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Irreducible Specht modules for Hecke algebras of type A

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Abstract

Let $\mathbb{F}$ be a field, n a non-negative integer, λ a partition of n and S^λ the corresponding Specht module for the Iwahori–Hecke algebra $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$. James and Mathas conjecture a necessary and sufficient condition on λ for S^λ to be irreducible. We prove the sufficiency of this condition in the case where $\mathbb{F}$ has infinite characteristic and also in the case where $q = 1$.

1 Introduction

In the representation theory of finite groups, it is useful to know which ordinary irreducible representations remain irreducible modulo a prime p. For the symmetric group $\mathfrak{S}_n$, this amounts to determining which Specht modules are irreducible over a field of characteristic p. The group algebra of $\mathfrak{S}_n$ is a special case (namely, the case $q = 1$) of the Iwahori–Hecke algebra $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$, and we may ask the more general question of which Specht modules for the latter algebra are irreducible modules. James and Mathas conjecture a necessary and sufficient condition in the case $q \neq -1$, and the purpose of this paper is to prove the sufficiency of this condition in the case $q = 1$ and in the case where $\mathbb{F}$ has infinite characteristic.

Given a field $\mathbb{F}$, let q be an invertible element of $\mathbb{F}$. For a positive integer n, define the *Iwahori–Hecke algebra* $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ to be associative $\mathbb{F}$-algebra with generators $T_1, \dots, T_{n-1}$ and relations

$$\begin{aligned} T_i^2 &= q + (q-1)T_i && (1 \leqslant i \leqslant n-1)\\ T_iT_j &= T_jT_i && (1 \leqslant i < j-1 \leqslant n-2)\\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && (1 \leqslant i \leqslant n-2). \end{aligned}$$

We write $\mathcal{H}_n$ for $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$; of course, if $q = 1$ then $\mathcal{H}_n$ is isomorphic to the group algebra $\mathbb{F}\mathfrak{S}_n$.

Let p be the characteristic of $\mathbb{F}$, and define

$$e = \inf\{d > 0 \mid 1 + q + \dots + q^{d-1} = 0\};$$

thus e equals p if $q = 1$, and e is the multiplicative order of q in $\mathbb{F}$ otherwise. For a positive integer h, define

$$\nu_{e,p}(h) = \begin{cases} 0 & (e = \infty \text{ or } e \nmid h)\\ (1 + \nu_p(\frac{h}{e})) & (\infty > e \mid h). \end{cases}$$

As usual, $\nu_p(h)$ denotes the largest power of p dividing h if p is finite, while $\nu_\infty(h)$ is always taken to be zero.

If e is infinite, then $\mathcal{H}_n$ is semi-simple [13, Corollary 3.44], so we are concerned with the case where e is finite. For each partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n, one defines a *Specht module* S^λ for $\mathcal{H}_n$; when $q = 1$, the Specht modules are p-modular reductions of the ordinary irreducible representations of $\mathfrak{S}_n$. In this paper we are interested in determining whether S^λ is irreducible as an $\mathcal{H}_n$-module.

Given a node (a, b) of the Young diagram $[\lambda]$ (i.e. a pair of positive integers such that $b \leqslant \lambda_a$), we define $h_\lambda(a, b)$ to be the (a, b)-hook length in $[\lambda]$, i.e.

$$h_\lambda(a, b) = \lambda_a - b + \lambda'_b - a + 1.$$

Then the conjecture made by James and Mathas may be stated as follows.

Conjecture 1.1. [13, Conjecture 5.47] *Suppose $e \neq 2$, and let λ be a partition of n. Then the Specht module S^λ for $\mathcal{H}_n$ is reducible if and only if the Young diagram $[\lambda]$ contains nodes (a, b), (a, y) and (x, b) such that*

$$\nu_{e,p}(h_\lambda(a, b)) > 0$$

and

$$\nu_{e,p}(h_\lambda(x, b)) \neq \nu_{e,p}(h_\lambda(a, b)) \neq \nu_{e,p}(h_\lambda(a, y)).$$

We shall refer to a partition not satisfying the condition of Conjecture 1.1 an *(e, p)-JM-partition*. Carter's Criterion [13, Proposition 5.40] says that Conjecture 1.1 holds in the case where λ is e-regular; combining this with [13, Exercise 3.14], we find that the conjecture also holds when λ is e-restricted. We shall refer to an e-regular (e, p)-JM-partition as an *(e, p)-Carter partition*, and to an e-restricted (e, p)-JM-partition as a *conjugate (e, p)-Carter partition*.

The representation theory of $\mathcal{H}_n$ is closely related to that of another $\mathbb{F}$-algebra, namely the *q-Schur algebra $S_q(d, n)$*. For the latter algebra one defines a *Weyl module* $\Delta(\lambda)$ for each partition λ of n, and one may ask when $\Delta(\lambda)$ is irreducible. This question is better understood than the corresponding question for Specht modules; this is due to the fact that there are simple $S_q(d, n)$-modules $L(\lambda)$ for all partitions λ, and $L(\lambda)$ occurs as the cosocle of $\Delta(\lambda)$. We quote the classification of irreducible Weyl modules here, since we shall need it later.

Proposition 1.2. [13, Proposition 5.39] *Let λ be a partition of n, and let $\Delta(\lambda)$ be the corresponding Weyl module for the q-Schur algebra $S_q(n, n)$. Then $\Delta(\lambda)$ is irreducible if and only if λ is a conjugate (e, p)-Carter partition.*

We comment briefly on the 'if' half of Conjecture 1.1. Building on the work of Lyle [11], the present author has proved this [6] in the case $q = 1$. This would generalise for arbitrary q given appropriate q-analogues of the Carter–Payne theorem [2, p. 425, Theorem] and the column removal theorem [7, Theorem 2.3] for homomorphisms between Specht modules. Lyle and Mathas [12] have submitted a proof of the latter.

The question of which Specht modules are irreducible in the case $e = 2$ seems to be a rather different one. In the case where λ is 2-regular, Carter's Criterion still applies, and this gives an answer for the case where λ is 2-restricted as well. But very little seems to be known about whether S^λ is reducible when λ is neither 2-regular nor 2-restricted, except in the symmetric group case $p = 2$, where James and Mathas [9] show that the only such λ for which S^λ is irreducible is (2^2).

We now indicate the layout of this paper. For the remainder of this introduction, we summarise the background details we shall need. In Section 2 we translate the combinatorial condition in Conjecture 1.1 to the context of the abacus. This enables us to show in Section 3 that the Specht module corresponding to a partition satisfying the *ir*reducibility criterion of Conjecture 1.1 may be induced to a Rouquier block in a nice way. Finally in Section 4 we prove the 'only if' part of Conjecture

1.1 for Rouquier blocks in the cases $q = 1$ and $p = \infty$ by examining the decomposition numbers for these blocks in these cases. The results of Sections 2 and 3 hold for arbitrary e and p, and it is only in Section 4 where we require that $e \neq 2$ and that either $e = p$ or $p = \infty$. Section 4 would generalise to all e, p (with $e \neq 2$) given an appropriate q-analogue of Turner's result [14] describing the decomposition matrix for a Rouquier block.

1.1 Representation theory of $\mathcal{H}_n$

Mathas's book [13] provides an excellent introduction to the representation theory of $\mathcal{H}_n$; we take most of our notation from there, although we use the Specht modules given by Dipper and James in [5]; the 'Specht module' S^λ defined in [13] is in fact the dual of the Specht module $S^{\lambda'}$ defined in [5].

For now, $\mathbb{F}$ and q are arbitrary, with e and p be as above; in Section 4 we shall exclude the case where $e = 2$. In the case where e is infinite, $\mathcal{H}_n$ is semi-simple, the Specht modules are irreducible, and $\{S^\lambda \mid \lambda \vdash n\}$ is a complete set of irreducible modules for $\mathcal{H}_n$. In the case where e is finite, the Specht modules are no longer necessarily irreducible. If λ is an e-regular partition (that is, does not have e equal non-zero parts), then S^λ has an irreducible cosocle D^λ, and the set $\{D^\lambda \mid \lambda \text{ an } e\text{-regular partition of } n\}$ is a complete set of irreducible modules for $\mathcal{H}_n$. The *decomposition matrix* for $\mathcal{H}_n$ records the composition multiplicities $d_{\lambda\mu} = [S^\lambda : D^\mu]$.

Throughout this paper we shall write λ' to denote the partition conjugate to λ, and $\varnothing$ for the partition of zero.

1.1.1 The abacus

Since all Specht modules are irreducible in the case where e is infinite, we assume from now on that e is finite; partitions are then conveniently represented on an abacus. Given a partition λ and an integer r greater than or equal to the number of non-zero parts of λ, we define the *beta-numbers*

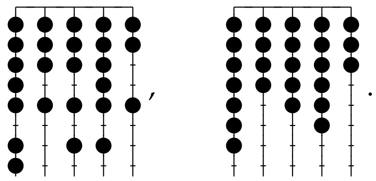
$$\beta_i = \lambda_i + r - i$$

for $i = 1, \dots, r$. Now we take an abacus with e vertical runners, numbered $0, \dots, e-1$ from left to right, and with the ith position on runner j (counting from the top) labelled with the integer $ej + i$. Given a set of beta-numbers for λ, we place a bead at position β_i on the abacus for each i, and we call the resulting configuration an abacus display for λ.

Given an abacus display for λ, let κ be the partition whose abacus display is obtained by moving all the beads as far up their runners as they will go. κ is called the *e-core* of λ, and Nakayama's Conjecture [13, Corollary 5.38] says that the Specht modules S^λ and S^μ lie in the same block of $\mathcal{H}_n$ if and only if λ and μ have the same e-core. If S^λ lies in a block B of $\mathcal{H}_n$, we abuse notation by saying that λ lies in B. κ is a partition of $n - ew$ for some w which we call the *e-weight* of λ; it is easy to see that two partitions lying in the same block have the same e-weight, and we say that the block has this e-weight as well. We shall talk of 'an abacus for the block B', meaning simply an abacus with fixed numbers of beads on the various runners, at unspecified positions on those runners. If an abacus for the block C has one bead fewer on runner i than the abacus for B and one bead more on runner j, then we shall refer to C as 'the block obtained by moving a bead from runner i to runner j'.

Given an abacus display for λ, we let $\lambda^{[i]}$ denote the number of beads on runner i. The e-core of λ, and hence the block in which S^λ lies, are determined by the integers $\lambda^{[0]}, \dots, \lambda^{[e-1]}$. We also write λ^i_j for the number of unoccupied positions above the jth lowest bead on runner i; then $\lambda(i) = (\lambda^i_1, \lambda^i_2, \dots)$ is a partition, and we refer to the sequence $(\lambda(0), \dots, \lambda(e-1))$ as the *e-quotient* of λ. Note that λ is determined by its e-core and e-quotient, and that the sum of all the parts of the e-quotient of λ equals the e-weight of λ.

Example. Suppose $e = 5$, and let $\lambda = (11, 10^2, 9, 4^5, 3, 1)$. Then the abacus displays for λ and its e-core on an abacus with 25 beads are



So the 5-core of λ is $(6, 5, 3, 2^2, 1)$, and the 5-quotient is $((1^2), (1), (2, 1), (1), (2))$.

Note that different choices of abacus display for a partition λ (that is, different choices of r) will give different values of $\lambda^{[i]}$ and different e-quotients. Specifically, if the display for λ on an abacus with r beads gives values

$$(\lambda^{[0]}, \ldots, \lambda^{[e-1]}) = (l_0, \ldots, l_{e-1})$$

and e-quotient $(\sigma_0, \ldots, \sigma_{e-1})$, then the display for λ on an abacus with $r + es + t$ beads (where $s \geqslant 0$ and $e > t \geqslant 0$) will give

$$(\lambda^{[0]}, \ldots, \lambda^{[e-1]}) = (l_{e-t} + s + 1, \ldots, l_{e-1} + s + 1, l_0 + s, \ldots, l_{e-t-1} + s)$$

and e-quotient $(\sigma_{e-t}, \ldots, \sigma_{e-1}, \sigma_0, \ldots, \sigma_{e-t-1})$.

Given an abacus display for a partition λ, it is easy to find an abacus display for λ': simply take a sufficiently large integer a, and use an abacus in which the number of beads on runner i is $a - \lambda^{[e-1-i]}$ for each i. Then λ' has a display on this abacus, with e-quotient $(\lambda(e-1)', \ldots, \lambda(0)')$. To express this another way, take an abacus display for λ, replace each bead with an empty space and vice versa, and then rotate the abacus through 180°.

1.1.2 The Branching Rule

$\mathcal{H}_{n-r}$ is a subalgebra of $\mathcal{H}_n$ in a natural way, and in fact $\mathcal{H}_n$ is free as an $\mathcal{H}_{n-r}$-module; accordingly, there are induction and restriction functors

$$\mathrm{Ind}^{\mathcal{H}_n} : \mathcal{H}_{n-r}\text{-mod} \longrightarrow \mathcal{H}_n\text{-mod}, \qquad \mathrm{Res}_{\mathcal{H}_{n-r}} : \mathcal{H}_n\text{-mod} \longrightarrow \mathcal{H}_{n-r}\text{-mod}.$$

If B is a block of $\mathcal{H}_{n-r}$, then we write $\mathrm{Res}_B M$ to denote the projection of $\mathrm{Res}_{\mathcal{H}_{n-r}} M$ onto B; similarly, we write $\mathrm{Ind}^B M$.

The effect of these functors on Specht modules is well understood. We cite the relevant result in the case $r = 1$; the general case may be obtained by applying this recursively. Suppose that B is a block of $\mathcal{H}_n$, and take an abacus for B. Given $0 \leqslant i \leqslant e - 1$, let C be the block of $\mathcal{H}_{n-1}$ whose abacus is obtained from that of B by moving a bead from runner i to runner $i - 1$ (where we reduce modulo e). Given a partition λ in B, say that a bead in the abacus display for λ is *i-removable* if it lies on runner i (at position $i + ae$, say) and there is no bead at position $i + ae - 1$. Let $\lambda^1, \ldots, \lambda^s$ be the distinct partitions that may be obtained from λ by moving an i-removable bead at position $i + ea$ to position $i + ea - 1$ for some a. Similarly, given a partition μ in C, say that a bead is *i-addable* if it lies on runner $i - 1$ (at position $i + ae - 1$, say) and there is no bead at position $i + ae$. Let $\mu^{(1)}, \ldots, \mu^{(t)}$ be the distinct partitions that may be obtained by moving an i-addable bead from position $i + ea - 1$ to position $i + ea$.

Theorem 1.3. The branching rule [13, Corollary 6.2] *Let λ and μ be as above. Then:*

1. *the module* $\mathrm{Res}_C S^\lambda$ *has a filtration in which the factors are* $S^{\lambda^1}, \ldots, S^{\lambda^s}$*;*
2. *the module* $\mathrm{Ind}^B S^\mu$ *has a filtration in which the factors are* $S^{\mu^{(1)}}, \ldots, S^{\mu^{(t)}}$.

1.1.3 Rouquier blocks

We say that a block B of $\mathcal{H}_n$ is *Rouquier* if it has an abacus display in which the number of beads on runner i exceeds the number of beads on runner $i-1$ by at least $w-1$ for $i=1,\dots,e-1$, where w is the e-weight of B. Rouquier blocks are well understood; in particular, their decomposition numbers are known in the case $p=\infty$ and the case $q=1$.

Remark. Our definition of Rouquier blocks is slightly unusual. Some authors (e.g. Chuang and Tan in [4]) define a Rouquier block to have exactly $w-1$ more beads on runner i than runner $i-1$, so that there is exactly one Rouquier block for each weight w. Others define a Rouquier block to be any block with an abacus such that, for $0 \leqslant i < j \leqslant e-1$, either there are at least $w-1$ more beads on runner j than runner i or there are at least w more beads on runner i than runner j; with this definition, the Rouquier blocks from a class under the Scopes equivalence. The blocks we have specified all lie in this equivalence class, and our definition is more convenient for our purposes.

Given a Rouquier block B, fix an abacus for B as above, and let λ be a partition in B, with e-quotient $(\lambda(0),\dots,\lambda(e-1))$. Note that λ is e-regular if and only if $\lambda(0)=\varnothing$.

Given partitions α,β,γ, let $c^{\alpha}_{\beta\gamma}$ be the corresponding Littlewood–Richardson coefficient, which we interpret as zero if $|\alpha| \neq |\beta|+|\gamma|$.

The following is due to Chuang and Tan [3, Theorem 1.1], and independently Leclerc and Miyachi [10, Corollary 10]; in the case $e=2$, the result was first proved by James and Mathas [8, Theorem 2.5].

Theorem 1.4. *Suppose B is a Rouquier block of $\mathcal{H}_n$ with $p=\infty$, and that λ and μ are partitions in B with e-quotients $(\lambda(0),\lambda(1),\dots,\lambda(e-1))$ and $(\varnothing,\mu(1),\dots,\mu(e-1))$ respectively. Then*

$$[S^\lambda : D^\mu] = \sum \prod_{i=0}^{e-1} c^{\lambda(i)}_{\tau(i)\sigma(i+1)} \prod_{j=1}^{e-1} c^{\mu(j)}_{\sigma(j)'\tau(j)'}$$

where the sum is over all choices of partitions $\sigma(1),\dots,\sigma(e-1),\tau(1),\dots,\tau(e-1)$, and we interpret $\tau(0)$ and $\sigma(e)$ as $\varnothing$.

For the case where $q=1$ and $p>w$, the decomposition numbers are known to be the same as those described by Theorem 1.4 [4]. In the case $p \leqslant w$ (where the defect group of the block is non-abelian), Turner has found the decomposition matrix; this is found by post-multiplying the decomposition matrix by an 'adjustment matrix' which is itself the decomposition matrix for a certain tensor product of Schur algebras.

Theorem 1.5. [14, Theorem 29] *Suppose $q=1$, and suppose B is a Rouquier block of $\mathcal{H}_n=\mathbb{F}\mathfrak{S}_n$. If λ and μ are partitions in B with e-quotients $(\lambda(0),\lambda(1),\dots,\lambda(e-1))$ and $(\varnothing,\mu(1),\dots,\mu(e-1))$ respectively, define*

$$\delta_{\lambda\mu} = \sum \prod_{i=0}^{e-1} c^{\lambda(i)}_{\tau(i)\sigma(i+1)} \prod_{j=1}^{e-1} c^{\mu(j)}_{\sigma(j)'\tau(j)'}$$

as in Theorem 1.4. If ν is an e-regular partition in B with e-quotient $(\varnothing,\nu(1),\dots,\nu(e-1))$, define

$$\epsilon_{\mu\nu} = \begin{cases} \prod_{i=1}^{e-1} [\Delta(\mu(i)') : L(\nu(i)')] & (\text{if } |\mu(i)| = |\nu(i)| \text{ for all } i) \\ 0 & (\text{otherwise}), \end{cases}$$

where $\Delta(\mu(i)')$ and $L(\nu(i)')$ are modules for the Schur algebra $S(n,n)$ over $\mathbb{F}$. Then

$$[S^\lambda : D^\nu] = \sum_{\mu} \delta_{\lambda\mu}\epsilon_{\mu\nu},$$

summing over all e-regular partitions μ in B.

2 JM-partitions on the abacus

Given an (e,p)-JM-partition, we examine its abacus display; this turns out to be easy to describe. Let λ be a partition, and take an abacus display for λ.

Proposition 2.1. *The following are equivalent.*

1. *λ is an (e,p)-JM-partition.*
2. *There exist some i and j such that:*
   (a) *$\lambda(k) = \varnothing$ whenever $i \neq k \neq j$;*
   (b) *if position $i + ea$ on runner i is unoccupied, then any position $b > i + ea$* not *on runner i is unoccupied;*
   (c) *if position $j + ec$ on runner j is occupied, then any position $d < j + ec$* not *on runner j is occupied;*
   (d) *$\lambda(i)$ is a (p,p)-Carter partition;*
   (e) *$\lambda(j)$ is a conjugate (p,p)-Carter partition.*

Proof. We begin by proving that (1) implies (2). Suppose λ is an (e,p)-JM-partition. Suppose that there exist i,a,c such that $c > a$, position $i + ea$ is unoccupied and position $i + ec$ is occupied. Then we claim that either

1. every position $b > i + ea$ not on runner i is unoccupied, or
2. every position $d < i + ec$ not on runner i is occupied.

Suppose that the bead at position $i + ec$ corresponds to the beta-number β_x, while the empty space at position $i + ea$ lies between the beads corresponding to the beta-numbers β_y and β_{y+1}. This means that if we set

$$z = y - x - e(a - c) + \lambda_x + 1,$$

then $\lambda_{y+1} < z \leqslant \lambda_y$ and the (x,z)-hook length in $[\lambda]$ is $e(c-a)$; in particular, it is divisible by e. Since λ is an (e,p)-JM-partition, either $h_\lambda(w,z)$ is divisible by e for $w = 1,2,\dots,y$, or $h_\lambda(x,w)$ is divisible by e for $w = 1,2,\dots,\lambda_x$. We shall assume that former; the latter case is dealt with by replacing λ with λ'.

Now the hook length $h_\lambda(i,z)$ equals $\lambda_i - z + y - i + 1$, which equals $\beta_i - r + y - z + 1$, and so the fact that the hook lengths $h_\lambda(1,z),\dots,h_\lambda(y,z)$ are all congruent modulo e implies that the beta-numbers $\beta_1,\dots,\beta_y$ are all congruent modulo e. So the corresponding beads (which are precisely those beads at positions greater than $i+ea$) all lie on the same runner, which must be runner i. This proves the claim.

Clearly there can be at most one value of i such that possibility (1) above occurs, and at most one value i such that possibility (2) occurs. So we have proved (2a)–(2c), and it remains to prove the claims concerning $\lambda(i)$ and $\lambda(j)$. If $p = \infty$, then there is nothing to prove, since any partition is an (∞,∞)-Carter partition; so assume that p is finite. We need only address $\lambda(i)$; $\lambda(j)$ may be dealt with by replacing λ with its conjugate.

Write $\tau = \lambda(i)$, and suppose for a contradiction that τ is not a (p,p)-Carter partition. This means that there are nodes (a,c) and (b,c) of $[\tau]$ such that

$$\nu_p(h_\tau(a,c)) \neq \nu_p(h_\tau(b,c)),$$

i.e.

$$\nu_p(\tau_a - c + \tau'_c - a + 1) \neq \nu_p(\tau_b - c + \tau'_c - b + 1).$$

Suppose τ has l non-zero parts, and let N be the number of unoccupied spaces less than the position occupied by the bead corresponding to β_l. Then

$$\begin{aligned}\lambda_l &= N,\\ \lambda_{l-1} &= N + e(\tau_{l-1} - \tau_l) + e - 1,\\ \lambda_{l-2} &= N + e(\tau_{l-2} - \tau_l) + 2(e-1),\\ &\vdots\\ \lambda_1 &= N + e(\tau_1 - \tau_l) + (l-1)(e-1).\end{aligned}$$

We write

$$d = N - e\tau_l + (e-1)(l - \tau'_c - 1) + ec,$$

and we claim that

$$\nu_{e,p}(h_\lambda(a,d)) \neq \nu_{e,p}(h_\lambda(b,d)).$$

First of all we need to know that $\lambda'_d = \tau'_c$; this follows easily from the definition of d. Hence we have

$$\begin{aligned}h_\lambda(a,d) &= \lambda_a - d + \tau'_c - a + 1\\ &= e(\tau_a - c + \tau'_c - a + 1)\\ &= e(h_\tau(a,c))\end{aligned}$$

and similarly

$$h_\lambda(b,d) = e(h_\tau(b,c)).$$

And so

$$\nu_{e,p}(h_\lambda(a,d)) = 1 + \nu_p(h_\tau(a,c)) \neq 1 + \nu_p(h_\tau(b,c)) = \nu_{e,p}(h_\lambda(b,d)).$$

This completes the proof that (1) implies (2).

For the other direction, suppose that λ has an abacus configuration as described in (2), and suppose that $h_\lambda(a,c)$ is divisible by e, say $h_\lambda(a,c) = es$. This means that there is an unoccupied space exactly s spaces above the bead corresponding to the beta-number β_a on the same runner. Hence this bead must lie either on runner i or runner j. We shall suppose that it lies on runner i (the case where it lies on runner j may be addressed by replacing λ with its conjugate). We claim that, for $b = 1, \dots, \lambda'_c$ we have

$$\nu_{e,p}(h_\lambda(b,c)) = \nu_{e,p}(h_\lambda(a,c)).$$

Write $d = \lambda'_c$. By condition (2b), we find that the beads corresponding to $\beta_1, \dots, \beta_d$ all lie on runner i. So, if we let M be the number of unoccupied spaces less than β_d on the abacus, then we have

$$\begin{aligned}\lambda_d &= M,\\ \lambda_{d-1} &= M + e(\tau_{d-1} - \tau_d) + e - 1,\\ &\vdots\\ \lambda_1 &= M + e(\tau_1 - \tau_d) + (d-1)(e-1).\end{aligned}$$

Put $y = \tau_a - s + d - a + 1$; then we claim that, for $x = 1, \dots, d$,

$$h_\lambda(x,c) = eh_\tau(x,y);$$

this will then be sufficient, since τ is a (p,p)-Carter partition, so we have

$$\nu_{e,p}(h_\lambda(x,c)) = 1 + \nu_p(h_\tau(x,y)) = 1 + \nu_p(h_\tau(a,y)) = \nu_{e,p}(h_\lambda(a,c)).$$

First we claim that $\tau'_y = d$; this follows easily from the fact that $\lambda'_c = d$. Verifying the above equality is then a formality. □

3 Induction to Rouquier blocks

In this section we show that the Specht module corresponding to a JM-partition induces 'nicely' to some Rouquier block.

Suppose $r > 0$ and that an abacus for a block B of $\mathcal{H}_n$ has r more beads on runner $i-1$ than on runner i (or, in the case $i = 0$, $r-1$ more beads on runner $e-1$ than runner i). Thus if λ is any partition in B then

$$\lambda^{[i]} = \begin{cases} \lambda^{[i-1]} - r & (i \geqslant 1) \\ \lambda^{[e-1]} - r + 1 & (i = 0). \end{cases}$$

Let C be the block of $\mathcal{H}_{n+r}$ whose abacus is obtained from that for B by moving r beads from runner $i-1$ to runner i. That is, if λ is any partition in B and μ any partition in C then

$$\mu^{[j]} = \begin{cases} \lambda^{[j]} + r & (j = i) \\ \lambda^{[j]} - r & (j \equiv i-1 \pmod e) \\ \lambda^{[j]} & (\text{otherwise}). \end{cases}$$

We say that B and C are *adjacent*; it is clear that adjacency is independent of the abacus display chosen for B. The following easy lemma will be crucial.

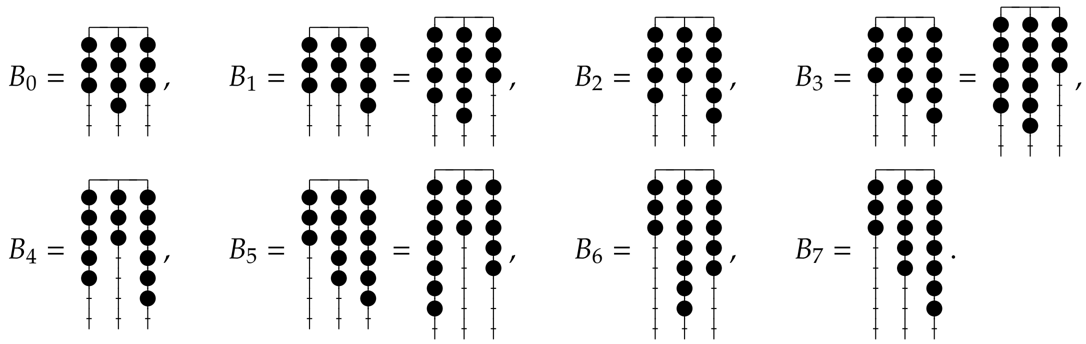
Lemma 3.1. *Let B be a block of $\mathcal{H}_n$. Then there exist $n = n_0 < n_1 < \dots < n_s$ and blocks $B = B_0, B_1, \dots, B_s$ of $\mathcal{H}_{n_0}, \mathcal{H}_{n_1}, \dots, \mathcal{H}_{n_s}$ respectively such that B_{i-1} and B_i are adjacent for $i = 1, \dots, s$ and B_s is a Rouquier block.*

Proof. Choose an abacus display for B and suppose that for some $1 \leqslant i \leqslant e-1$ there are more beads on runner $i-1$ than on runner i. We may swap runners $i-1$ and i to reach the abacus display of a block to which B is adjacent. We can repeat this procedure until the numbers of beads on the runners of the abacus display increase from left to right; suppose the number of beads on runner j is now b_j for each j. If $b_i - b_{i-1}$ is strictly less than $w-1$ for some i, then take a display for the same block but with $e-i$ more beads; the number of beads on runner j is now $b_{j+i}+1$ for $j = 0, \dots, e-i-1$, and b_{j-e+i} for $j = e-i, \dots, e-1$. Now we repeat the procedure at the start of this proof, and successively pass to adjacent blocks so that the numbers of beads on the runners increase from left to right. The number of beads of runner j of the abacus for the resulting block will then be b_j if $j \leqslant i-1$, or $b_j + 1$ if $j \geqslant i$. We may repeat this process until the number of beads on runner i exceeds the number of beads on runner $i-1$ by at least $w-1$ for each i. □

Example. Suppose $e = 3$, and that B is the block of weight 3 with 3-core (1). An abacus display for this block is

.

We construct a sequence $B = B_0, \dots, B_7$ as follows:



Now we look at the induction between adjacent blocks of Specht modules corresponding to JM-partitions. For modules M, N, we write $M \sim mN$ to indicate that M has a filtration with m factors all isomorphic to N.

Lemma 3.2. *Suppose that B and C are blocks of $\mathcal{H}_n$ and $\mathcal{H}_{n+r}$ respectively and are adjacent. If λ is an (e, p)-JM-partition lying in B, then there is an (e, p)-JM-partition μ in C such that*

$$\operatorname{Ind}^C S^\lambda \sim r!S^\mu, \qquad \operatorname{Res}_B S^\mu \sim r!S^\lambda.$$

Proof. Suppose that C is obtained from B by moving r beads from runner $k-1$ to runner k. Choose integers i and j as in Proposition 2.1(2). We claim that λ has no k-removable beads, which will guarantee that it has exactly r k-addable beads. Since there are more beads on runner $k-1$ than on runner k, we cannot have $k = i$ or $k - 1 = j$. Now suppose that there is a k-removable bead at position $k + ea$. If there is an unoccupied space at position $k + eb$ for some $b < a$, then we have $\lambda(k) \neq \varnothing$, so that $k = j$. But then by (2c) of Proposition 2.1 there is a bead at position $k + ea - 1$; contradiction. If there is a bead at position $k + ec - 1$ for some $c > a$, then $\lambda(k-1) \neq \varnothing$, so $k - 1 = i$. But then by (2b) of Proposition 2.1 there is no bead at position $k+ea$; contradiction. If there is neither an unoccupied space on runner k above $k + ea$ nor a bead on runner $k - 1$ below $k + ea - 1$, then $\lambda^{[k]} > \lambda^{[k-1]}$; contradiction.

So λ has r k-addable beads and no k-removable beads; let μ be the partition obtained by moving each of the k-addable beads one place to its right. Then μ has r k-removable beads and no k-addable beads, and so by Theorem 1.3 we have

$$\operatorname{Ind}^C S^\lambda \sim r!S^\mu, \qquad \operatorname{Res}_B S^\mu \sim r!S^\lambda.$$

Propoerties (2a)–(2e) in Proposition 2.1 may easily be checked for μ, and the proof is complete. □

Lemma 3.3. *Suppose B and C are blocks of $\mathcal{H}_n$ and $\mathcal{H}_{n+r}$ respectively, and M and N are modules such that*

$$\operatorname{Ind}^C M \sim mN, \qquad \operatorname{Res}_B N \sim mM.$$

Then M is irreducible if and only if N is.

Proof. Suppose that S is a composition factor of $\operatorname{cosoc}(N)$. Then, by Frobenius Reciprocity [1, Proposition 3.3.1],

$$\operatorname{Hom}_{\mathcal{H}_n}(M, \operatorname{Res}_B S) \cong \operatorname{Hom}_{\mathcal{H}_{n+r}}(\operatorname{Ind}^C M, S) \neq 0;$$

in particular, $\operatorname{Res}_B S \neq 0$. Similarly any composition factor T of $\operatorname{soc}(N)$ has $\operatorname{Res}_B T \neq 0$. If N is reducible, then the socle and cosocle of N between them contain at least two composition factors, and so $\operatorname{Res}_B N \sim mM$ has at least $2m$ composition factors, by the exactness of Res_B. So M is reducible. Similarly, M reducible implies N reducible. □

Corollary 3.4. *The 'only if' part of Conjecture 1.1 holds if and only if it holds for Rouquier blocks.*

Proof. Suppose λ is an (e,p)-JM-partition lying in a block B, and let $B_1,\dots,B_s$ be as in Lemma 3.1. By Lemma 3.2, there are (e,p)-JM-partitions $\lambda = \lambda^0, \lambda^1, \dots, \lambda^s$ lying in blocks $B_1,\dots,B_s$ respectively, such that

$$\mathrm{Ind}^{B_i} S^{\lambda^{i-1}} \sim r_i! S^{\lambda^i}, \qquad \mathrm{Res}_{B_{i-1}} S^{\lambda^i} \sim r_i! S^{\lambda^{i-1}}$$

for $i = 1,\dots,s$, where $r_1,\dots,r_s$ are some positive integers. By Lemma 3.3, S^λ is irreducible if and only if S^{λ^s} is. □

4 Conjecture 1.1 holds for Rouquier blocks

To complete the proof of the 'only if' part of Conjecture 1.1 in the case $p = \infty$ and the case $q = 1$, we use Theorems 1.4 and 1.5. Let B be a Rouquier block of $\mathcal{H}_n$, with an abacus as described in Section 1.1.3.

Lemma 4.1. *Let λ be a partition in B, with e-quotient $(\lambda(0),\dots,\lambda(e-1))$. Then λ is an (e,p)-JM-partition if and only if $\lambda(0)$ is a conjugate (p,p)-Carter partition, $\lambda(e-1)$ is a (p,p)-Carter partition and $\lambda(i) = \varnothing$ for $0 < i < e-1$.*

Proof. This is immediate from Proposition 2.1. □

Proposition 4.2. *Suppose that $p = \infty$, $e > 2$ and λ is an (e,∞)-JM-partition lying in B. Then S^λ is irreducible.*

Proof. We need to show that there is an e-regular partition ν in B such that $[S^\lambda : D^\mu]$ is 1 when $\mu = \nu$ and 0 otherwise. Now

$$[S^\lambda : D^\mu] = \sum \prod_{i=0}^{e-1} c^{\lambda(i)}_{\tau(i)\sigma(i+1)} \prod_{j=1}^{e-1} c^{\mu(j)}_{\sigma(j)'\tau(j)};$$

if this is non-zero, then, since $\lambda(1) = \dots = \lambda(e-2) = \varnothing$, we must have

$$\tau(1) = \sigma(2) = \tau(2) = \sigma(3) = \dots = \tau(e-2) = \sigma(e-1) = \varnothing.$$

Since $e > 2$, we must then have

$$\mu(1) = \sigma(1)' = \lambda(0)', \qquad \mu(e-1) = \tau(e-1) = \lambda(e-1)$$

and $\mu(i) = \varnothing$ for $1 < i < e-1$. This defines μ uniquely, and we get $[S^\lambda : D^\mu] = 1$ in this case. □

Proposition 4.3. *Suppose that $q = 1$, $e > 2$ and λ is an (e,p)-JM-partition lying in B. Then S^λ is irreducible.*

Proof. For an e-regular partition ν, we have

$$[S^\lambda : D^\nu] = \sum_\mu \delta_{\lambda\mu}\epsilon_{\mu\nu},$$

as in Theorem 1.5. By the argument in the proof of Proposition 4.2, there is a unique e-regular partition μ such that $\delta_{\lambda\mu} > 0$, namely the partition with e-quotient $(\varnothing, \lambda(0)', \varnothing, \dots, \varnothing, \lambda(e-1))$, and for this partition μ we have $\delta_{\lambda\mu} = 1$. We must show that there is a unique e-regular partition ν such that $\epsilon_{\mu\nu} > 0$, and that $\epsilon_{\mu\nu} = 1$ in this case.

We have

$$\epsilon_{\mu\nu} = \begin{cases} \prod_{i=1}^{e-1}[\Delta(\mu(i)') : L(\nu(i)')] & (\text{if } |\mu(i)| = |\nu(i)| \text{ for all } i) \\ 0 & (\text{otherwise}), \end{cases}$$

so for $\epsilon_{\mu\nu} > 0$ we require $|\nu(i)| = |\mu(i)|$ for all i. If this holds, then

$$\epsilon_{\mu\nu} = [\Delta(\nu(1)') : L(\mu(1)')][\Delta(\nu(e-1)') : L(\mu(e-1)')].$$

Now by Lemma 4.1 $\mu(1)' = \lambda(0)$ and $\mu(e-1)' = \lambda(e-1)'$ are conjugate (p,p)-Carter partitions, and so by Proposition 1.2 the product of these decomposition numbers is 1 if $\nu(1) = \mu(1)$ and $\nu(e-1) = \mu(e-1)$, and 0 otherwise. The proof is complete. □

References

[1] D. Benson, *Representations and cohomology I*, Cambridge Studies in Advanced Mathematics **30**, Cambridge University Press, Cambridge, 1998.

[2] R. Carter & M. Payne, 'On homomorphisms between Weyl modules and Specht modules', *Math. Proc. Cambridge Philos. Soc.* **87** (1980), 419–425.

[3] J. Chuang & K. M. Tan, 'Some canonical basis vectors in the basic $U_q(\widehat{\mathfrak{sl}}_n)$-module', *J. Algebra* **248** (2002), 765–779.

[4] J. Chuang & K. M. Tan, 'Filtrations in Rouquier blocks of symmetric groups and Schur algebras', *Proc. London Math. Soc. (3)* **86** (2003), 685–706.

[5] R. Dipper & G. James, 'Representations of Hecke algebras of general linear groups', *Proc. London Math. Soc. (3)* **52** (1986), 20–52.

[6] M. Fayers, 'Reducible Specht modules', *J. Algebra* **280** (2004) 500–504.

[7] M. Fayers & S. Lyle, 'Row and column removal theorems for homomorphisms between Specht modules', *J. Pure Appl. Algebra* **185** (2003), 147–164.

[8] G. James & A. Mathas, 'Hecke algebras of type A with $q = -1$', *J. Algebra* **184** (1996), 102–158.

[9] G. James & A. Mathas, 'The irreducible Specht modules in characteristic 2', *Bull. London Math. Soc.* **31** (1999), 457–462.

[10] B. Leclerc & H. Miyachi, 'Some closed formulas for canonical bases of Fock spaces', *Represent. Theory* **6** (2002), 290–312.

[11] S. Lyle, 'Some reducible Specht modules', *J. Algebra* **269** (2003), 536–543.

[12] S. Lyle & A. Mathas, 'Row and column removal theorems for homomorphisms of Specht modules and Weyl modules', *J. Algebraic Combin.* **22** (2005) 151–179.

[13] A. Mathas, 'Iwahori–Hecke algebras and Schur algebras of the symmetric group', *University lecture series* **15**, American Mathematical Society, Providence, RI, 1999.

[14] W. Turner, 'Rock blocks of symmetric groups: a global–local theorem', preprint.