$ notation. For example, if $e = 3$ then the partition λ above lies in the weight 5 block with the $\langle 5, 4^2\rangle$ notation.

We note that we can also easily read addable and removable nodes from an abacus display. Take an abacus display for λ, with r beads, say, and choose $i \in \mathbb{Z}/e\mathbb{Z}$. Let $j \in \{0, \dots, e-1\}$ be given by $j \equiv i + r \pmod e$. Then removable nodes of λ of residue i correspond to beads on runner j of the abacus with no bead immediately to the left, while addable nodes of residue i correspond to unoccupied spaces on runner j with beads immediately to the left. Here we adopt the convention that position $x - 1$ is immediately to the left of position x even when $x \equiv 0 \pmod e$, and that position 0 always has a bead immediately to the left. Removing a removable node corresponds to moving the corresponding bead one space to the left, while adding a node corresponds to moving the corresponding bead one space to the right.

2.4 Adjustment matrices and James's Conjecture

James's Conjecture concerns the comparison of the decomposition matrix for $\mathcal{H}_n$ with that of an Iwahori–Hecke algebra defined over $\mathbb{C}$. Fix a primitive eth root of unity ζ in $\mathbb{C}$, and write $\mathcal{H}_n^0$ for the Iwahori–Hecke algebra $\mathcal{H}_{\mathbb{C},\zeta}(\mathfrak{S}_n)$. Many of the theorems that we state involving modules for $\mathcal{H}_n$ we shall use with $\mathcal{H}_n$ replaced by $\mathcal{H}_n^0$ (that is, in the special case where $\mathbb{F} = \mathbb{C}$ and $q = \zeta$). For example, applying Theorem 2.2 with $\mathbb{F}, q$ replaced by $\mathbb{C}, \zeta$, one sees that two partitions λ and μ lie in the same block of $\mathcal{H}_n$ if and only if they lie in the same block of $\mathcal{H}_n^0$. Therefore, given a block B of $\mathcal{H}_n$, we may define the *block of $\mathcal{H}_n^0$ corresponding to B* to be the block B^0 with same e-core as B. Now we have the following theorem, due to Geck.

Theorem 2.3. [14, Theorem 6.35] *Let B be a block of $\mathcal{H}_n$, and B^0 the corresponding block of $\mathcal{H}_n^0$. Let D and D^0 be the decomposition matrices for these blocks, each with rows indexed by partitions in B and columns indexed by e-regular partitions in B. Then there exists a square matrix A, with non-negative integer entries, such that $D = D^0A$.*

The matrix A in this theorem is called the *adjustment matrix* for B, and is the main object of study in this paper. It arises from a 'decomposition map' from the category of $\mathcal{H}_n^0$-modules to the category of $\mathcal{H}_n$-modules. The (λ, μ)-entry of A gives the composition multiplicity of $D^\mu_{\mathbb{F},q}$ in the image under this map of the simple module $D^{\lambda}_{\mathbb{C},\zeta}$. An excellent introduction to decomposition maps can be found in the article by Geck [7]. Since the decomposition numbers for $\mathcal{H}^0_n$ can be computed by the LLT algorithm, determining the decomposition matrices in arbitrary characteristic is equivalent to computing adjustment matrices. However, not a great deal is known about adjustment matrices; our motivating conjecture is the following.

Conjecture 2.4. (James's Conjecture) *Suppose B is a block of $\mathcal{H}_n$, and that the weight of B is less than* $\mathrm{char}(\mathbb{F})$. *Then the adjustment matrix for B is the identity matrix.*

We remark that we have recently extended this conjecture to give a necessary and sufficient condition for the adjustment matrix of a block B to be the identity matrix [6]; however, in this paper we restrict attention to the version above. Some progress has been made in proving this for blocks of small weight. We shall use the following result.

Theorem 2.5. [16, 4] *Suppose* $\mathrm{char}(\mathbb{F}) \geqslant 5$, *and that B is a block of $\mathcal{H}_n$ of weight at most* 3. *Then James's Conjecture holds for B, and if λ, μ are partitions in B with μ e-regular, then* $[S^\lambda : D^\mu] \leqslant 1$.

If λ and μ are e-regular partitions lying in a block B of $\mathcal{H}_n$, we write $\mathrm{adj}_{\lambda\mu}$ for the (λ, μ)-entry of the adjustment matrix for B. Our main theorem is as follows.

Theorem 2.6. *James's Conjecture holds for blocks of weight* 4. *That is, if* $\mathrm{char}(\mathbb{F}) \geqslant 5$ *and λ, μ are e-regular partitions lying in a block B of $\mathcal{H}_n$ of weight* 4, *then* $\mathrm{adj}_{\lambda\mu} = \delta_{\lambda\mu}$.

2.5 The dominance order, the Jantzen–Schaper formula and Ryom-Hansen's theorem

The following is one of the simplest results in the representation theory of $\mathcal{H}_n$; it arises from the fact that $\mathcal{H}_n$ is cellular.

Theorem 2.7. *Suppose λ and μ are partitions of n, with μ e-regular. Then:*

- $[S^\mu : D^\mu] = 1$*;*
- *if* $[S^\lambda : D^\mu] > 0$, *then* $\mu \trianglerighteq \lambda$.

Of course, this theorem also applies in the special case $(\mathbb{F}, q) = (\mathbb{C}, \zeta)$, and comparing the two statements yields the following consequence for adjustment matrices.

Corollary 2.8. *Suppose λ and μ are e-regular partitions lying in a block B of $\mathcal{H}_n$. Then:*

- $\mathrm{adj}_{\mu\mu} = 1$*;*
- *if* $\mathrm{adj}_{\lambda\mu} > 0$, *then* $\mu \trianglerighteq \lambda$.

The order $\trianglerighteq$ in these results is the usual dominance order, but the results can be strengthened by using the Jantzen–Schaper formula (of which the version for Iwahori–Hecke algebras is proved in [10]). This is described in detail elsewhere, so we give just the details relevant to the discussion. For any ordered pair (λ, ν) of partitions of n, one defines an integer $J_{\mathbb{F},q}(\lambda, \nu)$, which we call a *Jantzen–Schaper coefficient*. The Jantzen–Schaper formula states that for any $\lambda \neq \mu$ we have

$$[S^\lambda : D^\mu] \leqslant \sum_{\nu} J_{\mathbb{F},q}(\lambda, \nu)[S^\nu : D^\mu],$$

and that the left-hand side of the above inequality is zero if and only if the right-hand side is zero. The crucial property of the Jantzen–Schaper coefficients is that $J_{\mathbb{F},q}(\lambda,\nu)$ is zero unless λ and ν lie in the same block and $\lambda \lhd \nu$. This means that the Jantzen–Schaper formula is a useful tool for calculating decomposition numbers for a particular block of $\mathcal{H}_n$ recursively.

We can use the Jantzen–Schaper formula to refine Theorem 2.7. If we redefine the dominance order by putting $\lambda \trianglelefteqslant \nu$ whenever $J_{\mathbb{F},q}(\lambda,\nu) \neq 0$ and extending transitively and reflexively, then we get a partial order, which is a refinement of the usual dominance order, and which we call the *Jantzen–Schaper dominance order*. It is an immediate consequence of the Jantzen–Schaper formula that Theorem 2.7 and Corollary 2.8 hold with this new dominance order. For the rest of this paper, we use $\trianglelefteqslant$ to denote the Jantzen–Schaper dominance order. Although the order in depends in general on $\mathbb{F}$ and q, the following lemma (which underpins James's Conjecture) shows that for our purposes we need only consider e (which will always be implicit).

Lemma 2.9. *Suppose λ and ν are partitions lying in a block B of $\mathcal{H}_n$ of weight $w < \operatorname{char}(\mathbb{F})$. Then*

$$J_{\mathbb{F},q}(\lambda,\nu) = J_{\mathbb{C},\zeta}(\lambda,\nu).$$

Proof. This is a straightforward consequence of the definition of $J_{\mathbb{F},q}(\lambda,\nu)$; it rests on the fact that because $w < \operatorname{char}(\mathbb{F})$, the Young diagram of a partition in B cannot have a rim hook of length divisible by $e\operatorname{char}(\mathbb{F})$. □

Since in this paper we shall be concerned entirely with blocks of weight less than $\operatorname{char}(\mathbb{F})$, we write $J(\lambda,\nu)$ in place of $J_{\mathbb{F},q}(\lambda,\nu)$ or $J_{\mathbb{C},\zeta}(\lambda,\nu)$ without fear of ambiguity. We shall use the Jantzen–Schaper formula to calculate decomposition numbers via the following result, which is an immediate consequence of the formula and Theorem 2.7.

Proposition 2.10. *Suppose λ and μ are partitions of n with μ e-regular. If we have*

$$\sum_{\lambda \lhd \nu \trianglelefteqslant \mu} J_{\mathbb{F},q}(\lambda,\nu)[S^\nu : D^\mu] \leqslant 1,$$

then

$$[S^\lambda : D^\mu] = \sum_{\lambda \lhd \nu \trianglelefteqslant \mu} J_{\mathbb{F},q}(\lambda,\nu)[S^\nu : D^\mu].$$

We use Proposition 2.10 several times later, to show that certain decomposition numbers $[S^\lambda : D^\mu]$ are independent of the underlying characteristic. In each case, we give a table of all the coefficients $J(\nu,\xi)$ for $\lambda \trianglelefteqslant \nu \lhd \xi \trianglelefteqslant \mu$; we then use Proposition 2.10 recursively to show that the Jantzen–Schaper formula determines the decomposition numbers $[S^\nu_{\mathbb{F},q} : D^\mu_{\mathbb{F},q}]$ and $[S^\lambda_{\mathbb{C},\zeta} : D^\mu_{\mathbb{C},\zeta}]$, and that they are equal. (In one case, the bound we get for each of these decomposition numbers is 2, and we use an independent argument to show that both decomposition numbers equal 1.)

Ryom-Hansen [17] demonstrated an important connection between the Jantzen–Schaper formula and v-decomposition numbers. (In fact, his paper contains a slight error, which is corrected by Yvonne in [20].) His result may be stated using our notation as follows.

Theorem 2.11. [17, Theorem 1] *Suppose λ and μ are partitions of n, with μ e-regular, and let $d^{(e)\prime}_{\lambda\mu}(v)$ denote the derivative with respect to v of the v-decomposition number $d^{(e)}_{\lambda\mu}(v)$. Then*

$$\sum_{\lambda \lhd \nu \trianglelefteqslant \mu} J_{\mathbb{C},\zeta}(\lambda,\nu) d^{(e)}_{\nu\mu}(1) = d^{(e)\prime}_{\lambda\mu}(1).$$

This result will prove useful to us in estimating decomposition numbers $[S^\lambda : D^\mu]$ when we know that $\mathrm{adj}_{\nu\mu} = 0$ for $\lambda \lhd \nu \lhd \mu$ but we do not know the decomposition numbers $[S^\nu : D^\mu]$. We use the following corollary.

Corollary 2.12. *Suppose λ and μ are e-regular partitions lying in a block B of $\mathcal{H}_n$ of weight $w <$ char($\mathbb{F}$). Suppose that $\mathrm{adj}_{\nu\mu} = 0$ for all e-regular partitions ν such that $\lambda \lhd \nu \lhd \mu$, and that the v-decomposition number $d^{(e)}_{\lambda\mu}(v)$ equals 0 or v. Then $\mathrm{adj}_{\lambda\mu} = 0$.*

Proof. Since $\mathrm{adj}_{\nu\mu} = 0$ for all e-regular partitions ν satisfying $\lambda \lhd \nu \lhd \mu$, we have $[S^\nu_{\mathbb{F},q} : D^\mu_{\mathbb{F},q}] = [S^\nu_{\mathbb{C},\zeta} : D^\mu_{\mathbb{C},\zeta}]$ for all partitions ν such that $\lambda \lhd \nu \trianglelefteq \mu$. Hence by Lemma 2.9 the bound b given by the Jantzen–Schaper formula for $[S^\lambda_{\mathbb{F},q} : D^\mu_{\mathbb{F},q}]$ is the same as the bound for $[S^\lambda_{\mathbb{C},\zeta} : D^\mu_{\mathbb{C},\zeta}]$. By Theorem 2.1 and Theorem 2.11 b equals either 0 or 1 (as $d^{(e)}_{\lambda\mu}(v)$ equals 0 or v, respectively), so by Proposition 2.10 we have

$$[S^\lambda_{\mathbb{F},q} : D^\mu_{\mathbb{F},q}] = b = [S^\lambda_{\mathbb{C},q;} : D^\mu_{\mathbb{C},\zeta}],$$

and hence $\mathrm{adj}_{\lambda\mu} = 0$. □

2.6 The row removal theorem

The following is a very useful theorem in the representation theory of $\mathcal{H}_n$.

Theorem 2.13. [8, Corollary 6.18] *Suppose λ and μ are partitions of n, with μ e-regular, and that for some r we have*

$$\lambda_1 + \cdots + \lambda_r = \mu_1 + \cdots + \mu_r.$$

Define

$$\lambda^1 = (\lambda_1, \ldots, \lambda_r), \qquad \mu^1 = (\mu_1, \ldots, \mu_r),$$
$$\lambda^2 = (\lambda_{r+1}, \lambda_{r+2}, \ldots), \qquad \mu^2 = (\mu_{r+1}, \mu_{r+2}, \ldots).$$

Then

$$[S^\lambda : D^\mu] = [S^{\lambda^1} : D^{\mu^1}].[S^{\lambda^2} : D^{\mu^2}].$$

The case $r = 1$ of this theorem is known as the 'row removal theorem', and has the following consequence for adjustment matrices.

Corollary 2.14. *Suppose λ and μ are e-regular partitions of n of weight $w <$ char($\mathbb{F}$) and that $\lambda_1 = \mu_1$. Define*

$$\lambda^2 = (\lambda_2, \lambda_3, \ldots), \qquad \mu^2 = (\mu_2, \mu_3, \ldots),$$

and suppose that James's Conjecture holds for the block containing λ^2 and μ^2. Then $\mathrm{adj}_{\lambda\mu} = \delta_{\lambda\mu}$.

Proof. By Theorems 2.13 and 2.7 we have

$$[S^\lambda_{\mathbb{F},q} : D^\mu_{\mathbb{F},q}] = [S^{\lambda^2}_{\mathbb{F},q} : D^{\mu^2}_{\mathbb{F},q}], \qquad [S^\lambda_{\mathbb{C},\zeta} : D^\mu_{\mathbb{C},\zeta}] = [S^{\lambda^2}_{\mathbb{C},\zeta} : D^{\mu^2}_{\mathbb{C},\zeta}].$$

Since James's Conjecture holds for the block containing λ^2 and μ^2, we have

$$[S^{\lambda^2}_{\mathbb{F},q} : D^{\mu^2}_{\mathbb{F},q}] = [S^{\lambda^2}_{\mathbb{C},\zeta} : D^{\mu^2}_{\mathbb{C},\zeta}],$$

and we deduce

$$[S^{\lambda}_{\mathbb{F},q} : D^{\mu}_{\mathbb{F},q}] = [S^{\lambda}_{\mathbb{C},\zeta} : D^{\mu}_{\mathbb{C},\zeta}],$$

from which the result follows. □

We shall use this corollary frequently in the rest of the paper. Typically, we shall be considering a pair of e-regular partitions λ and μ of weight 4 for which we wish to show that $\mathrm{adj}_{\lambda\mu} = 0$, and it will happen that in the abacus display for λ the bead corresponding to its largest part has $s \geqslant 1$ empty spaces above it on the same runner. Constructing the partition $\lambda^2 = (\lambda_2, \lambda_3, \dots)$ corresponds to replacing this bead with an empty space, and this means that the weight of λ^2 is $4 - s \leqslant 3$; so if $\mu_1 = \lambda_1$ we can apply Corollary 2.14 and Theorem 2.5 to get $\mathrm{adj}_{\lambda\mu} = 0$.

2.7 The runner removal theorems

Here we note two theorems which will help us to compute v-decomposition numbers more efficiently by relating v-decomposition numbers for different values of e.

Theorem 2.15. [11, Theorem 3.2] *Suppose that $e \geqslant 3$, and that λ and μ are partitions of n with μ e-regular, and take abacus displays for λ and μ with r beads. Suppose that there is some i such that in both displays the last bead on runner i occurs before every unoccupied space on the abacus. Define two abacus displays with $e - 1$ runners by deleting runner i from each display, and let λ^- and μ^- be the partitions defined by these displays. If μ^- is $(e-1)$-regular, then*

$$d^{(e)}_{\lambda\mu}(v) = d^{(e-1)}_{\lambda^-\mu^-}(v).$$

Theorem 2.16. [5, Theorem 4.1] *Suppose that $e \geqslant 3$, and that λ and μ are partitions of n with μ e-regular, and take abacus displays for λ and μ with r beads. Suppose that there is some i such that in both displays the first unoccupied space on runner i occurs after every bead on the abacus. Define two abacus displays with $e - 1$ runners by deleting runner i from each display, and let λ^- and μ^- be the partitions defined by these displays. If μ^- is $(e-1)$-regular, then*

$$d^{(e)}_{\lambda\mu}(v) = d^{(e-1)}_{\lambda^-\mu^-}(v).$$

We refer to these theorems as the 'runner removal theorems'.

2.8 The modular branching rules

For any $\kappa > 0$, $\mathcal{H}_n$ is naturally a subalgebra of $\mathcal{H}_{n+\kappa}$, and there are well-behaved induction and restriction functors between the module categories of these algebras. Given blocks B, C of $\mathcal{H}_n$ and $\mathcal{H}_{n+\kappa}$ respectively, and given an $\mathcal{H}_n$-module M and an $\mathcal{H}_{n+\kappa}$-module N, we write $M{\uparrow^C}$ for the projection onto C of $M{\uparrow^{\mathcal{H}_{n+1}}}$, and we write $N{\downarrow_B}$ for the projection onto B of $N{\downarrow_{\mathcal{H}_n}}$.

We are most interested in the effect of these functors on the simple modules D^μ. This is to some extent well-understood, through the 'modular branching rules' of Brundan and Kleshchev [2]. We shall not describe these in detail here, but we note that the modular branching rules tells us precisely when $D^\mu{\uparrow^C}$ is non-zero and when it is semi-simple, and the isomorphism type of its socle, and that this information depends only on μ, e and C, not $\mathbb{F}$ and q. A similar statement applies to restricted simple modules. The particular cases of the modular branching rules that we need will be summarised in the next two sections, and Appendix A.

2.9 Lowerable pairs of simple modules

In this section, we consider restricting and inducing simple modules from blocks of weight 4 to blocks of weight less than 4.

Proposition 2.17. *Suppose* $\mathrm{char}(\mathbb{F}) \geqslant 5$, *$B$ is a block of $\mathcal{H}_n$ of weight 4, and C is a block of $\mathcal{H}_{n-1}$ or $\mathcal{H}_{n+1}$ of weight less than 4. Suppose λ and μ are distinct e-regular partitions lying in B such that $D^\mu{\downarrow_C}$ (or $D^\mu{\uparrow^C}$, respectively) is non-zero, while $D^\lambda{\downarrow_C}$ (or $D^\lambda{\uparrow^C}$, respectively) is either zero or simple. Then* $\mathrm{adj}_{\lambda\mu} = 0$.

Proof. This is very similar to [4, Lemma 4.4]. We assume that C is a block of $\mathcal{H}_{n-1}$; the proof in the other case is essentially identical. Recalling the notation of Section 2.4, let B^0 and C^0 be the blocks of $\mathcal{H}^0_n$ and $\mathcal{H}^0_{n-1}$ respectively corresponding to B and C.

The modular branching rules [2] imply that there is an e-regular partition $\hat{\mu}$ in C such that $D^\mu_{\mathbb{F},q}{\downarrow_C}$ is an indecomposable module with simple socle $D^{\hat{\mu}}_{\mathbb{F},q}$, while $D^\mu_{\mathbb{C},\zeta}{\downarrow_C}$ is an indecomposable module with simple socle $D^{\hat{\mu}}_{\mathbb{C},\zeta}$. Moreover, we have $[D^\lambda_{\mathbb{C},\zeta}{\downarrow_C} : D^{\hat{\mu}}_{\mathbb{C},\zeta}] = 0$; for $D^\lambda_{\mathbb{C},\zeta}{\downarrow_{C^0}}$ is either simple or zero, and if it is simple then the modular branching rules show that it is a simple module other than $D^{\hat{\mu}}_{\mathbb{C},\zeta}$.

Let T be the 'simple branching matrix' from B to C, with rows indexed by e-regular partitions in B and columns by e-regular partitions in C, and with the (ν,ξ)-entry being the composition multiplicity $[D^\nu_{\mathbb{F},q}{\downarrow_C} : D^\xi_{\mathbb{F},q}]$. Let T^0 be the simple branching matrix from B^0 to C^0, defined analogously. Using the fact that restriction is an exact functor together with the fact that James's Conjecture holds for C shows that $T^0 = AT$, where A is the adjustment matrix for B. Comparing the $(\lambda,\hat{\mu})$-entries of both sides, we have

$$\begin{aligned} 0 = [D^\lambda_{\mathbb{C},\zeta}{\downarrow_{C^0}} : D^{\hat{\mu}}_{\mathbb{C},\zeta}] &= \sum_\nu \mathrm{adj}_{\lambda\nu}[D^\nu_{\mathbb{F},q}{\downarrow_C} : D^{\hat{\mu}}_{\mathbb{F},q}] \\ &= \mathrm{adj}_{\lambda\mu}[D^\mu_{\mathbb{F},q}{\downarrow_C} : D^{\hat{\mu}}_{\mathbb{F},q}] + \sum_{\nu\neq\mu} \mathrm{adj}_{\lambda\nu}[D^\nu_{\mathbb{F},q}{\downarrow_C} : D^{\hat{\mu}}_{\mathbb{F},q}]; \end{aligned}$$

since every term on the right-hand side is non-negative and $[D^\mu_{\mathbb{F},q}{\downarrow_C} : D^{\hat{\mu}}_{\mathbb{F},q}] > 0$, we have $\mathrm{adj}_{\lambda\mu} = 0$. □

If λ and μ are e-regular partitions lying in B as above, and satisfying the hypotheses of Proposition 2.17 for some C, then we say that the pair (λ,μ) is *lowerable*.

Note that the condition that $D^\lambda{\downarrow_C}$ (or $D^\lambda{\uparrow^C}$) is zero or simple is very mild. We know that an abacus for C may be obtained from an abacus for B by moving a bead between two adjacent runners, say $i-1$ and i. Using the modular branching rules, we find that $D^\lambda{\downarrow_C}$ (or $D^\lambda{\uparrow^C}$) is reducible if and only if there are equal numbers of beads on these two runners (so that C has weight 3) and λ is the partition $\langle i_{2^2}\rangle$.

The condition for $D^\mu{\downarrow_C}$ (or $D^\mu{\uparrow^C}$) to be non-zero can be checked from the abacus display for μ using the modular branching rule; in Appendix A we illustrate all possible configurations of runners $i-1$ and i in the abacus display which give $D^\mu{\downarrow_C}$ (or $D^\mu{\uparrow^C}$) non-zero. Specifically, we have the following statement.

- Suppose C is a block of $\mathcal{H}_{n-1}$, and an abacus for C is obtained from an abacus for B by moving a bead from runner i to runner $i-1$. If μ is an e-regular partition in B, then $D^\mu{\downarrow_C} \neq 0$ if and only if the configuration of runners $i-1$ and i in the abacus display for μ appears in A.1 or A.2.

- Suppose C is a block of $\mathcal{H}_{n+1}$, and an abacus for C is obtained from an abacus for B by moving a bead from runner $i-1$ to runner i. If μ is an e-regular partition in B, then $D^\mu{\uparrow_C} \neq 0$ if and only if the configuration of runners $i-1$ and i in the abacus display for μ appears in A.1 or A.3.

2.10 $[4:\kappa]$-pairs

Scopes [18] pioneered the study of the relationship between different blocks of the same weight, in the symmetric group context; her work was generalised to Iwahori–Hecke algebras by Jost [12]. We use the framework developed by Scopes in this paper, and we summarise the important points here.

Suppose B is a block of $\mathcal{H}_n$ of weight w, and suppose that in some abacus display for B there are κ more beads on runner i than on runner $i-1$. Let A be the weight w block of $\mathcal{H}_{n-\kappa}$ whose abacus is obtained by moving κ beads from runner i to runner $i-1$. We say that A forms a *$[w:\kappa]$-pair* with B. We abuse notation by including the case $i=0$ here: in this case we actually need $\kappa+1$ more beads on runner 0 than on runner $e-1$. Equivalently, two weight w blocks A and B form a $[w:\kappa]$-pair if the core of A is obtained from the core of B by removing all the removable nodes of a given residue, and there are exactly κ such nodes.

Blocks forming $[w:\kappa]$-pairs have very similar representation theories. There is a bijection Φ between the set of e-regular partitions in A and the set of e-regular partitions in B, with the property that for any e-regular μ in A, $D^{\mu}\uparrow^{B}$ is semi-simple if and only if $D^{\Phi(\mu)}\downarrow_{A}$ is, and in this case we have

$$D^{\mu}\uparrow^{B}\cong (D^{\Phi(\mu)})^{\oplus\kappa!},\qquad D^{\Phi(\mu)}\downarrow_{A}\cong (D^{\mu})^{\oplus\kappa!}.$$

We say that D^{μ} and $D^{\Phi(\mu)}$ (or μ and $\Phi(\mu)$) are *non-exceptional* for the $[w:\kappa]$-pair (A,B) if this happens. If $w\leqslant\kappa$, then in fact every μ is non-exceptional, and we say that A and B are *Scopes equivalent*; this implies in particular that they are Morita equivalent.

Whether an e-regular partition in A or B is exceptional can be deduced from the modular branching rule; we provide a description of such partitions in Appendix A. Specifically, suppose A and B form a $[4:\kappa]$-pair, with an abacus for A obtained from an abacus for B by moving κ beads from runner i to runner $i-1$. Then:

- if μ is an e-regular partition in A, then μ is exceptional for the $[4:\kappa]$-pair (A,B) if and only if $\kappa\leqslant 3$ and the configuration of runners $i-1$ and i in the abacus display for μ appears in A.2;
- if μ is an e-regular partition in B, then μ is exceptional for the $[4:\kappa]$-pair (A,B) if and only if $\kappa\leqslant 3$ and the configuration of runners $i-1$ and i in the abacus display for μ appears in A.3.

The map Φ on non-exceptional e-regular partitions is also an instance of the modular branching rule, and we give an explicit description for the case $w=4$. Suppose μ is a non-exceptional e-regular partition in A.

- Suppose every bead on runner i of the abacus display for μ has a bead immediately to its left on runner $i-1$. Then $\Phi(\mu)$ is obtained simply by interchanging runners $i-1$ and i.
- Otherwise, $\kappa\leqslant 3$ and the configurations of runners $i-1$ and i in the abacus displays for μ and $\Phi(\mu)$ form one of the pairs listed in A.4.

The importance for us of non-exceptional partitions is the following.

Proposition 2.18. *Suppose A and B are weight 4 blocks forming a $[4:\kappa]$-pair as above, and that λ,μ are e-regular partitions in A.*

1. *If μ is exceptional, then both pairs (λ,μ) and $(\Phi(\lambda),\Phi(\mu))$ are lowerable.*
2. *If λ is non-exceptional, then $\operatorname{adj}_{\lambda\mu}=\operatorname{adj}_{\Phi(\lambda)\Phi(\mu)}$.*

Proof.

1. Let C be the block of $\mathcal{H}_{n-1}$ of weight $3-\kappa$ whose abacus is obtained from that for B by moving a bead from runner i to runner $i-1$. The fact that μ appears in Appendix A.2 means that $D^{\mu}\downarrow_C \neq 0$ (see the discussion in §2.9 above). $D^{\lambda}\downarrow_C$ is either zero or irreducible (again, by the discussion in §2.9), and so the pair (λ,μ) is lowerable. A similar argument applies to $(\Phi(\lambda),\Phi(\mu))$.

2. If μ is exceptional, then $\mathrm{adj}_{\lambda\mu} = 0 = \mathrm{adj}_{\Phi(\lambda)\Phi(\mu)}$ by (1) and Proposition 2.17; if μ is non-exceptional, then we may copy the proof of [4, Lemma 4.3(2)].

□

We now set up some notation to help us use this result. Take a weight 4 block A of $\mathcal{H}_n$ and $i \in \mathbb{Z}/e\mathbb{Z}$, and suppose that the core of A has $\kappa \geqslant 1$ addable nodes of residue i. Let B be the weight 4 block whose core is obtained from the core of A by adding all the addable nodes of residue i. Then A and B form a $[4:\kappa]$-pair. We define a partial function $\mathfrak{f}_i$ on the set of e-regular partitions in A, by putting $\mathfrak{f}_i(\xi) = \mu$ if ξ is non-exceptional for this $[4:\kappa]$-pair and $\Phi(\xi) = \mu$, and leaving $\mathfrak{f}_i(\xi)$ undefined if ξ is exceptional. Doing this for every weight 4 block A, we obtain a partial function $\mathfrak{f}_i$ on the set of e-regular partitions of weight 4 (if the core of a partition ξ has no addable nodes of residue i, then we leave $\mathfrak{f}_i(\xi)$ undefined). We have the following corollary of Proposition 2.18.

Proposition 2.19. *Suppose λ and μ are e-regular partitions of weight 4 lying in the same block of $\mathcal{H}_n$. Suppose $i_1,\dots,i_r \in \mathbb{Z}/e\mathbb{Z}$, and let $\mathfrak{f}$ be the partial function $\mathfrak{f}_{i_1}\dots\mathfrak{f}_{i_r}$. If $\mathfrak{f}(\lambda)$ and $\mathfrak{f}(\mu)$ are both defined, then* $\mathrm{adj}_{\lambda\mu} = \mathrm{adj}_{\mathfrak{f}(\lambda)\mathfrak{f}(\mu)}$. *If $\mathfrak{f}(\lambda)$ is defined but $\mathfrak{f}(\mu)$ is not, then* $\mathrm{adj}_{\lambda\mu} = 0$.

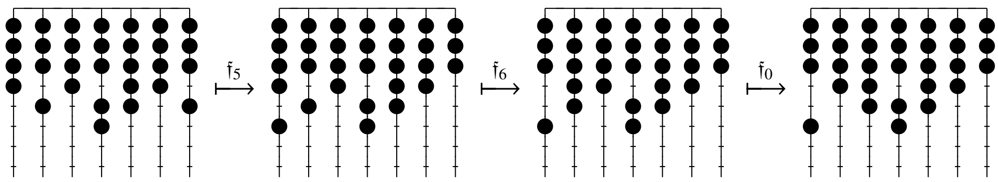
Example. Suppose $e = 7$, and

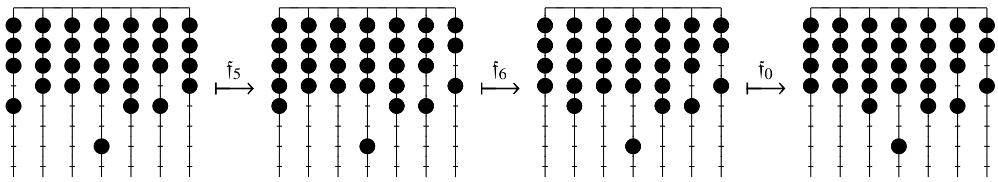
$$\lambda = \langle 1, 3_{1^2}, 6 \mid 4^3, 5^2, 4^2\rangle, \qquad \mu = \langle 0, 3_2, 5 \mid 4^3, 5^2, 4^2\rangle.$$

Let $\mathfrak{f} = \mathfrak{f}_0\mathfrak{f}_6\mathfrak{f}_5$. Then we have

$$\mathfrak{f}(\lambda) = \langle 0_2, 3_{1^2} \mid 4^2, 5^3, 4, 3\rangle, \qquad \mathfrak{f}(\mu) = \langle 3_2, 5, 6 \mid 4^2, 5^3, 4, 3\rangle,$$

as we see from the following abacus diagrams:

$\lambda =$  $= \mathfrak{f}(\lambda)$,

$\mu =$  $= \mathfrak{f}(\mu)$.

We deduce that $\mathrm{adj}_{\lambda\mu} = \mathrm{adj}_{\mathfrak{f}(\lambda)\mathfrak{f}(\mu)}$, and in fact this equals 0, since the pair $(\mathfrak{f}(\lambda),\mathfrak{f}(\mu))$ is lowerable. Examples of this type will be used extensively in Sections 5–8.

We may also use this formalism without making the function $\mathfrak{f}$ explicit. Given e-regular weight 4 partitions $\lambda^1,\dots,\lambda^t$ lying in a block B and $\overline{\lambda}^1,\dots,\overline{\lambda}^t$ lying in a block C, we may write $(\lambda^1,\dots,\lambda^t)\sim(\overline{\lambda}^1,\dots,\overline{\lambda}^t)$ to indicate that there exist $i_1,\dots,i_r\in\mathbb{Z}/e\mathbb{Z}$ such that $\mathfrak{f}_{i_1}\dots\mathfrak{f}_{i_r}(\lambda^i)$ is defined and equals $\overline{\lambda}^i$, for $i=1,\dots,t$. Where we use this notation, we hope that it will not be hard for the reader to construct an appropriate sequence $i_1,\dots,i_r$.

2.11 The Mullineux map

Let $T_1,\dots,T_{n-1}$ be the standard generators of $\mathcal{H}_n$. Let $\sharp:\mathcal{H}_n\to\mathcal{H}_n$ be the involutory automorphism sending T_i to $q-1-T_i$. Given a module M for $\mathcal{H}_n$, define $M^\sharp$ to be the module with the same underlying vector space and with action

$$h\cdot m=h^\sharp m.$$

Then the functor $M\mapsto M^\sharp$ is a self-equivalence of the category of $\mathcal{H}_n$-modules. In the symmetric group case $q=1$, $M^\sharp$ is simply $M\otimes\mathrm{sgn}$, where sgn is the 1-dimensional signature representation.

The fact that $M\mapsto M^\sharp$ defines a category equivalence means that it gives a map on the blocks of $\mathcal{H}_n$. That is, if B is a block of $\mathcal{H}_n$, then there is another block $B^\sharp$ such that a module M lies in B if and only if $M^\sharp$ lies in $B^\sharp$. We call $B^\sharp$ the *conjugate block* to B; it is easy to show that if B has weight w and core γ, then $B^\sharp$ has weight w and core γ'.

If λ is an e-regular partition of n, then $(D^\lambda)^\sharp$ must be a simple module, and we write $\lambda^\diamond$ for the e-regular partition such that $(D^\lambda)^\sharp\cong D^{\lambda^\diamond}$. Then $\lambda\mapsto\lambda^\diamond$ is an involutory bijection from the set of e-regular partitions of n to itself. This bijection is given combinatorially by Mullineux's algorithm [15], which we shall use extensively. We note that given an e-regular partition λ, the partition $\lambda^\diamond$ depends only on λ and e, not on $\mathbb{F}$ and q.

The fact that $M\mapsto M^\sharp$ is a category equivalence, combined with the fact that the map $\lambda\mapsto\lambda^\diamond$ does not depend on the underlying characteristic, yields the following.

Proposition 2.20. [4, Lemma 4.2] *If λ and μ are e-regular partitions of n, then* $\mathrm{adj}_{\lambda\mu}=\mathrm{adj}_{\lambda^\diamond\mu^\diamond}$.

We also examine the relationship between conjugate blocks and Scopes pairs. If A and B are blocks forming a $[w:\kappa]$-pair, then $A^\sharp$ and $B^\sharp$ also form a $[w:\kappa]$-pair, and if λ is an e-regular partition lying in B, then λ is exceptional for the pair (A,B) if and only if $\lambda^\diamond$ is exceptional for the pair $(A^\sharp,B^\sharp)$.

2.12 Rouquier blocks

A certain class of blocks of Hecke algebras singled out by Rouquier has attracted a great deal of attention in recent years. Suppose B is a weight w block of $\mathcal{H}_n$ which may be represented with the $\langle b_0,\dots,b_{e-1}\rangle$ abacus notation. We say that B is *Rouquier* if for every $0\leqslant i<j\leqslant e-1$ either $b_i-b_j\geqslant w$ or $b_j-b_i\geqslant w-1$. The Rouquier blocks form a single class under the Scopes equivalence, and are very well understood in many ways. For our purposes, the most important result is the following.

Theorem 2.21. [9, Corollary 3.15] *James's Conjecture holds for Rouquier blocks. That is, if B is a Rouquier block of $\mathcal{H}_n$ of weight $w<\mathrm{char}(\mathbb{F})$, then $\mathrm{adj}_{\lambda\mu}=\delta_{\lambda\mu}$ for all pairs (λ,μ) of e-regular partitions in B.*

Suppose B is any weight w block of $\mathcal{H}_n$, and that λ is an e-regular partition in B. We say that λ *induces semi-simply to a Rouquier block* if we have $\lambda\sim\overline{\lambda}$, where $\overline{\lambda}$ is an e-regular partition lying in some Rouquier block, and the relation $\sim$ is as in §2.10. The following is a direct consequence of Proposition 2.19 and Theorem 2.21.

Proposition 2.22. *Suppose λ and μ are e-regular partitions lying in a weight 4 block B of $\mathcal{H}_n$, and that λ induces semi-simply to a Rouquier block. Then* $\mathrm{adj}_{\lambda\mu} = \delta_{\lambda\mu}$.

We shall frequently make use of this result later, by noting that certain partitions induce semi-simply to Rouquier blocks. To justify these statements, we shall give the partition $\bar{\lambda}$ lying in the weight 4 Rouquier block with $\langle 4, 7, \dots, 3e+1\rangle$ notation, and invite the reader to verify that $\lambda \sim \bar{\lambda}$ in each case.

2.13 Outline of the proof of James's Conjecture for weight 4 blocks

In this section, we outline the proof of Theorem 2.6. From now on, we assume that the characteristic of $\mathbb{F}$ is at least 5. The proof of Theorem 2.6 is by induction on n, with the initial case being the unique weight 4 block of $\mathcal{H}_{4e}$. This is dealt with in Section 4.

For the inductive step, we use $[4:\kappa]$-pairs. If B is a weight 4 block of $\mathcal{H}_n$ and $n > 4e$, then there is at least one block A forming a $[4:\kappa]$-pair with B. Suppose in fact that there are distinct weight 4 blocks $A_1, \dots, A_r$ such that A_j and B form a $[4:\kappa_j]$-pair, for each j.

If λ is an e-regular partition in B which is non-exceptional for the pair (A_j, B), then by Proposition 2.18 and by induction we have $\mathrm{adj}_{\lambda\mu} = \delta_{\lambda\mu}$ for all e-regular partitions μ in B. So we may assume that λ is exceptional for each of the pairs (A_j, B). Take an abacus display for B and suppose that an abacus display for A_j is obtained by moving κ_j beads from runner i_j to runner $i_j - 1$, for each j. Then $i_1, \dots, i_r$ are distinct, and we see from the displays in A.3 that $|\lambda(i_j)| \geqslant \kappa_j + 1$ for each j. So we have $(\kappa_1 + 1) + \dots + (\kappa_r + 1) \leqslant |\lambda(i_1)| + \dots + |\lambda(i_r)| \leqslant 4$, which implies that either $r = 1$ and $\kappa_1 \leqslant 3$, or $r = 2$ and $\kappa_1 = \kappa_2 = 1$. The blocks B satisfying these conditions are dealt with in the remaining sections of the paper.

3 A special case

In this section, we deal with a special case which will arise in three later sections. Suppose that $2 \leqslant a \leqslant e-2$, and let B be the weight 4 block with core $((4a)^{e-a-2}, (3a-2)^{e-a}, (2a-2)^{e-a}, (a-2)^{e-a+2})$, for which we use the $\langle 4^{a-2}, 5^2, 7^2, 8^{e-a-2}\rangle$ notation. We fix

$$\begin{aligned} \lambda^1 &= \langle a\text{–}2_2, a_2\rangle, \\ \lambda^2 &= \langle a\text{–}1_4\rangle, \\ \lambda^3 &= \langle a\text{–}2_4\rangle, \\ \mu &= \langle a_2, a\text{+}1_2\rangle. \end{aligned}$$

Our goal is to show the following.

Proposition 3.1. *With $\lambda^1, \lambda^2, \lambda^3, \mu$ as above, we have*

$$\mathrm{adj}_{\lambda^1\mu} = \mathrm{adj}_{\lambda^2\mu} = \mathrm{adj}_{\lambda^3\mu} = 0.$$

We begin by calculating the relevant part of the canonical basis element $G(\mu)$; that is, we compute the v-decomposition numbers $d^{(e)}_{\nu\mu}(v)$ for those partitions ν which dominate λ^1, λ^2 or λ^3. Note that by using the runner removal theorems it suffices to perform this computation in the case $a = e - a = 2$. Using the LLT algorithm for this case and then applying the runner removal theorems yields the following.

Lemma 3.2. *Let μ be as above, and suppose ν is a partition dominating λ^1, λ^2 or λ^3. Then we have*

$$d^{(e)}_{\nu\mu} = \begin{cases} 1 & (\nu = \langle a_2, a{+}1_2\rangle) \\ v & (\nu = \langle a{+}1_{2^2}\rangle, \langle a_{1^2}, a{+}1_2\rangle, \langle a_2, a{+}1_{1^2}\rangle, \langle a{-}1, a_2, a{+}1\rangle \text{ or } \langle a{-}2_4\rangle) \\ v^2 & (\nu = \langle a{+}1_{2,1^2}\rangle, \langle a_{2,1}, a{+}1\rangle \text{ or } \langle a{-}1, a_{2,1}\rangle) \\ v + v^3 & (\nu = \langle a_{2^2}\rangle) \\ 0 & (\text{otherwise}). \end{cases}$$

Lemma 3.3. *Define*

$$\nu^1 = \langle a{+}1_{2,1^2}\rangle, \qquad \nu^2 = \langle a_{2,1}, a{+}1\rangle.$$

Then

$$\mathrm{adj}_{\nu^1\mu} = \mathrm{adj}_{\nu^2\mu} = 0.$$

Proof. We have $\nu^1_1 = \mu_1 = 5a + 2$, so we may apply Corollary 2.14 to get $\mathrm{adj}_{\nu^1\mu} = 0$. For ν^2, we can simply check that

$$\nu_2 \sim \langle a_{3,1} \mid 5, 8, 11, \dots, 3e + 2\rangle,$$

so ν^2 induces semi-simply to a Rouquier block; so we have $\mathrm{adj}_{\nu^2\mu} = 0$ by Proposition 2.22. □

Lemma 3.4. *Define*

$$\nu^3 = \langle a{-}2_2, a, a{+}1\rangle, \qquad \nu^4 = \langle a{-}2, a{+}1_{1^3}\rangle.$$

Then

$$\mathrm{adj}_{\nu^3\mu} = \mathrm{adj}_{\nu^4\mu} = 0.$$

Proof. We induce μ, ν^3, ν^4 to the block with the $\langle 4^{a-2}, 5^2, 7, 10, 11^{e-a-2}\rangle$ notation. We find that

$$(\nu^3, \nu^4, \mu) \sim (\bar\nu^3, \bar\nu^4, \bar\mu),$$

where

$$\begin{aligned} \bar\nu^3 &= \langle a{-}2_2, a_2 \mid 4^{a-2}, 5^2, 7, 10, 11^{e-a-2}\rangle, \\ \bar\nu^4 &= \langle a{-}2_2, a, a{+}1 \mid 4^{a-2}, 5^2, 7, 10, 11^{e-a-2}\rangle, \\ \bar\mu &= \langle a_4 \mid 4^{a-2}, 5^2, 7, 10, 11^{e-a-2}\rangle. \end{aligned}$$

We have $\bar\mu \not\trianglerighteq \bar\nu^4$, so that $\mathrm{adj}_{\bar\nu^4\bar\mu} = 0$ by Corollary 2.8. To show $\mathrm{adj}_{\bar\nu^3\bar\mu} = 0$, we use the Jantzen–Schaper formula and Proposition 2.10 to show that $[S^{\bar\nu^3} : D^{\bar\mu}] = 1$ independently of the underlying characteristic. The table of Jantzen–Schaper coefficients $J(\xi, \pi)$ for those partitions ξ, π with $\bar\mu \trianglerighteq \xi \triangleright \pi \trianglerighteq \bar\nu^3$ is as follows.

	$\langle a_4\rangle$	$\langle a{-}1, a_3\rangle$	$\langle a{-}2, a_3\rangle$	$\langle a{-}1_2, a_2\rangle$	$\langle a{-}2_2, a_2\rangle$	$[S^\xi : D^{\bar\mu}]$
$\langle a_4\rangle$	·	·	·	·	·	1
$\langle a{-}1, a_3\rangle$	1	·	·	·	·	1
$\langle a{-}2, a_3\rangle$	−1	1	·	·	·	0
$\langle a{-}1_2, a_2\rangle$	−1	1	·	·	·	0
$\langle a{-}2_2, a_2\rangle$	1	0	1	1	·	1

(Recall that we are using the Jantzen–Schaper dominance order here; this means that the partition $\langle a_{3,1}\rangle$ does not appear, for example.)

We deduce $\mathrm{adj}_{\nu^i\mu} = \mathrm{adj}_{\bar{\nu}^i\bar{\mu}} = 0$ for $i = 3, 4$. □

Lemma 3.5. *Define*

$$\nu^5 = \langle a_{2^2}\rangle, \qquad \nu^6 = \langle a{-}1, a_{2,1}\rangle.$$

Then

$$\mathrm{adj}_{\nu^5\mu} = \mathrm{adj}_{\nu^6\mu} = 0.$$

Proof. Writing $\hat{a} = e - a$ and applying the Mullineux map, we have

$$\begin{aligned}(\nu^5)^\diamond &= \langle \hat{a}{-}2_2, \hat{a}, \hat{a}{+}1 \mid 4^{\hat{a}-2}, 5^2, 7^2, 8^{e-\hat{a}}\rangle,\\ (\nu^6)^\diamond &= \langle \hat{a}{-}2, \hat{a}{+}1_{1^3} \mid 4^{\hat{a}-2}, 5^2, 7^2, 8^{e-\hat{a}}\rangle,\\ \mu^\diamond &= \langle \hat{a}_2, \hat{a}{+}1_2 \mid 4^{\hat{a}-2}, 5^2, 7^2, 8^{e-\hat{a}}\rangle.\end{aligned}$$

By Lemma 3.4 (replacing a with $\hat{a}$) we have $\mathrm{adj}_{\mu^\diamond(\nu^5)^\diamond} = \mathrm{adj}_{\mu^\diamond(\nu^6)^\diamond} = 0$, and applying Proposition 2.20 gives the result. □

Now we can deduce Proposition 3.1.

Proof of Proposition 3.1. We prove that $\mathrm{adj}_{\nu\mu} = 0$ for all $\mu \rhd \nu \unrhd \lambda^3$ inductively. For those ν for which $d^{(e)}_{\nu\mu}(v)$ equals 0 or v, we may appeal to Corollary 2.12. The remaining partitions are ν^1, ν^2 of Lemma 3.3 and ν^5, ν^6 of Lemma 3.5. □

4 The principal block of $\mathcal{H}_{4e}$

In this section, we show that James's Conjecture holds for the weight 4 block B with empty core, which we display on an abacus with the $\langle 4^e\rangle$ notation. First we observe the following.

Lemma 4.1. *If μ is an e-regular partition in B, then there is some block C of $\mathcal{H}_{n-1}$ of weight less than* 4 *such that $D^\mu\downarrow_C$ is non-zero.*

Proof. Since $\mathcal{H}_{4e-1}$ is a unital subalgebra of $\mathcal{H}_{4e}$, we have $D^\mu\downarrow_{\mathcal{H}_{4e-1}} \neq 0$; in particular, $D^\mu\downarrow_C \neq 0$ for some block C of $\mathcal{H}_{4e-1}$. Every block of $\mathcal{H}_{4e-1}$ has weight less than 4, and the result follows. □

Now suppose λ and μ are e-regular partitions in B. In proving that $\mathrm{adj}_{\lambda\mu} = \delta_{\lambda\mu}$, we may assume by Proposition 2.17 that the pair (λ, μ) is not lowerable. Given Lemma 4.1, this is a very restrictive assumption: by the discussion in §2.9, there must be some $i \in \{1, \dots, e-1\}$ such that $\lambda = \langle i_{2^2}\rangle$, and $D^\mu\downarrow_D = 0$ for every block D of $\mathcal{H}_{n-1}$ other than the weight 3 block C with the $\langle 4^{i-1}, 5, 3, 4^{e-1-i}\rangle$ notation (i.e. the block C for which $D^\lambda\downarrow_C$ is reducible).

We may also assume, in view of Corollaries 2.8 and 2.14, that $\mu \unrhd \lambda$ and $\mu_1 > \lambda_1$. The possibilities for μ are now as follows:

- $\langle i_4\rangle$;
- $\langle i_3, i{+}1\rangle$ (if $i \leqslant e-2$);
- $\langle 0, i_3\rangle$;
- $\langle i_2, i{+}1_2\rangle$ (if $i \leqslant e-2$).

Given Proposition 2.20, we may also assume that $\mu^\diamond \trianglerighteq \lambda^\diamond$ and $\mu^\diamond_1 > \lambda^\diamond_1$; to exploit this assumption, we compute $\lambda^\diamond$ and $\mu^\diamond$ for each of the possible μ above. Writing $\hat{\imath} = e - i$, we have

$$\lambda^\diamond = \langle \hat{\imath}_{2^2}\rangle,$$

while the possibilities for $\mu^\diamond$ are given in Table 4.1.

μ	conditions	$\mu^\diamond$
$\langle i_4\rangle$	$i \leqslant e-4$	$\langle 0,1,2,\hat{\imath}\rangle$
$\langle i_4\rangle$	$2 \leqslant i = e-3$	$\langle 0,1,\hat{\imath},\hat{\imath}{+}1\rangle$
$\langle i_4\rangle$	$1 = i = e-3$	$\langle 0_2,1,\hat{\imath}\rangle$
$\langle i_4\rangle$	$3 \leqslant i = e-2$	$\langle 0,\hat{\imath},\hat{\imath}{+}1,\hat{\imath}{+}2\rangle$
$\langle i_4\rangle$	$2 = i = e-2$	$\langle 0_2,\hat{\imath},\hat{\imath}{+}1\rangle$
$\langle i_4\rangle$	$1 = i = e-2$	$\langle 0_2,\hat{\imath}_2\rangle$
$\langle i_4\rangle$	$4 \leqslant i = e-1$	$\langle \hat{\imath},\hat{\imath}{+}1,\hat{\imath}{+}2,\hat{\imath}{+}3\rangle$
$\langle i_4\rangle$	$3 = i = e-1$	$\langle \hat{\imath}_2,\hat{\imath}{+}1,\hat{\imath}{+}2\rangle$
$\langle i_4\rangle$	$2 = i = e-1$	$\langle \hat{\imath}_2,\hat{\imath}{+}1_2\rangle$
$\langle i_4\rangle$	$1 = i = e-1$	$\langle \hat{\imath}_4\rangle$
$\langle i_3, i{+}1\rangle$	$2 \leqslant i$	$\langle 0,\hat{\imath}_{1^2},\hat{\imath}{+}1\rangle$
$\langle i_3, i{+}1\rangle$	$1 = i$	$\langle 0_2,\hat{\imath}_{1^2}\rangle$
$\langle 0, i_3\rangle$	$i \leqslant e-2$	$\langle 0,1,\hat{\imath}_2\rangle$
$\langle 0, i_3\rangle$	$2 \leqslant i = e-1$	$\langle 0,\hat{\imath}_2,\hat{\imath}{+}1\rangle$
$\langle 0, i_3\rangle$	$1 = i = e-1$	$\langle 0,\hat{\imath}_3\rangle$
$\langle i_2, i{+}1_2\rangle$	$2 \leqslant i \leqslant e-4$	$\langle 0,1,\hat{\imath},\hat{\imath}{+}1\rangle$
$\langle i_2, i{+}1_2\rangle$	$1 = i \leqslant e-4$	$\langle 0_2,1,\hat{\imath}\rangle$
$\langle i_2, i{+}1_2\rangle$	$3 \leqslant i = e-3$	$\langle 0,\hat{\imath},\hat{\imath}{+}1,\hat{\imath}{+}2\rangle$
$\langle i_2, i{+}1_2\rangle$	$2 = i = e-3$	$\langle 0_2,\hat{\imath},\hat{\imath}{+}1\rangle$
$\langle i_2, i{+}1_2\rangle$	$1 = i = e-3$	$\langle 0_2,\hat{\imath}_2\rangle$
$\langle i_2, i{+}1_2\rangle$	$4 \leqslant i = e-2$	$\langle \hat{\imath},\hat{\imath}{+}1,\hat{\imath}{+}2,\hat{\imath}{+}3\rangle$
$\langle i_2, i{+}1_2\rangle$	$3 = i = e-2$	$\langle \hat{\imath}_2,\hat{\imath}{+}1,\hat{\imath}{+}2\rangle$
$\langle i_2, i{+}1_2\rangle$	$2 = i = e-2$	$\langle \hat{\imath}_2,\hat{\imath}{+}1_2\rangle$
$\langle i_2, i{+}1_2\rangle$	$1 = i = e-2$	$\langle \hat{\imath}_4\rangle$

Table 4.1

We now find that there are only five cases satisfying all of our assumptions:

1. $e = 2, i = 1, \mu = \langle 1_4\rangle$;
2. $e = 2, i = 1, \mu = \langle 0, 1_3\rangle$;
3. $e = 3, i = 2, \mu = \langle 2_4\rangle$;
4. $e = 3, i = 1, \mu = \langle 1_2, 2_2\rangle$;
5. $e = 4, i = 2, \mu = \langle 2_2, 3_2\rangle$.

We deal with these using the LLT algorithm. In each case, we know that $\mathrm{adj}_{\nu\mu} = \delta_{\nu\mu}$ whenever $\nu \neq \lambda$, so by Corollary 2.12 it suffices to show that the v-decomposition number $d^{(e)}_{\lambda\mu}(v)$ is either 0 or v.

Applying the LLT algorithm, we find that in cases 1 and 3 we have $d^{(e)}_{\lambda\mu}(v) = 0$, while in cases 2 and 5 we have $d^{(e)}_{\lambda\mu}(v) = v$. This just leaves case 4; in this case, we have

$$\lambda^{\diamond} = \langle 2_{2^2}\rangle, \qquad \mu^{\diamond} = \langle 2_4\rangle.$$

From case 3 we have $\mathrm{adj}_{\lambda^{\diamond}\mu^{\diamond}} = 0$, and so $\mathrm{adj}_{\lambda\mu} = 0$ by Proposition 2.20.

5 Blocks forming a [4 : 1]-pair

Our purpose in this section is to prove the following result.

Proposition 5.1. *Suppose A and B are weight 4 blocks of $\mathcal{H}_{n-1}$ and $\mathcal{H}_n$ respectively, forming a [4 : 1]-pair. Suppose that there is no block other than A forming a [4 : κ]-pair with B, for any κ. If James's Conjecture holds for A, then it holds for B.*

The condition on B means that the core of B has the form (a^{b-a}) for some $0 < a < b \leqslant e$. We use the $\langle 4^a, 5^{b-a}, 4^{e-b}\rangle$ notation for partitions in B. The conjugate block $B^{\sharp}$ has core $(\hat{a}^{b-\hat{a}})$, where $\hat{a} = b - a$, and we employ the $\langle 4^{\hat{a}}, 5^{b-\hat{a}}, 4^{e-b}\rangle$ notation for $B^{\sharp}$. The block $A^{\sharp}$ forms a [4 : 1]-pair with $B^{\sharp}$, and by Proposition 2.20 and our hypothesis on A, James's Conjecture holds for $A^{\sharp}$. So the hypotheses of Proposition 5.1 apply also to $A^{\sharp}$ and $B^{\sharp}$; since James's Conjecture holds for B if and only if it holds for $B^{\sharp}$, we will often be able to interchange B and $B^{\sharp}$ in the proof that follows.

Suppose that λ and μ are e-regular partitions lying in B. If λ is non-exceptional for the [4 : 1]-pair (A, B), then we have $\mathrm{adj}_{\lambda\mu} = \delta_{\lambda\mu}$ by Proposition 2.18 and our hypothesis on A. So we can assume that λ is exceptional; that is, λ is one of the following partitions:

$$\begin{array}{ll}
\langle a_{3,1}\rangle; & \\
\langle a_{2^2}\rangle; & \\
\langle a_{2,1}, i\rangle & (0 \leqslant i \leqslant e-1, \quad i \notin \{a-1, a\}); \\
\langle a_{1^3}, i\rangle & (1 \leqslant i \leqslant a-1); \\
\langle a_{1^2}, i_2\rangle & (0 \leqslant i \leqslant e-1, \quad i \notin \{a-1, a\}); \\
\langle a_{1^2}, i_{1^2}\rangle & (a+1 \leqslant i \leqslant e-1); \\
\langle a_{1^2}, i, j\rangle & (0 \leqslant i < j \leqslant e-1, \quad i, j \notin \{a-1, a\}).
\end{array}$$

Appealing to Corollaries 2.8 and 2.14, we assume that $\mu \trianglerighteq \lambda$ and $\mu_1 > \lambda_1$. Note that one of the exceptional partitions is lower than all the others in the dominance order, namely

$$\lambda^0 = \begin{cases} \langle 1, a_{1^3}\rangle & (a \geqslant 2) \\ \langle a_{1^2}, b_{1^2}\rangle & (a = 1, \quad b < e) \\ \langle a_{1^2}, a{+}1_{1^2}\rangle & (a = 1, \quad a+1 \leqslant b = e) \\ \langle a_{2^2}\rangle & (e = 2). \end{cases}$$

(Actually, since we are using the Jantzen–Schaper dominance order, this is not strictly true when $e = 2$, since then $\langle a_{3,1}\rangle$ dominates $\langle a_{2^2}\rangle$ in the usual dominance order but not the Jantzen–Schaper dominance order. But this technicality makes no practical difference to the arguments that follow.) So our assumptions $\mu \trianglerighteq \lambda$ and $\mu_1 > \lambda_1$ imply that $\mu \trianglerighteq \lambda^0$ and $\mu_1 > \lambda^0_1$.

This restricts the range of possibilities for μ; using Proposition 2.17 we assume additionally that the pair (λ, μ) is not lowerable, and we deduce that μ must be one of the partitions shown in Table 5.1. We now consider two separate cases.

	partition	conditions		partition	conditions
1	$\langle a_4\rangle$	—	11	$\langle a,a{+}1,b_2\rangle$	$b-a=2,\ e-b\geqslant 1$
2	$\langle a_3,a{+}1\rangle$	$b-a\geqslant 2$	12	$\langle a,a{+}1,b,b{+}1\rangle$	$b-a\geqslant 2,\ e-b\geqslant 2$
3	$\langle a_2,a{+}1_2\rangle$	$b-a\geqslant 2$	13	$\langle a,b_2,b{+}1\rangle$	$b-a=1,\ e-b\geqslant 2$
4	$\langle a_2,a{+}1,a{+}2\rangle$	$b-a\geqslant 3$	14	$\langle 0,a_2,b\rangle$	$a\geqslant 2,\ e-b\geqslant 1$
5	$\langle a,a{+}1,a{+}2,a{+}3\rangle$	$b-a\geqslant 4$	15	$\langle 0,a,a{+}1,b\rangle$	$a\geqslant 2,\ b-a\geqslant 2,\ e-b\geqslant 1$
6	$\langle a_3,b\rangle$	$e-b\geqslant 1$	16	$\langle 0,a,b_2\rangle$	$a\geqslant 2,\ b-a=1,\ e-b\geqslant 1$
7	$\langle a_2,a{+}1,b\rangle$	$b-a\geqslant 2,\ e-b\geqslant 1$	17	$\langle 0_2,a_2\rangle$	$a\geqslant 2,\ e-b=0$
8	$\langle a,a{+}1,a{+}2,b\rangle$	$b-a\geqslant 3,\ e-b\geqslant 1$	18	$\langle 0_2,a,a{+}1\rangle$	$a\geqslant 2,\ b-a\geqslant 2,\ e-b=0$
9	$\langle a_2,b_2\rangle$	$b-a=1,\ e-b\geqslant 1$	19	$\langle 0_3,a\rangle$	$a\geqslant 2,\ b-a=1,\ e-b=0$
10	$\langle a_2,b,b{+}1\rangle$	$e-b\geqslant 2$			

Table 5.1

5.1 The case $e-b>0$

For the case $b<e$, we begin by defining two partial functions (compare the example following Proposition 2.19).

1. Define $\mathfrak{a}=\mathfrak{f}_{2a-b-1}\dots\mathfrak{f}_{a-b+1}\mathfrak{f}_{a-b}$. We find that the pair $(\mathfrak{a}(\lambda),\mathfrak{a}(\mu))$ is defined and lowerable unless one of the following happens:
   - $\mathfrak{a}(\lambda)$ is undefined, i.e. $a\geqslant 2$ and $\lambda=\langle i_2,a_{1^2}\rangle$ for some $0\leqslant i\leqslant a-2$;
   - $\lambda=\langle a_{2^2}\rangle$; or
   - μ is in case 8, 9 or 11 with $e-b=1$, or in case 12 or 13 with $e-b=2$.
2. Define $\mathfrak{b}=\mathfrak{f}_{2a-b+1}\dots\mathfrak{f}_{a-1}\mathfrak{f}_a$. The pair $(\mathfrak{b}(\lambda),\mathfrak{b}(\mu))$ is defined and lowerable unless one of the following occurs:
   - $\mathfrak{b}(\lambda)$ is undefined, i.e. $b-a\geqslant 2$ and $\lambda=\langle a_{1^2},b_2\rangle$ or $\langle a_{1^2},i,b\rangle$ for some $a+2\leqslant i\leqslant b-1$;
   - $\lambda=\langle a_{1^2},b_{1^2}\rangle$; or
   - μ is in one of cases 1–5.

If either of the pairs $(\mathfrak{a}(\lambda),\mathfrak{a}(\mu))$ or $(\mathfrak{b}(\lambda),\mathfrak{b}(\mu))$ is defined and lowerable, then we have $\mathrm{adj}_{\lambda\mu}=0$ by Proposition 2.19 and Proposition 2.17. So we assume otherwise, and examine four cases.

(i) Suppose that $\mathfrak{b}(\lambda)$ is not defined. Then we may assume that μ is in case 8 or 11 (with $e-b=1$) or 12 (with $e-b=2$). We get $\lambda_1\geqslant\mu_1$, contradicting our earlier assumption.

(ii) Suppose that $\mathfrak{a}(\lambda)$ is not defined. Applying the Mullineux map, we find that $\lambda^\diamond$ has the form $\langle i_2,\hat{a}_{1^2}\rangle$ for $0\leqslant i\leqslant\hat{a}-2$, so we may replace (B,λ,μ) with $(B^\sharp,\lambda^\diamond,\mu^\diamond)$ and appeal to case (i).

(iii) Suppose that $\lambda=\langle a_{1^2},b_{1^2}\rangle$, and that μ is in case 8, 9 or 11 (with $e-b=1$) or case 12 or 13 (with $e-b=2$). We will show that $\mathrm{adj}_{\lambda^\diamond\mu^\diamond}=0$, and hence $\mathrm{adj}_{\lambda\mu}=0$ by Proposition 2.20. Applying the Mullineux map, we find

$$\lambda^\diamond=\langle\hat{a}_{2^2}\rangle,$$

$$\mu^\diamond = \begin{cases} \langle \hat{a}, \hat{a}{+}1, \hat{a}{+}2, \hat{a}{+}3 \rangle & (a \geqslant 4) \\ \langle \hat{a}_2, \hat{a}{+}1, \hat{a}{+}2 \rangle & (a = 3) \\ \langle \hat{a}_2, \hat{a}{+}1_2 \rangle & (a = 2) \\ \langle \hat{a}_4 \rangle & (a = 1); \end{cases}$$

recall that $\hat{a} = b - a$, and we use the $\langle 4^{\hat{a}}, 5^a, 4^{e-b} \rangle$ notation for partitions in $B^\sharp$. Note that if $a \geqslant 3$, then we have $\mu^\diamond_1 \leqslant \lambda^\diamond_1$, and hence $\mathrm{adj}_{\lambda^\diamond\mu^\diamond} = 0$ by Corollary 2.14 of Corollary 2.8. So we may assume that $a \leqslant 2$. Now we compute the v-decomposition number $d^{(e)}_{\lambda^\diamond\mu^\diamond}(v)$; using the runner removal theorems and the fact that $a \leqslant 2$, we need to perform only a few computations with the LLT algorithm. We find that $d^{(e)}_{\lambda^\diamond\mu^\diamond}(v)$ equals 0 if $a = 1$, or v if $a = e - b = 2$. Since we know that $\mathrm{adj}_{\nu\mu^\diamond} = \delta_{\nu\mu^\diamond}$ by earlier arguments for all ν other than $\lambda^\diamond$, we have $\mathrm{adj}_{\lambda^\diamond\mu^\diamond} = 0$ by Corollary 2.12 in these cases.

This leaves only the case where $a = 2$ and $e - b = 1$. In this case, we induce $\lambda^\diamond$ and $\mu^\diamond$ up to the block with the $\langle 4^{e-4}, 5^2, 7^2 \rangle$ notation, and we find

$$(\lambda^\diamond, \mu^\diamond) \sim (\overline{\lambda}, \overline{\mu}),$$

where

$$\overline{\lambda} = \langle e{-}3_4 \mid 4^{e-4}, 5^2, 7^2 \rangle,$$
$$\overline{\mu} = \langle e{-}2_2, e{-}1_2 \mid 4^{e-4}, 5^2, 7^2 \rangle.$$

We have $\mathrm{adj}_{\overline{\lambda}\overline{\mu}} = 0$ by Proposition 3.1, and hence $\mathrm{adj}_{\lambda^\diamond\mu^\diamond} = 0$.

(iv) Finally, suppose $\lambda = \langle a_{2^2} \rangle$. We have

$$\lambda^\diamond = \langle \hat{a}_{1^2}, b_{1^2} \rangle,$$

and so we may appeal to case (iii) and Proposition 2.20.

5.2 The case $e - b = 0$

The case where $b = e$ is harder to deal with, even though many of the cases in Table 5.1 are not relevant. We begin with two partial functions.

1. Define $\mathfrak{a} = \mathfrak{f}_{2a-1} \dots \mathfrak{f}_{a+1}\mathfrak{f}_a$. If μ is in case 1 (with $e - a \geqslant 2$), 2, 3 (with $e - a \geqslant 3$) or 4 (with $e - a \geqslant 4$), then $(\mathfrak{a}(\lambda), \mathfrak{a}(\mu))$ is defined and lowerable. Hence $\mathrm{adj}_{\lambda\mu} = 0$ in these cases.
2. Define $\mathfrak{b} = \mathfrak{f}_{2a+1} \dots \mathfrak{f}_{a-1}\mathfrak{f}_a$. If μ is in one of cases 17–19 then $(\mathfrak{b}(\lambda), \mathfrak{b}(\mu))$ is defined and lowerable, so that $\mathrm{adj}_{\lambda\mu} = 0$.

We are left with one possible partition μ, namely

$$\mu = \begin{cases} \langle a, a{+}1, a{+}2, a{+}3 \rangle & (e - a \geqslant 4) \\ \langle a_2, a{+}1, a{+}2 \rangle & (e - a = 3) \\ \langle a_2, a{+}1_2 \rangle & (e - a = 2) \\ \langle a_4 \rangle & (e - a = 1). \end{cases}$$

Appealing to Proposition 2.20, we now assume that $\mu^\diamond \trianglerighteq \lambda^\diamond$ and $\mu^\diamond_1 > \lambda^\diamond_1$. In order to exploit this, we calculate $\mu^\diamond$ and $\lambda^\diamond$ for each of the possible λ that remain. First we compute $\mu^\diamond$; applying the Mullineux map, we find

$$\mu^\diamond = \begin{cases} \langle \hat{a}, \hat{a}{+}1, \hat{a}{+}2, \hat{a}{+}3\rangle & (a \geqslant 4)\\ \langle \hat{a}_2, \hat{a}{+}1, \hat{a}{+}2\rangle & (a = 3)\\ \langle \hat{a}_2, \hat{a}{+}1_2\rangle & (a = 2)\\ \langle \hat{a}_4\rangle & (a = 1). \end{cases}$$

(Recall that we use the $\langle 4^{\hat{a}}, 5^a\rangle$ notation for $B^\sharp$.) Now we calculate $\lambda^\diamond$ for each of the exceptional partitions λ such that $\mu \trianglerighteq \lambda$ and $\mu_1 > \lambda_1$. This is quite a tedious undertaking, and gives the possibilities listed in Table 5.2; we write $\hat{\imath} = e - i$, for any $i \in \{1, \dots, e-1\}$.

Given this information, we can can list the possible partitions λ meeting all our assumptions. There are between three and five of these, depending on the values of a and $\hat{a}$. We label these as follows (leaving λ^4 and λ^5 undefined if $e - a = 1$, and leaving λ^3 and λ^5 undefined if $a = 1$):

$$\lambda^1 = \begin{cases} \langle a_{1^2}, a{+}1, a{+}2\rangle & (e-a \geqslant 3)\\ \langle a_{2,1}, a{+}1\rangle & (e-a = 2)\\ \langle a_{3,1}\rangle & (e-a=1); \end{cases}$$

$$\lambda^2 = \begin{cases} \langle a_{1^2}, a{+}1_{1^2}\rangle & (e-a \geqslant 2)\\ \langle a_{2^2}\rangle & (e-a=1); \end{cases}$$

$$\lambda^3 = \begin{cases} \langle a{-}2, a_{1^2}, a{+}1\rangle & (a \geqslant 2,\ e-a \geqslant 2)\\ \langle a{-}2, a_{2,1}\rangle & (a \geqslant 2,\ e-a = 1); \end{cases}$$

$$\lambda^4 = \begin{cases} \langle a_{1^2}, a{+}2_{1^2}\rangle & (e-a \geqslant 3)\\ \langle a_{2^2}\rangle & (e-a=2); \end{cases}$$

$$\lambda^5 = \begin{cases} \langle a{-}2, a_{1^2}, a{+}2\rangle & (a \geqslant 2,\ e-a \geqslant 3)\\ \langle a{-}2, a_{2,1}\rangle & (a \geqslant 2,\ e-a = 2). \end{cases}$$

We address each of these possibilities. First we note that

$$\lambda^1 \sim \langle a_{3,1} \mid 4, 7, 10, \dots, 3e+1\rangle,$$
$$\lambda^2 \sim \langle a_{2^2} \mid 4, 7, 10, \dots, 3e+1\rangle,$$

i.e both λ^1 and λ^2 induce semi-simply to a Rouquier block, so by Proposition 2.22 we have $\mathrm{adj}_{\lambda\mu} = 0$ if $\lambda = \lambda^1$ or λ^2.

Next we deal with λ^3. We induce both λ^3 and μ up to the block with the $\langle 4^a, 5, 8^{e-a-1}\rangle$ notation, and we find that

$$(\lambda^3, \mu) \sim (\bar{\lambda}^3, \bar{\mu}),$$

where

$$\bar{\lambda}^3 = \langle a{-}2, a_{2,1} \mid 4^a, 5, 8^{e-a-1}\rangle,$$
$$\bar{\mu} = \langle a_4 \mid 4^a, 5, 8^{e-a-1}\rangle.$$

λ	conditions	$\lambda^\diamond$
$\langle a_{3,1}\rangle$	$a \geqslant 3,\ e-a=1$	$\langle \hat{a}_{1^2}, \hat{a}{+}1, \hat{a}{+}2\rangle$
$\langle a_{3,1}\rangle$	$a=2,\ e-a=1$	$\langle \hat{a}_{2,1}, \hat{a}{+}1\rangle$
$\langle a_{3,1}\rangle$	$a=1,\ e-a=1$	$\langle \hat{a}_{3,1}\rangle$
$\langle a_{2^2}\rangle$	$a \geqslant 2,\ e-a=2$	$\langle 0, \hat{a}_{1^2}, \hat{a}{+}1\rangle$
$\langle a_{2^2}\rangle$	$a \geqslant 2,\ e-a=1$	$\langle \hat{a}_{1^2}, \hat{a}{+}1_{1^2}\rangle$
$\langle a_{2^2}\rangle$	$a=1,\ e-a=2$	$\langle 0, \hat{a}_{2,1}\rangle$
$\langle a_{2^2}\rangle$	$a=1,\ e-a=1$	$\langle \hat{a}_{2^2}\rangle$
$\langle i, a_{2,1}\rangle$	$a \geqslant i+2 \geqslant 3,\ e-a=2$	$\langle 0, \hat{a}_{1^2}, \hat{\imath}\rangle$
$\langle i, a_{2,1}\rangle$	$a \geqslant i+2 \geqslant 3,\ e-a=1$	$\langle \hat{a}_{1^2}, \hat{\imath}_{1^2}\rangle$
$\langle 0, a_{2,1}\rangle$	$a \geqslant 2,\ e-a=2$	$\langle 0, \hat{a}_{2,1}\rangle$
$\langle 0, a_{2,1}\rangle$	$a \geqslant 2,\ e-a=1$	$\langle \hat{a}_{2^2}\rangle$
$\langle a_{2,1}, a{+}1\rangle$	$a \geqslant 3,\ e-a=2$	$\langle \hat{a}_{1^2}, \hat{a}{+}1, \hat{a}{+}2\rangle$
$\langle a_{2,1}, a{+}1\rangle$	$a=2,\ e-a=2$	$\langle \hat{a}_{2,1}, \hat{a}{+}1\rangle$
$\langle a_{2,1}, a{+}1\rangle$	$a=1,\ e-a=2$	$\langle \hat{a}_{3,1}\rangle$
$\langle i, a_{1^3}\rangle$	$a \geqslant i+1 \geqslant 2$	$\langle \hat{a}_{1^2}, \hat{\imath}_2\rangle$
$\langle 0_2, a_{1^2}\rangle$	$a \geqslant 3$	$\langle \hat{a}_{2,1}, \hat{a}{+}1\rangle$
$\langle 0_2, a_{1^2}\rangle$	$a=2$	$\langle \hat{a}_{3,1}\rangle$
$\langle i_2, a_{1^2}\rangle$	$a \geqslant i+3 \geqslant 4$	$\langle \hat{a}_{1^2}, \hat{a}{+}1, \hat{\imath}\rangle$
$\langle a{-}2_2, a_{1^2}\rangle$	$a \geqslant 4$	$\langle \hat{a}_{1^2}, \hat{a}{+}2, \hat{a}{+}3\rangle$
$\langle 1_2, a_{1^2}\rangle$	$a=3$	$\langle \hat{a}_{2,1}, \hat{a}{+}2\rangle$
$\langle a_{1^2}, a{+}2_{1^2}\rangle$	$a \geqslant 2,\ e-a \geqslant 3$	$\langle \hat{a}{-}2, \hat{a}_{1^2}, \hat{a}{+}1\rangle$
$\langle a_{1^2}, a{+}2_{1^2}\rangle$	$a=1,\ e-a \geqslant 3$	$\langle \hat{a}{-}2, \hat{a}_{2,1}\rangle$
$\langle a_{1^2}, a{+}1_{1^2}\rangle$	$a \geqslant 2,\ e-a \geqslant 2$	$\langle \hat{a}_{1^2}, \hat{a}{+}1_{1^2}\rangle$
$\langle a_{1^2}, a{+}1_{1^2}\rangle$	$a=1,\ e-a \geqslant 2$	$\langle \hat{a}_{2^2}\rangle$
$\langle i, j, a_{1^2}\rangle$	$a \geqslant j+2 \geqslant i+4 \geqslant 5$	$\langle \hat{a}_{1^2}, \hat{\jmath}, \hat{\imath}\rangle$
$\langle 0, i, a_{1^2}\rangle$	$a \geqslant i+2 \geqslant 4$	$\langle \hat{a}_{2,1}, \hat{\imath}\rangle$
$\langle i, i{+}1, a_{1^2}\rangle$	$a \geqslant i+3 \geqslant 5$	$\langle \hat{a}_{1^2}, \hat{\imath}, \hat{\imath}{+}1\rangle$
$\langle 1, 2, a_{1^2}\rangle$	$a \geqslant 4$	$\langle \hat{a}_{2,1}, e{-}1\rangle$
$\langle 0, 1, a_{1^2}\rangle$	$a \geqslant 3$	$\langle \hat{a}_{3,1}\rangle$
$\langle i, a_{1^2}, a{+}2\rangle$	$a \geqslant i+2 \geqslant 3,\ e-a \geqslant 3$	$\langle \hat{a}{-}2, \hat{a}_{1^2}, \hat{\imath}\rangle$
$\langle 0, a_{1^2}, a{+}2\rangle$	$a \geqslant 2,\ e-a \geqslant 3$	$\langle \hat{a}{-}2, \hat{a}_{2,1}\rangle$
$\langle i, a_{1^2}, a{+}1\rangle$	$a \geqslant i+2 \geqslant 3,\ e-a \geqslant 2$	$\langle \hat{a}_{1^2}, \hat{\imath}_{1^2}\rangle$
$\langle 0, a_{1^2}, a{+}1\rangle$	$a \geqslant 2,\ e-a \geqslant 2$	$\langle \hat{a}_{2^2}\rangle$
$\langle a_{1^2}, a{+}1, a{+}2\rangle$	$a \geqslant 3,\ e-a \geqslant 3$	$\langle \hat{a}_{1^2}, \hat{a}{+}1, \hat{a}{+}2\rangle$
$\langle a_{1^2}, a{+}1, a{+}2\rangle$	$a=2,\ e-a \geqslant 3$	$\langle \hat{a}_{2,1}, \hat{a}{+}1\rangle$
$\langle a_{1^2}, a{+}1, a{+}2\rangle$	$a=1,\ e-a \geqslant 3$	$\langle \hat{a}_{3,1}\rangle$

Table 5.2

We use the Jantzen–Schaper formula to show that $[S^{\bar{\lambda}^3} : D^{\bar{\mu}}] = 1$ irrespective of the underlying characteristic, so that $\mathrm{adj}_{\bar{\lambda}^3\bar{\mu}} = 0$, and hence $\mathrm{adj}_{\lambda^3\mu} = 0$. The table of Jantzen–Schaper coefficients $J(\nu, \xi)$ for partitions ν, ξ with $\bar{\mu} \trianglerighteq \nu \triangleright \xi \trianglerighteq \bar{\lambda}^3$ is as follows, and yields the result.

	$\langle a_4\rangle$	$\langle a{-}1, a_3\rangle$	$\langle a{-}2, a_3\rangle$	$\langle a{-}1_4\rangle$	$\langle a{-}2_4\rangle$	$\langle a{-}1_2, a_2\rangle$	$\langle a{-}2_2, a_2\rangle$	$\langle a{-}1, a_{2,1}\rangle$	$\langle a{-}2, a_{2,1}\rangle$	$[S^\nu : D^{\bar{\mu}}]$
$\langle a_4\rangle$	·	·	·	·	·	·	·	·	·	1
$\langle a{-}1, a_3\rangle$	1	·	·	·	·	·	·	·	·	1
$\langle a{-}2, a_3\rangle$	−1	1	·	·	·	·	·	·	·	0
$\langle a{-}1_4\rangle$	−1	1	·	·	·	·	·	·	·	0
$\langle a{-}2_4\rangle$	1	0	1	1	·	·	·	·	·	1
$\langle a{-}1_2, a_2\rangle$	1	−1	·	1	·	·	·	·	·	0
$\langle a{-}2_2, a_2\rangle$	−1	0	−1	0	1	1	·	·	·	0
$\langle a{-}1, a_{2,1}\rangle$	0	0	·	1	·	1	·	·	·	0
$\langle a{-}2, a_{2,1}\rangle$	0	0	0	0	1	0	1	1	·	1

Next we deal with λ^4. From Table 5.2, we see that $(\lambda^4)^\diamond$ is a partition of the form λ^3 in $B^\sharp$, so we may appeal to the case just studied.

Finally, we have to contend with λ^5. We induce both λ^5 and μ up to the block with the $\langle 4^{a-2}, 5^2, 7^2, 8^{e-a-2}\rangle$ notation. We have

$$(\lambda^5, \mu) \sim (\bar{\lambda}^5, \bar{\mu}),$$

where

$$\bar{\lambda}^5 = \langle a{-}2_2, a_2 \mid 4^{a-2}, 5^2, 7^2, 8^{e-a-2}\rangle,$$
$$\bar{\mu} = \langle a_2, a{+}1_2 \mid 4^{a-2}, 5^2, 7^2, 8^{e-a-2}\rangle.$$

By Proposition 3.1 we have $\mathrm{adj}_{\bar{\lambda}^5\bar{\mu}} = 0$, and hence $\mathrm{adj}_{\lambda^5\mu} = 0$.

6 Blocks forming a $[4:2]$-pair

In this section, we prove the following result.

Proposition 6.1. *Suppose A and B are weight 4 blocks of $\mathcal{H}_{n-2}$ and $\mathcal{H}_n$ respectively, forming a $[4:2]$-pair. Suppose that there is no block other than A forming a $[4:\kappa]$-pair with B, for any κ. If James's Conjecture holds for A, then it holds for B.*

In this situation, the core of B has the form $((2a+e-c)^{b-a}, a^{c-a})$ for some $0 < a < b \leqslant c \leqslant e$; we use the $\langle 4^a, 6^{b-a}, 5^{c-b}, 4^{e-c}\rangle$ notation for B. As in §5, we often replace B with the conjugate block $B^\sharp$, for which we use the $\langle 4^{b-a}, 6^a, 5^{e-c}, 4^{c-b}\rangle$ notation.

Suppose λ and μ are e-regular partitions in B. Using the inductive hypothesis and Proposition 2.18, we may assume that λ is one of the exceptional partitions for the pair (A, B). These are the partitions $\langle i, a_{1^3}\rangle$ for all $i \neq a, a-1$, together with the partition $\langle a_{2,1^2}\rangle$. Note that these are totally ordered by the

	partition	conditions		partition	conditions
1	$\langle a_4\rangle$	—	22	$\langle a, a{+}1, b, b{+}1\rangle$	$b-a\geqslant 2,\ c-b\geqslant 2$
2	$\langle a_{3,1}\rangle$	—	23	$\langle a, b_{2,1}\rangle$	$b-a=1,\ c-b\geqslant 1$
3	$\langle a_{2^2}\rangle$	$e\geqslant 3$	24	$\langle a, b_2, b{+}1\rangle$	$b-a=1,\ c-b\geqslant 2$
4	$\langle a_3, a{+}1\rangle$	$b-a\geqslant 2$	25	$\langle a_3, c\rangle$	$e-c\geqslant 1$
5	$\langle a_{2,1}, a{+}1\rangle$	$b-a\geqslant 2$	26	$\langle a_{2,1}, c\rangle$	$e-c\geqslant 1$
6	$\langle a_2, a{+}1_2\rangle$	$b-a\geqslant 2$	27	$\langle a_2, a{+}1, c\rangle$	$b-a\geqslant 2,\ e-c\geqslant 1$
7	$\langle a_{1^2}, a{+}1_{1^2}\rangle$	$b-a\geqslant 2,\ a-b+e\geqslant 2$	28	$\langle a_{1^2}, a{+}1, c\rangle$	$b-a\geqslant 2,\ e-c\geqslant 1$
8	$\langle a_2, a{+}1, a{+}2\rangle$	$b-a\geqslant 3$	29	$\langle a, a{+}1, a{+}2, c\rangle$	$b-a\geqslant 3,\ e-c\geqslant 1$
9	$\langle a_{1^2}, a{+}1, a{+}2\rangle$	$b-a\geqslant 3$	30	$\langle a_2, b, c\rangle$	$c-b\geqslant 1,\ e-c\geqslant 1$
10	$\langle a, a{+}1, a{+}2, a{+}3\rangle$	$b-a\geqslant 4$	31	$\langle a, a{+}1, b, c\rangle$	$b-a\geqslant 2,\ c-b\geqslant 1,\ e-c\geqslant 1$
11	$\langle a_3, b\rangle$	$c-b\geqslant 1$	32	$\langle a, b_2, c\rangle$	$b-a=1,\ c-b\geqslant 1,\ e-c\geqslant 1$
12	$\langle a_{2,1}, b\rangle$	$c-b\geqslant 1$	33	$\langle a_2, c_2\rangle$	$c-b=0,\ e-c\geqslant 1$
13	$\langle a_2, a{+}1, b\rangle$	$b-a\geqslant 2,\ c-b\geqslant 1$	34	$\langle a_2, c, c{+}1\rangle$	$e-c\geqslant 2$
14	$\langle a_{1^2}, a{+}1, b\rangle$	$b-a\geqslant 2,\ c-b\geqslant 1$	35	$\langle a, a{+}1, c_2\rangle$	$b-a\geqslant 2,\ c-b=0,\ e-c\geqslant 1$
15	$\langle a, a{+}1, a{+}2, b\rangle$	$b-a\geqslant 3,\ c-b\geqslant 1$	36	$\langle a, a{+}1, c, c{+}1\rangle$	$b-a\geqslant 2,\ e-c\geqslant 2$
16	$\langle a_2, b_2\rangle$	$b-a=1,\ c-b\geqslant 1$	37	$\langle a, c_3\rangle$	$b-a=1,\ c-b=0,\ e-c\geqslant 1$
17	$\langle a_2, b_{1^2}\rangle$	$c-b\geqslant 1$	38	$\langle 0, a_2, c\rangle$	$a\geqslant 2,\ e-c\geqslant 1$
18	$\langle a_2, b, b{+}1\rangle$	$c-b\geqslant 2$	39	$\langle 0, a, a{+}1, c\rangle$	$a\geqslant 2,\ b-a\geqslant 2,\ e-c\geqslant 1$
19	$\langle a_{1^2}, b_2\rangle$	$b-a=1,\ c-b\geqslant 1$	40	$\langle 0_2, a_2\rangle$	$a\geqslant 2,\ e-c=0$
20	$\langle a, a{+}1, b_2\rangle$	$b-a=2,\ c-b\geqslant 1$	41	$\langle 0_2, a, a{+}1\rangle$	$a\geqslant 2,\ b-a\geqslant 2,\ e-c=0$
21	$\langle a, a{+}1, b_{1^2}\rangle$	$b-a\geqslant 2,\ c-b\geqslant 1$			

Table 6.1

dominance order, with the least dominant being

$$\lambda^0=\begin{cases}\langle 0, a_{1^3}\rangle & (a\geqslant 2)\\ \langle a_{1^3}, c\rangle & (a=1,\ c<e)\\ \langle a_{1^3}, b\rangle & (a=1,\ b<c=e)\\ \langle a_{1^3}, a{+}1\rangle & (a=1,\ a+1<b=e)\\ \langle a_{2,1^2}\rangle & (e=2).\end{cases}$$

As before, we also assume that $\mu \trianglerighteq \lambda$ and $\mu_1 > \lambda_1$, and this implies $\mu \trianglerighteq \lambda^0$ and $\mu_1 > \lambda^0_1$. We can also assume that the pair (λ, μ) is not lowerable. Given all these assumptions, we find that μ must be one of the partitions listed in Table 6.1.

6.1 The case $c - b > 0,\ e - c > 0$

The situation where $b < c < e$ is straightforward to deal with.

1. Define $\mathfrak{a} = \mathfrak{f}_{3a-b-c-1}\dots\mathfrak{f}_{2a-b+1}\mathfrak{f}_{2a-b}$. If μ is in any of cases 1–10, 25–32, 34, 36, 38 or 39, then $(\mathfrak{a}(\lambda), \mathfrak{a}(\mu))$ is defined and lowerable, so that $\mathrm{adj}_{\lambda\mu} = 0$.

2. Define $\mathfrak{b} = \mathfrak{f}_{3a-b-c+1} \dots \mathfrak{f}_{2a-c-1}\mathfrak{f}_{2a-c}\mathfrak{f}_{3a-b-c-1}\mathfrak{f}_{3a-b-c-2} \dots \mathfrak{f}_{2a-b-c}$. If μ is in any of cases 11–24, then $(\mathfrak{b}(\lambda), \mathfrak{b}(\mu))$ is defined and lowerable, and so we have $\mathrm{adj}_{\lambda\mu} = 0$.

This deals with all possible μ.

6.2 The case $c - b > 0,\ e - c = 0$

For the situation where $b < c = e$, we begin with two partial functions.

1. Let $\mathfrak{a} = \mathfrak{f}_{3a-b-1} \dots \mathfrak{f}_{2a-b+1}\mathfrak{f}_{2a-b}$. If μ is in one of cases 1–14, 17–19, 40 or 41, then $(\mathfrak{a}(\lambda), \mathfrak{a}(\mu))$ is defined and lowerable, so that $\mathrm{adj}_{\lambda\mu} = 0$. The same applies in cases 15, 16 and 20 provided $e - b \geqslant 2$, and in cases 22 and 24 as long as $e - b \geqslant 3$.
2. Define $\mathfrak{b} = \mathfrak{f}_{3a-b+1} \dots \mathfrak{f}_{2a-1}\mathfrak{f}_{2a}$. If μ is in case 21 or 23 $(\mathfrak{b}(\lambda), \mathfrak{b}(\mu))$ is defined and lowerable.

Assuming that μ is not one of the partitions dealt with in 1 or 2, we must have $e - b \leqslant 2$, and

$$\mu = \begin{cases} \langle a, a{+}1, b, b{+}1\rangle & (e-b=2,\ b-a \geqslant 2)\\ \langle a, b_2, b{+}1\rangle & (e-b=2,\ b-a=1)\\ \langle a, a{+}1, a{+}2, b\rangle & (e-b=1,\ b-a \geqslant 3)\\ \langle a, a{+}1, b_2\rangle & (e-b=1,\ b-a=2)\\ \langle a_2, b_2\rangle & (e-b=1,\ b-a=1). \end{cases}$$

Now we add the assumptions that $\mu^\diamond \trianglerighteq \lambda^\diamond$ and $\mu^\diamond_1 > \lambda^\diamond_1$. To make use of these conditions, we compute

$$\mu^\diamond = \begin{cases} \langle \breve{a}, \breve{a}{+}1, \breve{a}{+}2, \breve{a}{+}3\rangle & (a \geqslant 4)\\ \langle \breve{a}_2, \breve{a}{+}1, \breve{a}{+}2\rangle & (a=3)\\ \langle \breve{a}_2, \breve{a}{+}1_2\rangle & (a=2)\\ \langle \breve{a}_4\rangle & (a=1), \end{cases}$$

where the partitions on the right are written in the $\langle 4^{b-a}, 6^a, 4^{e-b}\rangle$ notation, and we write $\breve{a} = b - a$. We also compute $\lambda^\diamond$ for each exceptional λ; the possibilities are listed in Table 6.2.

λ	conditions	$\lambda^\diamond$
$\langle a_{2,1^2}\rangle$	$b - a \geqslant 2$	$\langle 0, \breve{a}_{1^3}\rangle$
$\langle a_{2,1^2}\rangle$	$b - a = 1$	$\langle \breve{a}_{1^3}, b\rangle$
$\langle a_{1^3}, a{+}1\rangle$	$b - a \geqslant 2$	$\langle \breve{a}_{1^3}, b\rangle$
$\langle a_{1^3}, i\rangle$	$a + 2 \leqslant i \leqslant b - 1$	$\langle \breve{a}_{1^3}, b{-}i\rangle$
$\langle a_{1^3}, b\rangle$	$a \geqslant 2$	$\langle \breve{a}_{1^3}, \breve{a}{+}1\rangle$
$\langle a_{1^3}, b\rangle$	$a = 1$	$\langle \breve{a}_{2,1^2}\rangle$
$\langle a_{1^3}, e{-}1\rangle$	$e - b = 2$	$\langle \breve{a}_{1^3}, e{-}1\rangle$
$\langle 0, a_{1^3}\rangle$	$a \geqslant 2$	$\langle \breve{a}_{2,1^2}\rangle$
$\langle i, a_{1^3}\rangle$	$1 \leqslant i \leqslant a - 2$	$\langle \breve{a}_{1^3}, b{-}i\rangle$

Table 6.2

Armed with this, we can calculate all possible λ satisfying our assumptions. There are between two and four of these, depending on the values of a and b, and we label them as follows.

$$\begin{aligned}
\lambda^1 &= \langle a_{1^3}, a{+}1\rangle \quad (\text{if } e-b=1 \text{ and } b-a \geqslant 2),\\
\lambda^2 &= \langle a_{1^3}, b\rangle,\\
\lambda^3 &= \langle a{-}2, a_{1^3}\rangle \quad (\text{if } a \geqslant 2),\\
\lambda^4 &= \langle a_{1^3}, b{+}1\rangle \quad (\text{if } e-b=2).
\end{aligned}$$

First we note that

$$\begin{aligned}
\lambda^1 &\sim \langle a_{1^2}, a{+}1_2 \mid 4,7,10,\dots,3e+1\rangle,\\
\lambda^2 &\sim \langle a_{2,1}, a{+}e{-}b \mid 4,7,10,\dots,3e+1\rangle,
\end{aligned}$$

i.e. λ^1 (if it is defined) and λ^2 induce semi-simply to a Rouquier block, so we have $\mathrm{adj}_{\lambda\mu} = 0$ if $\lambda = \lambda^1$ or λ^2.

To deal with λ^3, we induce both λ^3 and μ up to the block with the $\langle 4^a, 5, 8^{e-a-1}\rangle$ notation. We find that

$$(\lambda^3, \mu) \sim (\bar\lambda^3, \bar\mu),$$

where

$$\begin{aligned}
\bar\lambda^3 &= \langle a{-}2, a_{1^2}, a{+}e{-}b \mid 4^a, 5, 8^{e-a-1}\rangle,\\
\bar\mu &= \langle a_4 \mid 4^a, 5, 8^{e-a-1}\rangle.
\end{aligned}$$

We have $\bar\mu \ntrianglerighteq \bar\lambda^3$, so that $\mathrm{adj}_{\bar\lambda^3\bar\mu} = 0$, and hence $\mathrm{adj}_{\lambda^3\mu} = 0$.

We are left only with λ^4, if $e-b=2$. We induce both λ^4 and μ up to the block with the $\langle 4^a, 5^2, 6^{b-a}\rangle$ notation. We have

$$(\lambda^4, \mu) \sim (\bar\lambda^4, \bar\mu),$$

where

$$\begin{aligned}
\bar\lambda^4 &= \langle a_{1^2}, a{+}1_2 \mid 4^a, 5^2, 6^{b-a}\rangle,\\
\bar\mu &= \langle a_2, a{+}1_2 \mid 4^a, 5^2, 6^{b-a}\rangle.
\end{aligned}$$

We have $\bar\lambda^4_1 = \bar\mu_1$, and so by Corollary 2.14 we have $\mathrm{adj}_{\lambda^4\mu} = \mathrm{adj}_{\bar\lambda^4\bar\mu} = 0$.

6.3 The case $c-b=0,\ e-c>0$

In the case where $b = c < e$, we know by the results of §6.2 that Proposition 6.1 holds for the conjugate block $B^\sharp$. So we may apply Proposition 2.20, and we find that Proposition 6.1 holds for B.

6.4 The case $c-b=0,\ e-c=0$

Now we consider the case where $b = c = e$.

1. Suppose $e-a \geqslant 2$, and let $\mathfrak{a} = \mathfrak{f}_{3a-1}\dots\mathfrak{f}_{2a+1}\mathfrak{f}_{2a}$. If μ is in one of cases 1–4, then $(\mathfrak{a}(\lambda), \mathfrak{a}(\mu))$ is defined and lowerable, so that $\mathrm{adj}_{\lambda\mu} = 0$. The same applies in cases 5 and 6 if $e-a \geqslant 3$, and case 8 if $e-a \geqslant 4$.

2. Now suppose $a \geqslant 2$, and define $\mathfrak{b} = \tilde{\mathfrak{f}}_{3a+1} \dots \tilde{\mathfrak{f}}_{2a-1}\tilde{\mathfrak{f}}_{2a}$. If μ is in case 40 or 41 we find that $(\mathfrak{b}(\lambda), \mathfrak{b}(\mu))$ is defined and lowerable, so that $\mathrm{adj}_{\lambda\mu} = 0$.

Assuming that μ does not fall into any of these cases, we are left with three possibilities for μ, which we label μ^1, μ^2, μ^3 as follows:

$$\mu^1 = \begin{cases} \langle a, a{+}1, a{+}2, a{+}3\rangle & (e-a \geqslant 4)\\ \langle a_2, a{+}1, a{+}2\rangle & (e-a = 3)\\ \langle a_2, a{+}1_2\rangle & (e-a=2)\\ \langle a_4\rangle & (e-a=1); \end{cases}$$

$$\mu^2 = \begin{cases} \langle a_{1^2}, a{+}1, a{+}2\rangle & (e-a \geqslant 3)\\ \langle a_{2,1}, a{+}1\rangle & (e-a=2)\\ \langle a_{3,1}\rangle & (e-a=1); \end{cases}$$

$$\mu^3 = \begin{cases} \langle a_{1^2}, a{+}1_{1^2}\rangle & (e-a \geqslant 2)\\ \langle a_{2^2}\rangle & (e-a=1). \end{cases}$$

Using the Mullineux map, we find

$$\mu^{1\diamond} = \begin{cases} \langle \hat{a}, \hat{a}{+}1, \hat{a}{+}2, \hat{a}{+}3\rangle & (a \geqslant 4)\\ \langle \hat{a}_2, \hat{a}{+}1, \hat{a}{+}2\rangle & (a = 3)\\ \langle \hat{a}_2, \hat{a}{+}1_2\rangle & (a=2)\\ \langle \hat{a}_4\rangle & (a=1), \end{cases}$$

$$\mu^{2\diamond} = \begin{cases} \langle \hat{a}_{1^2}, \hat{a}{+}1, \hat{a}{+}2\rangle & (a \geqslant 3)\\ \langle \hat{a}_{2,1}, \hat{a}{+}1\rangle & (a=2)\\ \langle \hat{a}_{3,1}\rangle & (a=1), \end{cases}$$

$$\mu^{3\diamond} = \begin{cases} \langle \hat{a}_{1^2}, \hat{a}{+}1_{1^2}\rangle & (a \geqslant 2)\\ \langle \hat{a}_{2^2}\rangle & (a=1), \end{cases}$$

where the partitions on the right are written with the $\langle 4^{e-a}, 6^a\rangle$ notation, and we put $\hat{a} = e - a$. We also calculate $\lambda^\diamond$ for each of the exceptional partitions λ, listing the various cases in Table 6.3. Now for each μ can calculate all possible λ for which $\mu \trianglerighteq \lambda$, $\mu^\diamond \trianglerighteq \lambda^\diamond$, $\mu_1 > \lambda_1$ and $\mu^\diamond_1 > \lambda^\diamond_1$.

If $\mu = \mu^1$, there are at most three such λ, which we label as follows:

$$\lambda^1 = \begin{cases} \langle a_{1^3}, a{+}1\rangle & (e-a \geqslant 2)\\ \langle a_{2,1^2}\rangle & (e-a=1); \end{cases}$$

$$\lambda^2 = \langle a{-}2, a_{1^3}\rangle \quad (a \geqslant 2);$$

$$\lambda^3 = \begin{cases} \langle a_{1^3}, a{+}2\rangle & (e-a \geqslant 3)\\ \langle a_{2,1^2}\rangle & (e-a=2). \end{cases}$$

(We leave λ^2 undefined if $a = 1$, and we leave λ^3 undefined if $e - a = 1$.)

λ	conditions	$\lambda^\diamond$
$\langle a_{2,1^2}\rangle$	$a \leqslant e-2$	$\langle 0, \hat{a}_{1^3}\rangle$
$\langle a_{2,1^2}\rangle$	$2 \leqslant a = e-1$	$\langle \hat{a}_{1^3}, \hat{a}{+}1\rangle$
$\langle a_{2,1^2}\rangle$	$e = 2$	$\langle \hat{a}_{2,1^2}\rangle$
$\langle a_{1^3}, i\rangle$	$a+2 \leqslant i \leqslant e-1$	$\langle \hat{a}_{1^3}, e{-}i\rangle$
$\langle a_{1^3}, a{+}1\rangle$	$2 \leqslant a \leqslant e-2$	$\langle \hat{a}_{1^3}, \hat{a}{+}1\rangle$
$\langle a_{1^3}, a{+}1\rangle$	$1 = a \leqslant e-2$	$\langle \hat{a}_{2,1^2}\rangle$
$\langle i, a_{1^3}\rangle$	$1 \leqslant i \leqslant a-2$	$\langle \hat{a}_{1^3}, e{-}i\rangle$
$\langle 0, a_{1^3}\rangle$	$2 \leqslant a$	$\langle \hat{a}_{2,1^2}\rangle$

Table 6.3

We have

$$\lambda^1 \sim \langle a_{2,1^2} \mid 4, 7, 10, \ldots, 3e+1\rangle,$$

i.e. λ^1 induces semi-simply to a Rouquier block, so $\mathrm{adj}_{\lambda\mu} = 0$ if $\lambda = \lambda^1$, by Proposition 2.22.

Now assume $a \geqslant 2$ and consider λ^2. We induce both λ^2 and μ^1 up to the block with the $\langle 4^a, 6, 9^{e-a-1}\rangle$ notation. We find that

$$(\lambda^2, \mu^1) \sim (\bar{\lambda}^2, \bar{\mu}^1),$$

where

$$\bar{\lambda}^2 = \langle a{-}2, a_{1^3} \mid 4^a, 6, 9^{e-a-1}\rangle,$$
$$\bar{\mu}^1 = \langle a_4 \mid 4^a, 6, 9^{e-a-1}\rangle.$$

We use the Jantzen–Schaper formula and Proposition 2.10 to show that $[S^{\bar{\lambda}^2} : D^{\bar{\mu}^1}] = 0$, independently of characteristic, which implies that $\mathrm{adj}_{\lambda^2\mu^1} = 0$. The table of Jantzen–Schaper coefficients for those partitions ν, ξ with $\bar{\mu}^1 \trianglerighteq \nu \triangleright \xi \trianglerighteq \bar{\lambda}^2$ is as follows.

	$\langle a_4\rangle$	$\langle a{-}1, a_3\rangle$	$\langle a{-}2, a_3\rangle$	$\langle a{-}1_2, a_2\rangle$	$\langle a{-}2_2, a_2\rangle$	$\langle a{-}1_4\rangle$	$\langle a{-}2_4\rangle$	$\langle a{-}1_3, a\rangle$	$\langle a{-}2_3, a\rangle$	$\langle a{-}1_2, a_{1^2}\rangle$	$\langle a{-}2_2, a_{1^2}\rangle$	$\langle a{-}1, a_{1^3}\rangle$	$\langle a{-}2, a_{1^3}\rangle$	$[S^\nu : D^{\bar{\mu}}]$
$\langle a_4\rangle$	·	·	·	·	·	·	·	·	·	·	·	·	·	1
$\langle a{-}1, a_3\rangle$	1	·	·	·	·	·	·	·	·	·	·	·	·	1
$\langle a{-}2, a_3\rangle$	−1	1	·	·	·	·	·	·	·	·	·	·	·	0
$\langle a{-}1_2, a_2\rangle$	−1	1	·	·	·	·	·	·	·	·	·	·	·	0
$\langle a{-}2_2, a_2\rangle$	1	0	1	1	·	·	·	·	·	·	·	·	·	1
$\langle a{-}1_4\rangle$	1	−1	·	1	·	·	·	·	·	·	·	·	·	0
$\langle a{-}2_4\rangle$	−1	0	−1	0	1	1	·	·	·	·	·	·	·	0
$\langle a{-}1_3, a\rangle$	−1	1	·	−1	·	1	·	·	·	·	·	·	·	0
$\langle a{-}2_3, a\rangle$	1	0	1	0	−1	0	1	1	·	·	·	·	·	0
$\langle a{-}1_2, a_{1^2}\rangle$	0	0	·	0	·	1	·	1	·	·	·	·	·	0
$\langle a{-}2_2, a_{1^2}\rangle$	0	0	0	0	0	0	1	0	1	1	·	·	·	0
$\langle a{-}1, a_{1^3}\rangle$	0	0	·	0	·	1	·	1	·	1	·	·	·	0
$\langle a{-}2, a_{1^3}\rangle$	0	0	0	0	0	0	1	0	1	0	1	1	·	0

We deduce that $\mathrm{adj}_{\bar{\lambda}^2\bar{\mu}^1} = 0$, and hence $\mathrm{adj}_{\lambda\mu} = 0$ when $\lambda = \lambda^2$ and $\mu = \mu^1$.

We are left with λ^3, if $e - a \geqslant 2$. From Table 6.3 we have

$$(\lambda^3)^\diamond = \langle \hat{a}\text{–}2, \hat{a}_{1^3} \mid 4^{\hat{a}}, 6^{e-\hat{a}} \rangle,$$

so that $(\lambda^3)^\diamond$ is the partition of the form λ^2 in the conjugate block $B^\sharp$. So using the above result for λ^2, we deduce $\mathrm{adj}_{\lambda^\diamond\mu^\diamond} = 0$, and hence $\mathrm{adj}_{\lambda\mu} = 0$ when $\lambda = \lambda^3$ and $\mu = \mu^1$.

Now we look at the case $\mu = \mu^2$. Here, the only exceptional partition λ satisfying $\mu \trianglerighteq \lambda$, $\mu^\diamond \trianglerighteq \lambda^\diamond$, $\mu_1 > \lambda_1$ and $\mu^\diamond_1 > \lambda^\diamond_1$ is the partition λ^1 given above. But as we have seen, λ^1 induces semi-simply to a Rouquier block, and so $\mathrm{adj}_{\lambda\mu} = 0$ when $\lambda = \lambda^1$ and $\mu = \mu^2$.

Finally, we look at the case $\mu = \mu^3$. Here the situation is even simpler; there are no exceptional λ satisfying the given conditions.

7 Blocks forming two $[4:1]$-pairs I

In this section and the next, we prove the following proposition.

Proposition 7.1. *Suppose that A_1 and A_2 are weight 4 blocks of $\mathcal{H}_{n-1}$, and B is a weight 4 block of $\mathcal{H}_n$. Suppose that both A_1 and A_2 form $[4:1]$-pairs with B, and that there is no block other than A_1 or A_2 forming a $[4:\kappa]$-pair with B, for any κ. If James's Conjecture holds for A_1 and A_2, then it holds for B.*

The conditions give two distinct types of block B. In this section we suppose that the core of B has the form $((a-b+c)^{d-c}, a^{b-a})$, where $0 < a < b < c < d \leqslant e$. B may be displayed on an abacus with the $\langle 4^a, 5^{b-a}, 4^{c-b}, 5^{d-c}, 4^{d-e} \rangle$ notation.

Suppose λ and μ are e-regular partitions in B. By Proposition 2.18 and the hypothesis on A_1 and A_2, we know that $\mathrm{adj}_{\lambda\mu} = \delta_{\lambda\mu}$ unless λ is the 'doubly exceptional' partition $\langle a_{1^2}, c_{1^2} \rangle$. So we assume that this is the case. We also assume as usual that $\mu \trianglerighteq \lambda$, $\mu_1 > \lambda_1$ and the pair (λ, μ) is not lowerable. The remaining possibilities for μ are listed in Table 7.1.

7.1 The case $e - d > 0$

In the case where $e > d$, we can show that $\mathrm{adj}_{\lambda\mu} = 0$ by using three different partial functions.

1. Consider $\mathfrak{a} = \mathfrak{f}_{a-b+2c-d-1} \dots \mathfrak{f}_{a+c-d+1}\mathfrak{f}_{a+c-d}$. In cases 20–29 we find that $(\mathfrak{a}(\lambda), \mathfrak{a}(\mu))$ is defined and lowerable, and so we have $\mathrm{adj}_{\lambda\mu} = 0$ by Proposition 2.19.

2. Now define $\mathfrak{b} = \mathfrak{f}_{2a-b+c-d-1} \dots \mathfrak{f}_{a-b+c-d+1}\mathfrak{f}_{a-b+c-d}$. We find that in cases 1–19, 30–33 and 41 $(\mathfrak{b}(\lambda), \mathfrak{b}(\mu))$ is defined and lowerable, and so $\mathrm{adj}_{\lambda\mu} = 0$.

3. Finally, let $\mathfrak{c} = \mathfrak{f}_{a-b+2c-d+1} \dots \mathfrak{f}_{a-b+c-1}\mathfrak{f}_{a-b+c}$. We find that in cases 34–40 and 42–48 $(\mathfrak{c}(\lambda), \mathfrak{c}(\mu))$ is defined and lowerable, so that $\mathrm{adj}_{\lambda\mu} = 0$.

This deals with every possible μ.

	partition	conditions		partition	conditions
1	$\langle a_4\rangle$	—	25	$\langle c_4\rangle$	—
2	$\langle a_3, a{+}1\rangle$	$b-a\geqslant 2$	26	$\langle c_3, c{+}1\rangle$	$d-c\geqslant 2$
3	$\langle a_2, a{+}1_2\rangle$	$b-a\geqslant 2$	27	$\langle c_2, c{+}1_2\rangle$	$d-c\geqslant 2$
4	$\langle a_2, a{+}1, a{+}2\rangle$	$b-a\geqslant 3$	28	$\langle c_2, c{+}1, c{+}2\rangle$	$d-c\geqslant 3$
5	$\langle a_3, b\rangle$	—	29	$\langle c, c{+}1, c{+}2, c{+}3\rangle$	$d-c\geqslant 4$
6	$\langle a_2, a{+}1, b\rangle$	$b-a\geqslant 2$	30	$\langle a_3, d\rangle$	$e-d\geqslant 1$
7	$\langle a_2, b_2\rangle$	$b-a=1$	31	$\langle a_2, a{+}1, d\rangle$	$b-a\geqslant 2,\ e-d\geqslant 1$
8	$\langle a_3, c\rangle$	—	32	$\langle a_2, b, d\rangle$	$e-d\geqslant 1$
9	$\langle a_2, a{+}1, c\rangle$	$b-a\geqslant 2$	33	$\langle a_2, c, d\rangle$	$e-d\geqslant 1$
10	$\langle a_2, b, c\rangle$	—	34	$\langle a, c_2, d\rangle$	$e-d\geqslant 1$
11	$\langle a_2, c_2\rangle$	—	35	$\langle a, c, c{+}1, d\rangle$	$d-c\geqslant 2,\ e-d\geqslant 1$
12	$\langle a_2, c, c{+}1\rangle$	$d-c\geqslant 2$	36	$\langle b, c_2, d\rangle$	$e-d\geqslant 1$
13	$\langle a, a{+}1, c_2\rangle$	$b-a\geqslant 2$	37	$\langle b, c, c{+}1, d\rangle$	$d-c\geqslant 2,\ e-d\geqslant 1$
14	$\langle a, a{+}1, c, c{+}1\rangle$	$b-a\geqslant 2,\ d-c\geqslant 2$	38	$\langle c_3, d\rangle$	$e-d\geqslant 1$
15	$\langle a, b, c_2\rangle$	—	39	$\langle c_2, c{+}1, d\rangle$	$d-c\geqslant 2,\ e-d\geqslant 1$
16	$\langle a, b, c, c{+}1\rangle$	$d-c\geqslant 2$	40	$\langle c, c{+}1, c{+}2, d\rangle$	$d-c\geqslant 3,\ e-d\geqslant 1$
17	$\langle a, c_3\rangle$	—	41	$\langle a_2, d, d{+}1\rangle$	$e-d\geqslant 2$
18	$\langle a, c_2, c{+}1\rangle$	$d-c\geqslant 2$	42	$\langle a, c, d_2\rangle$	$d-c=1,\ e-d\geqslant 1$
19	$\langle a, c, c{+}1, c{+}2\rangle$	$d-c\geqslant 3$	43	$\langle b, c, d_2\rangle$	$d-c=1,\ e-d\geqslant 1$
20	$\langle b, c_3\rangle$	—	44	$\langle c_2, d_2\rangle$	$d-c=1,\ e-d\geqslant 1$
21	$\langle b, c_{2,1}\rangle$	$c-b=1$	45	$\langle c_2, d, d{+}1\rangle$	$e-d\geqslant 2$
22	$\langle b, c_2, c{+}1\rangle$	$d-c\geqslant 2$	46	$\langle c, c{+}1, d_2\rangle$	$d-c=2,\ e-d\geqslant 1$
23	$\langle b, c_{1^2}, c{+}1\rangle$	$c-b=1,\ d-c\geqslant 2$	47	$\langle c, c{+}1, d, d{+}1\rangle$	$d-c\geqslant 2,\ e-d\geqslant 2$
24	$\langle b, c, c{+}1, c{+}2\rangle$	$d-c\geqslant 3$	48	$\langle c, d_2, d{+}1\rangle$	$d-c=1,\ e-d\geqslant 2$

Table 7.1

7.2 The case $e-d=0$

The case where $d=e$ is rather more complicated.

1. As above, we let $\mathfrak{a}=\mathfrak{f}_{a+2c-b-e-1}\dots\mathfrak{f}_{a+c-e+1}\mathfrak{f}_{a+c-e}$. In cases 10, 13, 15–18 and 20–29 we find that $(\mathfrak{a}(\lambda),\mathfrak{a}(\mu))$ is defined and lowerable, and so we have $\mathrm{adj}_{\lambda\mu}=0$. This also applies in cases 9 and 14 if $b-a\geqslant 3$, and in cases 11, 12 and 19 if $b-a\geqslant 2$.

2. We let $\mathfrak{b}=\mathfrak{f}_{2a+c-b-e-1}\dots\mathfrak{f}_{a+c-b-e+1}\mathfrak{f}_{a+c-b-e}$. This deals with cases 1–8. It also deals with cases 9 and 11 if $e-c\geqslant 2$, and with case 12 if $e-c\geqslant 3$.

Now assume that μ does not fall into one of the cases dealt with above. Then we must have $b-a\leqslant 2$ and

$$\mu=\begin{cases}\langle a, c, c{+}1, c{+}2\rangle & (b-a=1,\ e-c\geqslant 3)\\ \langle a_2, c, c{+}1\rangle & (b-a=1,\ e-c=2)\\ \langle a_2, c_2\rangle & (b-a=1,\ e-c=1)\\ \langle a, a{+}1, c, c{+}1\rangle & (b-a=2,\ e-c\geqslant 2)\\ \langle a_2, a{+}1, c\rangle & (b-a=2,\ e-c=1).\end{cases}$$

Now the conjugate block $B^\sharp$ may be displayed on an abacus with the $\langle 4^{e-c}, 5^{c-b}, 4^{b-a}, 5^{a}\rangle$ notation. If $c-b \geqslant 3$, then we have $\mathrm{adj}_{\nu\xi} = 0$ for all ξ, ν in $B^\sharp$, by the above arguments. Hence, using Proposition 2.20, we may assume that $c-b \leqslant 2$. We also assume (replacing B with $B^\sharp$ if necessary) that $c-b \geqslant b-a$.

Suppose that $b-a=1$, and induce λ and μ to the block with the $\langle 4^{c-1}, 6, 9^{e-c}\rangle$ notation. We get

$$(\lambda,\mu) \sim (\bar\lambda,\bar\mu),$$

where

$$\bar\lambda = \langle a_4 \mid 4^{c-1}, 6, 9^{e-c}\rangle,$$
$$\bar\mu = \langle c{-}1_4 \mid 4^{c-1}, 6, 9^{e-c}\rangle.$$

To show that $\mathrm{adj}_{\bar\lambda\bar\mu} = 0$ (and hence that $\mathrm{adj}_{\lambda\mu} = 0$), we apply the Jantzen–Schaper formula to determine that $[S^{\bar\lambda} : D^{\bar\mu}] = 0$, independently of the underlying characteristic. There are two tables of Jantzen-Schaper coefficients, according to the value of $c-b$. If $c-b=2$, we have

	$\langle c{-}1_4\rangle$	$\langle a{+}1, c{-}1_3\rangle$	$\langle a, c{-}1_3\rangle$	$\langle a{+}1_2, c{-}1_2\rangle$	$\langle a_2, c{-}1_2\rangle$	$\langle a{+}1_4\rangle$	$\langle a_4\rangle$	$[S^\nu : D^{\bar\mu}]$
$\langle c{-}1_4\rangle$	·	·	·	·	·	·	·	1
$\langle a{+}1, c{-}1_3\rangle$	1	·	·	·	·	·	·	1
$\langle a, c{-}1_3\rangle$	−1	1	·	·	·	·	·	0
$\langle a{+}1_2, c{-}1_2\rangle$	−1	1	·	·	·	·	·	0
$\langle a_2, c{-}1_2\rangle$	1	0	1	1	·	·	·	1
$\langle a{+}1_4\rangle$	1	−1	·	1	·	·	·	0
$\langle a_4\rangle$	−1	0	−1	0	1	1	·	0

,

while if $c-b=1$, we have

	$\langle c{-}1_4\rangle$	$\langle a, c{-}1_3\rangle$	$\langle a_2, c{-}1_2\rangle$	$\langle a_4\rangle$	$[S^\nu : D^{\bar\mu}]$
$\langle c{-}1_4\rangle$	·	·	·	·	1
$\langle a, c{-}1_3\rangle$	1	·	·	·	1
$\langle a_2, c{-}1_2\rangle$	−1	1	·	·	0
$\langle a_4\rangle$	1	−1	1	·	0

.

Next we suppose that $b-a=2$ (and hence, by our assumptions, that $c-b=2$). We induce to the block with the $\langle 4^a, 5^2, 7^2, 8^{e-c}\rangle$ notation, and we find that

$$(\lambda,\mu) \sim (\bar\lambda,\bar\mu),$$

where

$$\bar\lambda = \langle a_4 \mid 4^a, 5^2, 7^2, 8^{e-c}\rangle,$$
$$\bar\mu = \langle a{+}2_2, a{+}3_2 \mid 4^a, 5^2, 7^2, 8^{e-c}\rangle.$$

We have $\mathrm{adj}_{\bar\lambda\bar\mu} = 0$ by Proposition 3.1, and hence $\mathrm{adj}_{\lambda\mu} = 0$.

8 Blocks forming two [4 : 1]-pairs II

In this section, we complete the proof of Proposition 7.1, by considering a weight 4 block B whose core has the form $((a+b-d+e)^{c-b}, a^{d-a})$ for some $0 < a < b < c \leqslant d \leqslant e$. B may be displayed on an abacus with the $\langle 4^a, 5^{b-a}, 6^{c-b}, 5^{d-c}, 4^{e-d}\rangle$ notation.

Taking e-regular partitions λ and μ in B, we know that $\mathrm{adj}_{\lambda\mu} = \delta_{\lambda\mu}$ unless λ is exceptional for both the [4 : 1]-pairs formed by B. So we assume that $b-a \geqslant 2$ and that λ is the 'doubly exceptional' partition $\langle a_{1^2}, b_{1^2}\rangle$.

As usual, we also assume that $\mu \trianglerighteq \lambda$, $\mu_1 > \lambda_1$ and that (λ, μ) is not lowerable. The remaining possibilities for μ are listed in Table 8.1.

	partition	conditions		partition	conditions
1	$\langle a_4\rangle$	—	22	$\langle a, b, c_2\rangle$	$c-b=1,\ d-c\geqslant 1$
2	$\langle a_3, a{+}1\rangle$	—	23	$\langle b_2, c_2\rangle$	$c-b=1,\ d-c\geqslant 1$
3	$\langle a_3, b\rangle$	—	24	$\langle b_2, c_{1^2}\rangle$	$d-c\geqslant 1$
4	$\langle a_2, b_2\rangle$	—	25	$\langle b_2, c, c{+}1\rangle$	$d-c\geqslant 2$
5	$\langle a_2, b, b{+}1\rangle$	$c-b\geqslant 2$	26	$\langle b, b{+}1, c_2\rangle$	$c-b=2,\ d-c\geqslant 1$
6	$\langle a, a{+}1, b_2\rangle$	—	27	$\langle b, b{+}1, c_{1^2}\rangle$	$c-b\geqslant 2,\ d-c\geqslant 1$
7	$\langle a, a{+}1, b, b{+}1\rangle$	$c-b\geqslant 2$	28	$\langle b, b{+}1, c, c{+}1\rangle$	$c-b\geqslant 2,\ d-c\geqslant 2$
8	$\langle a, b_3\rangle$	—	29	$\langle b, c_{2,1}\rangle$	$c-b=1,\ d-c\geqslant 1$
9	$\langle a, b_2, b{+}1\rangle$	$c-b\geqslant 2$	30	$\langle b, c_2, c{+}1\rangle$	$c-b=1,\ d-c\geqslant 2$
10	$\langle a, b, b{+}1, b{+}2\rangle$	$c-b\geqslant 3$	31	$\langle a_3, d\rangle$	$e-d\geqslant 1$
11	$\langle b_4\rangle$	—	32	$\langle a, b_2, d\rangle$	$e-d\geqslant 1$
12	$\langle b_3, b{+}1\rangle$	$c-b\geqslant 2$	33	$\langle a, b, b{+}1, d\rangle$	$c-b\geqslant 2,\ e-d\geqslant 1$
13	$\langle b_2, b{+}1_2\rangle$	$c-b\geqslant 2$	34	$\langle b_3, d\rangle$	$e-d\geqslant 1$
14	$\langle b_2, b{+}1, b{+}2\rangle$	$c-b\geqslant 3$	35	$\langle b_2, b{+}1, d\rangle$	$c-b\geqslant 2,\ e-d\geqslant 1$
15	$\langle b, b{+}1, b{+}2, b{+}3\rangle$	$c-b\geqslant 4$	36	$\langle b, b{+}1, b{+}2, d\rangle$	$c-b\geqslant 3,\ e-d\geqslant 1$
16	$\langle a_3, c\rangle$	$d-c\geqslant 1$	37	$\langle b_2, c, d\rangle$	$d-c\geqslant 1,\ e-d\geqslant 1$
17	$\langle a, b_2, c\rangle$	$d-c\geqslant 1$	38	$\langle b, b{+}1, c, d\rangle$	$c-b\geqslant 2,\ d-c\geqslant 1,\ e-d\geqslant 1$
18	$\langle a, b, b{+}1, c\rangle$	$c-b\geqslant 2,\ d-c\geqslant 1$	39	$\langle b, c_2, d\rangle$	$c-b=1,\ d-c\geqslant 1,\ e-d\geqslant 1$
19	$\langle b_3, c\rangle$	$d-c\geqslant 1$	40	$\langle b_2, d_2\rangle$	$d-c=0,\ e-d\geqslant 1$
20	$\langle b_2, b{+}1, c\rangle$	$c-b\geqslant 2,\ d-c\geqslant 1$	41	$\langle b, b{+}1, d_2\rangle$	$c-b\geqslant 2,\ d-c=0,\ e-d\geqslant 1$
21	$\langle b, b{+}1, b{+}2, c\rangle$	$c-b\geqslant 3,\ d-c\geqslant 1$	42	$\langle b, d_3\rangle$	$c-b=1,\ d-c=0,\ e-d\geqslant 1$

Table 8.1

8.1 The case $d-c>0,\ e-d>0$

The case where $c<d<e$ can be dealt with using three partial functions.

1. Define $\mathfrak{a} = \mathfrak{f}_{2a+b-c-d-1}\dots\mathfrak{f}_{a+b-c-d+1}\mathfrak{f}_{a+b-c-d}$. In cases 1–10, 16–18 and 31 the pair $(\mathfrak{a}(\lambda), \mathfrak{a}(\mu))$ is defined and lowerable, and so $\mathrm{adj}_{\lambda\mu} = 0$.

2. Define $\mathfrak{b} = \mathfrak{f}_{a+2b-c-d-1}\dots\mathfrak{f}_{a+b-c+1}\mathfrak{f}_{a+b-c}$. In cases 11–15 and 32–39 we find that $(\mathfrak{b}(\lambda), \mathfrak{b}(\mu))$ is defined and lowerable.

3. Finally, let $\mathfrak{c} = \mathfrak{f}_{a+2b-c-d+1} \dots \mathfrak{f}_{a+b-d-1}\mathfrak{f}_{a+b-d}$. We find in cases 19–30 that $(\mathfrak{c}(\lambda), \mathfrak{c}(\mu))$ is defined and lowerable, and so $\mathrm{adj}_{\lambda\mu} = 0$.

8.2 The case $d - c = 0,\ e - d > 0$

Now we consider the case where $c = d < e$. As usual, we begin by trying to induce D^λ and D^μ to get a lowerable pair.

1. Define $\mathfrak{a} = \mathfrak{f}_{2a+b-2d-1} \dots \mathfrak{f}_{a+b-2d+1}\mathfrak{f}_{a+b-2d}$. In cases 1–10 $(\mathfrak{a}(\lambda), \mathfrak{a}(\mu))$ is defined and lowerable, so that $\mathrm{adj}_{\lambda\mu} = 0$.

2. Now suppose $d - b \geqslant 2$ and define $\mathfrak{b} = \mathfrak{f}_{a+2b-2d-1} \dots \mathfrak{f}_{a+b-d+1}\mathfrak{f}_{a+b-d}$. We find that $(\mathfrak{b}(\lambda), \mathfrak{b}(\mu))$ is defined and lowerable in cases 11, 12, 13 (provided $d - b \geqslant 3$) and 14 (provided $d - b \geqslant 4$); so $\mathrm{adj}_{\lambda\mu} = 0$ in these cases.

3. Next, define $\mathfrak{c} = \mathfrak{f}_{a+2b-2d+1} \dots \mathfrak{f}_{a+b-d-1}\mathfrak{f}_{a+b-d}$. In cases 40–42 $(\mathfrak{c}(\lambda), \mathfrak{c}(\mu))$ is defined and lowerable, so that $\mathrm{adj}_{\lambda\mu} = 0$.

4. Finally, define $\mathfrak{d} = \mathfrak{f}_{2a+b-2d+1} \dots \mathfrak{f}_{a+b-d-1}\mathfrak{f}_{a+b-d}$. In cases 31–36 $(\mathfrak{d}(\lambda), \mathfrak{d}(\mu))$ is defined and lowerable.

We are left with just one partition, namely

$$\mu = \begin{cases} \langle b, b{+}1, b{+}2, b{+}3\rangle & (d-b \geqslant 4) \\ \langle b_2, b{+}1, b{+}2\rangle & (d-b = 3) \\ \langle b_2, b{+}1_2\rangle & (d-b = 2) \\ \langle b_4\rangle & (d-b = 1). \end{cases}$$

Using the Mullineux map we find that

$$\mu^\diamond = \begin{cases} \langle \check{b}, \check{b}{+}1, \check{b}{+}2, \check{b}{+}3\rangle & (b-a \geqslant 4) \\ \langle \check{b}, \check{b}{+}1, \check{b}{+}2, d\rangle & (b-a = 3) \\ \langle \check{b}, \check{b}{+}1, d, d{+}1\rangle & (b-a = 2,\ e-d \geqslant 2) \\ \langle \check{b}_2, \check{b}{+}1, d\rangle & (b-a = 2,\ e-d = 1), \end{cases}$$

where the partitions on the right are written with the $\langle 4^{d-b}, 5^{b-a}, 6^a, 5^{e-d}\rangle$ notation, and $\check{b} = d - b$. We also find

$$\lambda^\diamond = \langle \check{b}_{1^2}, (d{-}a)_{1^2}\rangle.$$

So $\mu^\diamond \not\trianglerighteq \lambda^\diamond$, which implies that $\mathrm{adj}_{\lambda\mu} = 0$ by Corollary 2.8 and Proposition 2.20.

8.3 The case $d - c > 0,\ e - d = 0$

In the case where $c < d = e$, we examine the conjugate block $B^\sharp$. This block has the $\langle 4^{c-b}, 5^{b-a}, 6^a, 4^{e-c}\rangle$ notation, and so by the results of Section 8.2 we know that James's Conjecture holds for $B^\sharp$. Hence it holds for B.

8.4 The case $d - c = 0,\ e - d = 0$

Now we consider the case where $c = d = e$.

1. Define $\mathfrak{a} = \mathfrak{f}_{2a+b-1} \dots \mathfrak{f}_{a+b+1}\mathfrak{f}_{a+b}$. In cases 1 and 2, and also case 3 if $e - b \geqslant 2$, $(\mathfrak{a}(\lambda), \mathfrak{a}(\mu))$ is defined and lowerable, so that $\mathrm{adj}_{\lambda\mu} = 0$.
2. If $e - b \geqslant 2$, define $\mathfrak{b} = \mathfrak{f}_{a+2b-1} \dots \mathfrak{f}_{a+b+1}\mathfrak{f}_{a+b}$. In cases 4, 6, 8, 9 (if $e - b \geqslant 3$), 11, 12, 13 (if $e - b \geqslant 3$) and 14 (if $e - b \geqslant 4$) $(\mathfrak{b}(\lambda), \mathfrak{b}(\mu))$ is defined and lowerable.

Assuming that μ does not fit into one of these cases, there are four or five possible possibilities for μ, which we label as follows (leaving μ^1 undefined if $e - b > 1$):

$$\mu^1 = \langle a_2, b_2\rangle \quad (e - b = 1);$$

$$\mu^2 = \begin{cases} \langle b, b{+}1, b{+}2, b{+}3\rangle & (e - b \geqslant 4) \\ \langle b_2, b{+}1, b{+}2\rangle & (e - b = 3) \\ \langle b_2, b{+}1_2\rangle & (e - b = 2) \\ \langle b_4\rangle & (e - b = 1); \end{cases}$$

$$\mu^3 = \begin{cases} \langle a, b, b{+}1, b{+}2\rangle & (e - b \geqslant 3) \\ \langle a, b_2, b{+}1\rangle & (e - b = 2) \\ \langle a, b_3\rangle & (e - b = 1); \end{cases}$$

$$\mu^4 = \begin{cases} \langle a_2, b, b{+}1\rangle & (e - b \geqslant 2) \\ \langle a_3, b\rangle & (e - b = 1); \end{cases}$$

$$\mu^5 = \begin{cases} \langle a, a{+}1, b, b{+}1\rangle & (e - b \geqslant 2) \\ \langle a, a{+}1, b_2\rangle & (e - b = 1). \end{cases}$$

We eliminate some of these cases by making our usual assumption that $\mu_1^\diamond > \lambda_1^\diamond$. We calculate

$$\lambda^\diamond = \langle \hat{b}_{1^2}, \hat{a}_{1^2}\rangle,$$

$$(\mu^1)^\diamond = \langle 0_2, 1_2\rangle,$$

$$(\mu^2)^\diamond = \begin{cases} \langle \hat{b}, \hat{b}{+}1, \hat{b}{+}2, \hat{b}{+}3\rangle & (b - a \geqslant 4) \\ \langle \hat{b}_2, \hat{b}{+}1, \hat{b}{+}2\rangle & (b - a = 3) \\ \langle \hat{b}_2, \hat{b}{+}1_2\rangle & (b - a = 2), \end{cases}$$

$$(\mu^3)^\diamond = \begin{cases} \langle \hat{b}, \hat{b}{+}1, \hat{b}{+}2, \hat{a}\rangle & (b - a \geqslant 3) \\ \langle \hat{b}, \hat{b}{+}1, \hat{a}_{1^2}\rangle & (b - a = 2), \end{cases}$$

where all the partitions on the right are written with the $\langle 4^{e-b}, 5^{b-a}, 6^a\rangle$ notation, and we put $\hat{a} = e - a$, $\hat{b} = e - b$. We see that $(\mu^i)_1^\diamond \leqslant \lambda_1^\diamond$ for $i = 1, 2, 3$, so we cannot have $\mu = \mu^1, \mu^2$ or μ^3.

We are left to consider the cases $\mu = \mu^4$ and $\mu = \mu^5$. To deal with these, we induce to the block with the $\langle 4^a, 5^{b-a}, 8, 9^{e-b-1}\rangle$ notation. We have

$$(\lambda, \mu^4, \mu^5) \sim (\bar{\lambda}, \bar{\mu}^4, \bar{\mu}^5),$$

where

$$\bar{\lambda} = \langle a_{2,1}, b \mid 4^a, 5^{b-a}, 8, 9^{e-b-1}\rangle,$$
$$\bar{\mu}^4 = \langle a_4 \mid 4^a, 5^{b-a}, 8, 9^{e-b-1}\rangle,$$
$$\bar{\mu}^5 = \langle a, a{+}1, b_2 \mid 4^a, 5^{b-a}, 8, 9^{e-b-1}\rangle.$$

We have $\bar{\mu}^4 \not\trianglerighteq \bar{\lambda}$, so $\mathrm{adj}_{\lambda\mu^4} = \mathrm{adj}_{\bar{\lambda}\bar{\mu}^4} = 0$. To show that $\mathrm{adj}_{\bar{\lambda}\bar{\mu}^5} = 0$, we use the Jantzen–Schaper formula to show that $[S^{\bar{\lambda}} : D^{\bar{\mu}^5}] = 1$, independently of the underlying characteristic. First we need the following lemma.

Lemma 8.1. *Define* $\bar{\mu}^5$ *as above, and*

$$\nu = \langle a{+}1_{2,1}, b \mid 4^a, 5^{b-a}, 8, 9^{e-b-1}\rangle.$$

Then $[S^\nu : D^{\bar{\mu}^5}] \leqslant 1$.

Proof. We have

$$\bar{\mu}^5 = (a+5b+2, (a+4b+2)^{e-b-1}, (a+3b+1)^{e-b-1}, (a+2b)^{e-b}, (a+b)^{e-b}, (2a+2)^2, (a+2)^{e-a-2}),$$
$$\nu = ((a+4b+1)^{e-b}, (a+3b+1)^{e-b-1}, (a+2b)^{e-b}, 2a+b+2, (a+b+1)^{e-b}, (a+1)^{e-a}, 1^{e-a-2}).$$

Hence

$$\bar{\mu}^5_1 + \dots + \bar{\mu}^5_{4(e-b)} = (4a+10b+4)(e-b) + a - 2b + 1 = \nu_1 + \dots + \nu_{4(e-b)}.$$

So we may apply Theorem 2.13, and we find that $[S^\nu : D^{\bar{\mu}}]$ is a product of decomposition numbers for blocks of weight 1 or 2, and so is at most 1 by Theorem 2.5. □

Now we may perform the calculation with the Jantzen–Schaper formula. The table of Jantzen–Schaper coefficients for partitions ν such that $\bar{\mu}^5 \trianglerighteq \nu \trianglerighteq \bar{\lambda}$ is as follows, and the Jantzen–Schaper formula together with Proposition 2.10 and Lemma 8.1 gives $[S^{\bar{\lambda}} : D^{\bar{\mu}^5}] = 1$, so that $\mathrm{adj}_{\lambda\mu^5} = \mathrm{adj}_{\bar{\lambda}\bar{\mu}^5} = 0$.

	$\langle a, a{+}1, b_2\rangle$	$\langle a{+}1_{1^2}, b_2\rangle$	$\langle a_{1^2}, b_2\rangle$	$\langle a, a{+}1_2, b\rangle$	$\langle a{+}1_{2,1}, b\rangle$	$\langle a_2, a{+}1, b\rangle$	$\langle a_{2,1}, b\rangle$	$[S^\nu : D^{\bar{\mu}^5}]$
$\langle a, a{+}1, b_2\rangle$	·	·	·	·	·	·	·	1
$\langle a{+}1_{1^2}, b_2\rangle$	1	·	·	·	·	·	·	1
$\langle a_{1^2}, b_2\rangle$	−1	1	·	·	·	·	·	0
$\langle a, a{+}1_2, b\rangle$	1	·	·	·	·	·	·	1
$\langle a{+}1_{2,1}, b\rangle$	0	1	·	1	·	·	·	1
$\langle a_2, a{+}1, b\rangle$	−1	·	·	1	·	·	·	0
$\langle a_{2,1}, b\rangle$	0	0	1	0	1	1	·	1

9 Blocks forming a [4 : 3]-pair

In this section, we complete the proof of Theorem 2.6 by proving the following result.

Proposition 9.1. *Suppose A and B are weight 4 blocks of $\mathcal{H}_{n-3}$ and $\mathcal{H}_n$ respectively, forming a $[4:3]$-pair. Suppose that there is no block other than A forming a $[4:\kappa]$-pair with B, for any κ. If James's Conjecture holds for A, then it holds for B.*

The conditions of the proposition imply that the core of B has the form $((3a-c-d+2e)^{b-a},(2a-d+e)^{c-a},a^{d-a})$ for some $0<a<b\leqslant c\leqslant d\leqslant e$. B can be displayed on an abacus with the $\langle 4^a,7^{b-a},6^{c-b},5^{d-c},4^{e-d}\rangle$ notation.

If λ and μ are e-regular partitions in B, then we have $\mathrm{adj}_{\lambda\mu}=\delta_{\lambda\mu}$ by Proposition 2.18 except when λ is the unique exceptional partition for the $[4:3]$-pair (A,B), namely $\lambda=\langle a_{1^4}\rangle$. But in this case we can check that λ always induces semi-simply to a Rouquier block; in fact we have

$$\lambda\sim\begin{cases}\langle a_{1^4}\mid 4,7,10,\dots,3e+1\rangle & (b=e)\\ \langle a_{1^3},a{+}e{-}b\mid 4,7,10,\dots,3e+1\rangle & (b<c=e)\\ \langle a_{1^2},a{+}e{-}c_{1^2}\mid 4,7,10,\dots,3e+1\rangle & (b=c<d=e)\\ \langle a,a{+}e{-}d_{1^3}\mid 4,7,10,\dots,3e+1\rangle & (b=d<e)\\ \langle a_{1^2},a{+}e{-}c,a{+}e{-}b\mid 4,7,10,\dots,3e+1\rangle & (b<c<d=e)\\ \langle a,a{+}e{-}d_{1^2},a{+}e{-}b\mid 4,7,10,\dots,3e+1\rangle & (b<c=d<e)\\ \langle a,a{+}e{-}d,a{+}e{-}c_{1^2}\mid 4,7,10,\dots,3e+1\rangle & (b=c<d<e)\\ \langle a,a{+}e{-}d,a{+}e{-}c,a{+}e{-}b\mid 4,7,10,\dots,3e+1\rangle & (b<c<d<e).\end{cases}$$

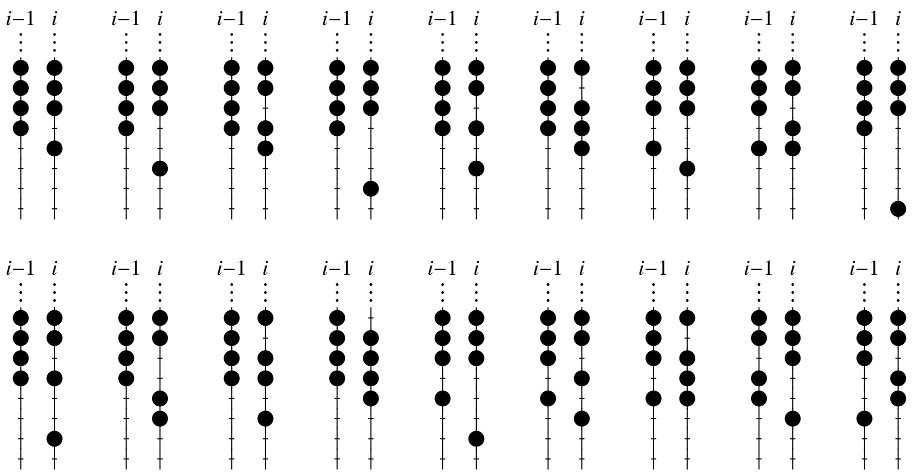
Hence $\mathrm{adj}_{\lambda\mu}=\delta_{\lambda\mu}$ by Proposition 2.22.

Given the discussion in §2.13 and the results of Sections 4–9, the proof of Theorem 2.6 is now complete.

A Induction and restriction of simple modules

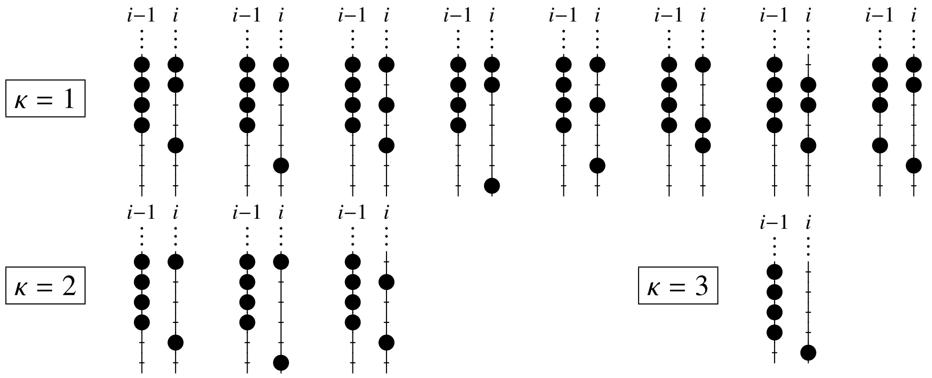
A.1 Induction and restriction to blocks of weight 3

(See §2.9 for an explanation.)



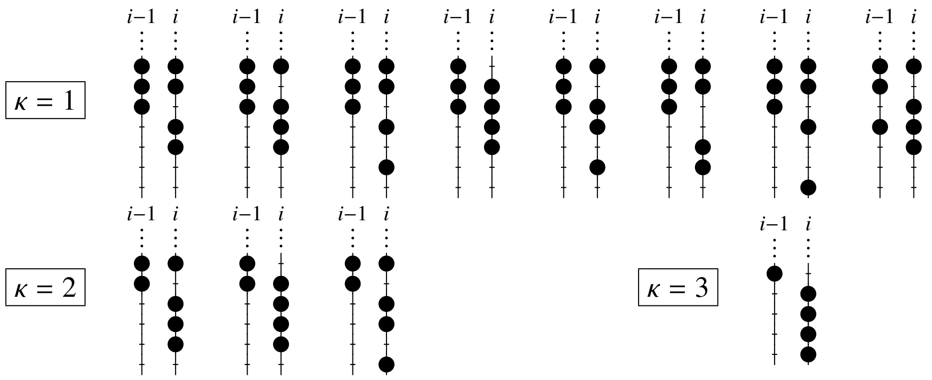
A.2 Restriction to blocks of weight less than 3, and exceptional partitions for $[4 : \kappa]$-pairs

(See §2.9 and §2.10 for an explanation.)



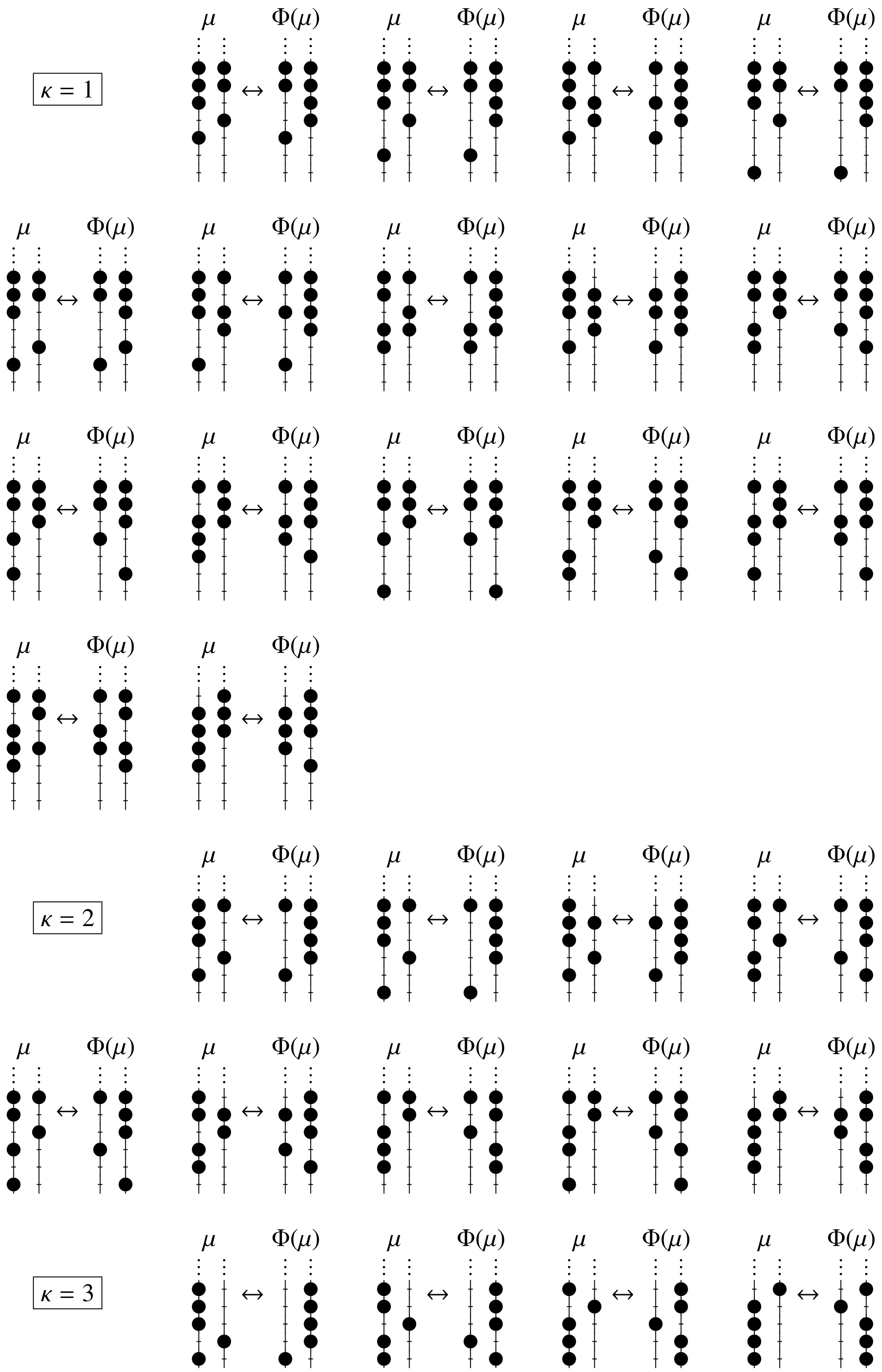
A.3 Induction to blocks of weight less than 3, and exceptional partitions for $[4 : \kappa]$-pairs

(See §2.9 and §2.10 for an explanation.)



A.4 Induction and restriction of non-exceptional simple modules in a $[4 : \kappa]$-pair

(See §2.10 for an explanation.)



$\kappa = 1$
$\kappa = 2$
$\kappa = 3$
μ ↔ $\Phi(\mu)$

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