A theorem concerning homomorphisms between Specht modules*

Matthew Fayers
Magdalene College, Cambridge, CB3 0AG, U.K.

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Abstract

Given partitions $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_r)$ of n with $\lambda_1, \mu_1 \leqslant s$, define $\check{\lambda} = (s - \lambda_r, \dots, s - \lambda_1)$, $\check{\mu} = (s - \mu_r, \dots, s - \mu_1)$. We prove that $\dim_{\mathbb{F}} \operatorname{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, S^\mu) = \dim_{\mathbb{F}} \operatorname{Hom}_{\mathbb{F}\mathfrak{S}_{rs-n}}(S^{\check{\lambda}}, S^{\check{\mu}})$, where $\mathbb{F}$ is any field of odd characteristic, $\mathfrak{S}_n$ is the symmetric group on n letters and S^λ is the Specht module for $\mathbb{F}\mathfrak{S}_n$. Our proof remains within the context of representations of $\mathfrak{S}_n$, via the combinatorics of Young tableaux.

1 Introduction

Throughout this paper we let n and r be fixed non-negative integers. Let $\mathfrak{S}_n$ denote the symmetric group on n letters. For any partition λ of n, one defines a *Specht module* S^λ over any field $\mathbb{F}$. If $\mathbb{F}$ has infinite characteristic, then the Specht modules are irreducible and afford all irreducible representations of $\mathbb{F}\mathfrak{S}_n$ as λ varies over the set of partitions of n. If the characteristic of $\mathbb{F}$ is finite, then the Specht modules are no longer irreducible in general; if λ is p-regular (that is, λ does not have p equal positive parts) and $\operatorname{char}(\mathbb{F}) = p$, then $S^\lambda_{\mathbb{F}}$ has an irreducible cosocle $D^\lambda_{\mathbb{F}}$, and every irreducible $\mathbb{F}\mathfrak{S}_n$-module arises in this way. The structures of the Specht modules in prime characteristic are of great interest, and in particular one would like to determine the decomposition numbers $[S^\lambda : D^\mu]$ and the spaces $\operatorname{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, S^\mu)$ of $\mathbb{F}\mathfrak{S}_n$-homomorphisms between Specht modules. These seem to be problems of similar difficulty, and neither of them is close to being solved. Many more results are known about the decomposition numbers than about Hom-spaces, which is in some sense surprising, since in principle we have an algorithm (see [6, p. 102]) for computing $\operatorname{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, S^\mu)$.

Particularly useful results concerning decomposition numbers are James's row and column removal theorems [7], and Donkin's generalisations of these [3]. These have been known for some time, but only recently have the analogous results for homomorphisms between Specht modules been proved. Specifically, the author and Lyle proved in [4] that if λ and μ have the same first part and we define $\bar{\lambda} = (\lambda_2, \lambda_3, \dots)$ and $\bar{\mu} = (\mu_2, \mu_3, \dots)$, then

$$\dim_{\mathbb{F}} \operatorname{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, S^\mu) = \dim_{\mathbb{F}} \operatorname{Hom}_{\mathbb{F}\mathfrak{S}_{n-\lambda_1}}(S^{\bar{\lambda}}, S^{\bar{\mu}}),$$

provided the characteristic of $\mathbb{F}$ is not 2. They proved a similar result for column removal, and a generalisation (analogous to Donkin's result). Another result which demonstrates the existence of homomorphisms between Specht modules in certain cases is the Carter–Payne theorem [2] (and slight generalisations due to the author and Martin [5]). But, apart from a few special cases, this seems to be all that

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is known about homomorphisms between Specht modules. In this paper we prove the following new result, which provides a new proof of the row and column removal results.

Theorem 1.1. *Suppose λ and μ are partitions of n with at most r parts and with $\lambda_1, \mu_1 \leqslant s$, and define*

$$\breve{\lambda} = (s-\lambda_r, \dots, s-\lambda_1), \qquad \breve{\mu} = (s-\mu_r, \dots, s-\mu_1).$$

Then, if $\mathbb{F}$ is any field whose characteristic is not 2, we have

$$\dim_{\mathbb{F}} \operatorname{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, S^\mu) = \dim_{\mathbb{F}} \operatorname{Hom}_{\mathbb{F}\mathfrak{S}_{rs-n}}(S^{\breve{\lambda}}, S^{\breve{\mu}}).$$

In fact, this result is quite easy to prove if we translate it to the context of representations of general linear groups; we outline this argument below. The object of the present paper is to provide an elementary proof entirely within the context of the symmetric group. As with the results in [4], we develop the combinatorics relating to semi-standard homomorphisms; however, the proof of the present result is slightly more subtle than for the results in [4], and in particular we are not able to provide an explicit bijection in terms of semi-standard homomorphisms.

Theorem 1.1 is false in characteristic 2: if we take $\lambda = (1,1)$, $\mu = (2,0)$, $r = 2$ and $s = 3$, then we have $\breve{\lambda} = (2,2)$, $\breve{\mu} = (3,1)$. There is a non-zero homomorphism from $S^{(1,1)}$ to $S^{(2,0)}$ over the field of two elements, but no such from $S^{(2,2)}$ to $S^{(3,1)}$.

A corresponding result is true (even in characteristic 2) for decomposition numbers, namely $[S^\lambda : D^\mu] = [S^{\breve{\lambda}} : D^{\breve{\mu}}]$ if μ and $\breve{\mu}$ are both (char($\mathbb{F}$))-regular. The author has not been able to find this result in print, and is grateful to Gordon James for communicating a proof.

1.1 A simple proof of Theorem 1.1 using rational representations of the general linear group

Here we briefly indicate how the main theorem may be proved by translating it to the general linear group setting; a similar method may be used to prove the decomposition number result. Let n, r, λ and μ be as above. Then we have the usual Weyl module $\Delta(\lambda)$ and dual Weyl module $\nabla(\lambda)$ for $\mathrm{GL}_r(\mathbb{F})$. Since we are assuming that char($\mathbb{F}$) is odd, we have

$$\dim_{\mathbb{F}} \operatorname{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, S^\mu) = \dim_{\mathbb{F}} \operatorname{Hom}_{\mathbb{F}\mathrm{GL}_r(\mathbb{F})}(\Delta(\lambda), \Delta(\mu))$$

by [1, Theorem 3.7]. So it remains to prove that

$$\dim_{\mathbb{F}} \operatorname{Hom}_{\mathbb{F}\mathrm{GL}_r(\mathbb{F})}(\Delta(\lambda), \Delta(\mu)) = \dim_{\mathbb{F}} \operatorname{Hom}_{\mathbb{F}\mathrm{GL}_r(\mathbb{F})}(\Delta(\breve{\lambda}), \Delta(\breve{\mu})).$$

In fact, $\Delta(\nu)$ and $\nabla(\nu)$ are defined for any r-tuple $(\nu_1, \dots, \nu_r)$ of integers with $\nu_1 \geqslant \dots \geqslant \nu_r$. If we let det denote the one-dimensional determinant representation of $\mathrm{GL}_r(\mathbb{F})$, then we have $\Delta(\nu_1+1, \dots, \nu_r+1) \cong \Delta(\nu) \otimes \det$. We also have

$$\Delta(\nu)^\circ \cong \nabla(\nu)$$

and

$$\Delta(\nu)^* \cong \nabla(-\nu_r, -\nu_{r-1}, \dots, -\nu_1),$$

where M° and M^* denote the contravariant dual and usual linear dual of a module respectively (corresponding to the anti-automorphisms $A \mapsto A^{\mathrm{T}}$ and $A \mapsto A^{-1}$ of $\mathrm{GL}_r(\mathbb{F})$). So we have

$$\Delta(\breve{\lambda}) \cong (\Delta(\lambda)^*)^\circ \otimes \det^{\otimes s}$$

and similarly for $\Delta(\check{\mu})$, and the result follows, since $(- \otimes \det)$ is a category equivalence on the rational $\mathrm{GL}_r(\mathbb{F})$-modules.

The author is very grateful to the referee for outlining this proof, which readily generalises to q-Schur algebras and Iwahori–Hecke algebras.

1.2 Background and notation

We take our notation and most of our results from James's book [6], whch remains the essential reference for the representation theory of the symmetric groups. Note, however, that we write maps on the left.

Given a composition λ of n, the *Young diagram* $[\lambda]$ for λ is defined to be the subset

$$\{(i, j) \mid j \leqslant \lambda_i\}$$

of $\mathbb{N}^2$, whose elements are usually called *nodes*. The Young diagram is usually drawn using a $\times$ for each node, with i increasing down the page and j increasing from left to right. A *λ-tableau* is a function from $[\lambda]$ to $\mathbb{N}$, and unless specified otherwise is a bijection from $[\lambda]$ to $\{1, \ldots, n\}$. We think of a λ-tableau as '$[\lambda]$ with nodes replaced by integers'. We define 'row equivalence' relation $\sim_{\mathrm{row}}$ on the set of λ-tableaux by saying that $\mathtt{s} \sim_{\mathrm{row}} \mathtt{t}$ if $\mathtt{t}$ can be obtained from $\mathtt{s}$ by permuting the entries of each row; similarly, we define the 'column equivalence' relation $\sim_{\mathrm{col}}$. We say that a λ-tableau is *standard* if its entries are increasing along rows and down columns.

An equivalence class of λ-tableaux under $\sim_{\mathrm{row}}$ is called a *λ-tabloid*; we denote by $\{\mathtt{t}\}$ the tabloid containing $\mathtt{t}$. The set of λ-tabloids is acted upon in an obvious way by $\mathfrak{S}_n$, and we define M^λ to be the corresponding permutation module. If λ is a partition, then for a tableau $\mathtt{t}$ we define the *polytabloid*

$$e_{\mathtt{t}} = \sum_{\mathtt{s} \sim_{\mathrm{col}} \mathtt{t}} (-1)^{\mathtt{st}} \{\mathtt{s}\} \in M^\lambda,$$

where $(-1)^{\mathtt{st}}$ is the sign of the permutation sending $\mathtt{s}$ to $\mathtt{t}$. We define the *Specht module* $S^\lambda \leqslant M^\lambda$ to be the span of the polytabloids.

Write λ' for the partition conjugate to λ, that is, $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ with $\lambda'_i = \max\{j \mid \lambda_j \geqslant i\}$.

1.2.1 The Garnir relations

Let λ be a partition, and $\mathtt{t}$ a λ-tableau. Let X be a subset of the ith column of $\mathtt{t}$, and Y a subset of the $(i+1)$th column of $\mathtt{t}$. We define $\mathfrak{S}_X$ to be the subgroup of $\mathfrak{S}_n$ fixing the complement of X pointwise, and make similar definitions for $\mathfrak{S}_Y$ and $\mathfrak{S}_{X \cup Y}$. We identify $\mathfrak{S}_X \times \mathfrak{S}_Y$ with the subgroup of $\mathfrak{S}_{X \cup Y}$ fixing X (and hence also Y) setwise.

Let $\sigma_1, \ldots, \sigma_k$ be right coset representatives of $\mathfrak{S}_X \times \mathfrak{S}_Y$ in $\mathfrak{S}_{X \cup Y}$, and define

$$G_{X,Y} = \sum_{i=1}^{k} (-1)^{\sigma_i} \sigma_i,$$

where $(-1)^\sigma$ is the sign of the permutation σ.

Theorem 1.2. [6, Theorem 7.2] *If $|X \cup Y| > \lambda'_i$, then $G_{X,Y} e_{\mathtt{t}} = 0$.*

1.2.2 Semi-standard homomorphisms

Let λ be a partition of n, and μ a composition of n. We now construct a basis for $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, M^\mu)$ in the case where $\mathbb{F}$ does not have characteristic 2. To do this, we consider tableaux with repeated entries. Specifically, we say that a λ-tableau has type μ if it contains μ_i entries equal to i, for each i. We let $\mathcal{T}(\lambda,\mu)$ denote the set of such tableaux. Note that we still use the equivalence relations $\sim_{\mathrm{row}}$ and $\sim_{\mathrm{col}}$ for tableaux in $\mathcal{T}(\lambda,\mu)$.

In order to construct homomorphisms from S^λ to M^μ, we use an isomorphic copy of M^μ depending on a fixed λ-tableau $\mathfrak{t}$. Define a bijection from the set of μ-tabloids to $\mathcal{T}(\lambda,\mu)$ by mapping

$$\{\mathfrak{s}\} \longmapsto \mathtt{T},$$

where, for each node x of $[\lambda]$, $\mathtt{T}(x)$ is the number of the row of $\mathfrak{s}$ in which $\mathfrak{t}(x)$ lies. By means of this bijection, we may take $\mathcal{T}(\lambda,\mu)$ as a basis for M^μ.

Now, given $\mathtt{T} \in \mathcal{T}(\lambda,\mu)$, define $\Theta_{\mathtt{T}} : M^\lambda \to M^\mu$ by

$$\{\mathfrak{t}\} \longmapsto \sum_{\mathtt{S} \sim_{\mathrm{row}} \mathtt{T}} \mathtt{S},$$

extending homomorphically. Define $\hat{\Theta}_{\mathtt{T}}$ to be the restriction of $\Theta_{\mathtt{T}}$ to S^λ.

We call a tableau $\mathtt{T} \in \mathcal{T}(\lambda,\mu)$ *semi-standard* if its entries are weakly increasing along the rows and strictly increasing down the columns, and we write $\mathcal{T}_0(\lambda,\mu)$ for the set of semi-standard λ-tableaux of type μ. We then have the following, due to Carter and Lusztig [1].

Theorem 1.3. [6, Theorem 13.13] *If the characteristic of $\mathbb{F}$ is not 2, then the set*

$$\{\hat{\Theta}_{\mathtt{T}} \mid \mathtt{T} \in \mathcal{T}_0(\lambda,\mu)\}$$

is a basis for $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, M^\mu)$.

1.2.3 Multipolytabloids

Suppose that μ is a partition of n, and that $I = \{I_1,\dots,I_m\}$ is a partition of the set $\{1,\dots,n\}$. We define an equivalence relation $\sim_I$ on the set of μ-tableaux by putting $\mathfrak{s} \sim_I \mathfrak{u}$ if and only if $\mathfrak{u} = \pi\mathfrak{s}$ where $\pi \in \mathfrak{S}_n$ preserves the partition I. (We shall be considering the case where I_i is the set of entries in the ith column of a λ-tableau $\mathfrak{t}$; then we shall have $\mathfrak{s} \sim_I \mathfrak{u}$ if and only if $\mathfrak{u} = \pi\mathfrak{s}$ for π in the column stabiliser of $\mathfrak{t}$.)

Given a μ-tableau $\mathfrak{s}$, define

$$C_I(\mathfrak{s}) = \{\mathfrak{u} \mid \mathfrak{s} \sim_I \mathfrak{u}\}$$

and let $\{\mathfrak{u}_1,\dots,\mathfrak{u}_a\}$ be a transversal of the $\sim_{\mathrm{col}}$ equivalence classes in $C_I(\mathfrak{s})$. Now define the *multipolytabloid*

$$e^I_{\mathfrak{s}} = \sum_{j=1}^{a} (-1)^{\mathfrak{s}\mathfrak{u}_j} e_{\mathfrak{u}_j}.$$

We may use multipolytabloids to describe the images of homomorphisms between Specht modules, but first we make a more restrictive definition, slightly differently from [4]. Let λ be a partition, and let $\mathfrak{t}^\lambda$ be the λ-tableau with entries $1,\dots,\lambda'_1$ down the first column, $\lambda'_1+1,\dots,\lambda'_2$ down the second column and so on. Let I_i be the set of numbers appearing in the ith column of $\mathfrak{t}^\lambda$, and say that the multipolytabloid $e^I_{\mathfrak{s}}$ is *rectified* if $\mathfrak{s}$ is a standard tableau with no two elements of I_i in the same row, for any i.

Given an element x of an $\mathbb{F}\mathfrak{S}_n$-module M and a subset S of $\{1,\dots,n\}$, we say that x is *alternating in the elements of S* if for any $a, b \in S$ we have $(a\ b)x = -x$.

Proposition 1.4. *Suppose that an element x of S^μ is alternating in the elements of each I_i. Then x is a linear combination of rectified multipolytabloids.*

Proof. We may use the proof of [4, Proposition 4.4], even though the statement of that result is weaker. By expressing x in terms of standard polytabloids, we guarantee that each multipolytabloid $e^I_{\mathfrak{s}}$ we subtract from x has $\mathfrak{s}$ standard; the fact that the elements of each I_i occur in distinct rows of $\mathfrak{s}$ is also already given. □

2 An isomorphic copy of $S^{\breve{\mu}}$

In order to prove Theorem 1.1, it will turn out to be much easier to work with a copy of $S^{\breve{\mu}}$ (contained in a copy of $M^{\breve{\mu}}$). Define μ° to be the composition $(s-\mu_1,\dots,s-\mu_r)$, i.e. $\breve{\mu}$ with parts in reverse order. Then $M^{\mu^\circ}\cong M^{\breve{\mu}}$, and we construct a copy of $S^{\breve{\mu}}$ inside M^{μ° as follows. Let $\lfloor\mu^\circ\rceil$ be the subset

$$\{(i,j)\mid i\leqslant r,\ \mu_i<j\leqslant s\}$$

of $\mathbb{N}^2$, i.e. the Young diagram of $\breve{\mu}$ rotated through 180° about $\left(\frac{r+1}{2},\frac{s+1}{2}\right)$. We define μ°-tableaux, -tabloids and -polytabloids using this diagram exactly as for partitions, and we let $R^{\mu^\circ}\leqslant M^{\mu^\circ}$ be the span of the μ°-polytabloids. By [8, 2.1] we then have $R^{\mu^\circ}\cong S^{\breve{\mu}}$. We shall henceforth work entirely with μ° rather than $\breve{\mu}$.

We also fix from now on the λ-tableau $\mathtt{t}^\lambda$ as defined in 1.2.3, i.e. with the numbers $1,\dots,n$ arranged in order down successive columns. We define $\mathtt{t}^{\breve{\lambda}}$ similarly.

Example. Suppose $\lambda=(4,3)$ and $\mu=(3,3,1)$, and take $r=3$, $s=5$ so that $\breve{\lambda}=(5,2,1)$, $\breve{\mu}=(4,2,2)$. Then we have

$$\mathtt{t}^\lambda=\begin{array}{cccc}1&3&5&7\\2&4&6&\end{array},\qquad \mathtt{t}^{\breve{\lambda}}=\begin{array}{ccccc}1&4&6&7&8\\2&5&&&\\3&&&&\end{array},\qquad \lfloor\mu^\circ\rceil=\begin{array}{ccccc}&&&\times&\times\\&&&\times&\times\\&\times&\times&\times&\times\end{array}.$$

3 Tableau combinatorics

Recall the definition of semi-standard tableaux from 1.2.2. We now make some similar definitions.

A tableau is *column standard* if its entries are strictly increasing down columns. We write $\mathcal{T}_{\mathrm{c}}(\lambda,\mu)$ for the set of column standard λ-tableaux of type μ.

A tableau is *row standard* if its entries are strictly increasing along rows. We write $\mathcal{T}_{\mathrm{r}}(\lambda,\mu)$ for the set of row standard λ-tableaux of type μ.

A tableau is *transpose semi-standard* if its entries are strictly increasing along rows and weakly increasing down columns. We write $\mathcal{T}_1(\lambda,\mu)$ for the set of transpose semi-standard λ-tableaux of type μ.

We shall be working with the sets $\mathcal{T}_{\mathrm{c}}(\lambda,\mu)$, $\mathcal{T}_{\mathrm{c}}(\breve{\lambda},\mu^\circ)$, $\mathcal{T}_{\mathrm{r}}(\mu,\lambda')$ and $\mathcal{T}_{\mathrm{r}}(\mu^\circ,(\breve{\lambda})')$. In particular, we wish to construct natural bijections between these four sets.

Given $\mathtt{T}\in\mathcal{T}_{\mathrm{c}}(\lambda,\mu)$, let $\mathtt{T}[i]\subseteq\{1,\dots,r\}$ be the set of entries in the ith column of $\mathtt{T}$. Now define $\breve{\mathtt{T}}[i]$ to be the complement in $\{1,\dots,r\}$ of $\mathtt{T}[s+1-i]$, and let $\breve{\mathtt{T}}$ be the column standard $\breve{\lambda}$-tableau with the elements of $\breve{\mathtt{T}}[i]$ in its ith column, for each i.

Given $\mathtt{T}\in\mathcal{T}_{\mathrm{r}}(\mu,\lambda')$, let $\mathtt{T}\langle i\rangle\subseteq\{1,\dots,s\}$ be the set of entries in the ith row of $\mathtt{T}$. Now define

$$\overline{\mathtt{T}\langle i\rangle}=\{s+1-a\mid a\in\mathtt{T}\langle i\rangle\},$$

and define $\breve{\mathtt{T}}$ to be the row standard μ°-tableau with

$$\breve{\mathtt{T}}\langle i\rangle=\{1,\dots,s\}=\overline{\mathtt{T}\langle i\rangle}$$

for each i.

Given $\mathtt{T} \in \mathcal{T}_{\mathrm{c}}(\lambda, \mu)$, define $\acute{\mathtt{T}}$ to be the row standard μ-tableau such that there is an entry equal to i in row j of $\acute{\mathtt{T}}$ if and only if there is an entry equal to j in column i of $\mathtt{T}$. Similarly, given $\mathtt{T} \in \mathcal{T}_{\mathrm{c}}(\breve{\lambda}, \mu^\circ)$, define $\grave{\mathtt{T}}$ to be the row standard μ°-tableau which has an i in its jth row if and only if $\mathtt{T}$ has a j in its ith column.

Lemma 3.1.

1. *The map* $\mathtt{T} \mapsto \acute{\mathtt{T}}$ *is a bijection from* $\mathcal{T}_{\mathrm{c}}(\lambda, \mu)$ *to* $\mathcal{T}_{\mathrm{r}}(\mu, \lambda')$.
2. *The map* $\mathtt{T} \mapsto \grave{\mathtt{T}}$ *is a bijection from* $\mathcal{T}_{\mathrm{c}}(\breve{\lambda}, \mu^\circ)$ *to* $\mathcal{T}_{\mathrm{r}}(\mu^\circ, (\breve{\lambda})')$.
3. *The map* $\mathtt{T} \mapsto \breve{\mathtt{T}}$ *is a bijection from* $\mathcal{T}_{\mathrm{c}}(\lambda, \mu)$ *to* $\mathcal{T}_{\mathrm{c}}(\breve{\lambda}, \mu^\circ)$ *which restricts to a bijection* $\mathcal{T}_0(\lambda, \mu) \to \mathcal{T}_0(\breve{\lambda}, \mu^\circ)$.
4. *The map* $\mathtt{T} \mapsto \check{\mathtt{T}}$ *is a bijection from* $\mathcal{T}_{\mathrm{r}}(\mu, \lambda')$ *to* $\mathcal{T}_{\mathrm{r}}(\mu^\circ, (\breve{\lambda})')$ *which restricts to a bijection* $\mathcal{T}_1(\mu, \lambda') \to \mathcal{T}_1(\mu^\circ, (\breve{\lambda})')$.
5. *For* $\mathtt{T} \in \mathcal{T}_{\mathrm{c}}(\lambda, \mu)$, *we have* $\grave{\breve{\mathtt{T}}} = \check{\acute{\mathtt{T}}}$.

Proof. The types of the tableaux $\acute{\mathtt{T}}, \grave{\mathtt{T}}, \breve{\mathtt{T}}, \check{\mathtt{T}}$ are easily verified, and the constructions are easily seen to be invertible. (5) is also easily checked, so it remains to prove the relative clauses in (3) and (4); we prove the first of these.

Suppose $\mathtt{T} \in \mathcal{T}_{\mathrm{c}}(\lambda, \mu)$, and for each i, j define $z^{\mathtt{T}}_{ij}$ to be the number of entries less than or equal to j in the ith column of $\mathtt{T}$. The condition that $\mathtt{T}$ is semi-standard is then equivalent to

$$z^{\mathtt{T}}_{ij} \geqslant z^{\mathtt{T}}_{(i+1)j}$$

for all $1 \leqslant i < s$ and $1 \leqslant j \leqslant r$. But this is the same as saying that the number of integers less than or equal to j which do *not* appear in the ith column of $\mathtt{T}$ is at most the number of integers less than or equal to j which do not appear in the $(i+1)$th column, i.e.

$$z^{\breve{\mathtt{T}}}_{(s+1-i)j} \leqslant z^{\breve{\mathtt{T}}}_{(s-i)j}$$

for all i and j. Hence $\breve{\mathtt{T}}$ is semi-standard. Similarly, $\mathtt{T}$ is semi-standard whenever $\breve{\mathtt{T}}$ is. □

Example. Take λ, μ, r and s as in the last example. Then, if

$$\mathtt{T} = \begin{array}{cccc} 1 & 1 & 2 & 1 \\ 2 & 2 & 3 & \end{array},$$

we have

$$\breve{\mathtt{T}} = \begin{array}{ccccc} 1 & 2 & 1 & 3 & 3 \\ 2 & 3 & & & \\ 3 & & & & \end{array}, \qquad \acute{\mathtt{T}} = \begin{array}{ccc} 1 & 2 & 4 \\ 1 & 2 & 3 \\ 3 & & \end{array}, \qquad \grave{\breve{\mathtt{T}}} = \check{\acute{\mathtt{T}}} = \begin{array}{cccc} & & 1 & 3 \\ & & 1 & 2 \\ 1 & 2 & 4 & 5 \end{array}.$$

Given a μ-tableau $\mathfrak{s}$ (of type (1^n)), define the μ-tableau $\mathtt{T}_{\mathfrak{s}}$ of type λ' by replacing each entry of $\mathfrak{s}$ with the number of the column in which it appears in $\mathfrak{t}^\lambda$. Now we choose a μ-tabloid corresponding to each $\mathtt{S} \in \mathcal{T}_{\mathrm{c}}(\lambda, \mu)$. Given $\mathtt{S}$, let $\mathfrak{r}_{\mathtt{S}}$ be the row standard μ-tableau such that $\{\mathfrak{r}_{\mathtt{S}}\}$ corresponds to $\mathtt{S}$ via $\mathfrak{t}^\lambda$ as in 1.2.2 (or, to put it another way, so that $\mathtt{T}_{\mathfrak{r}_{\mathtt{S}}} = \acute{\mathtt{S}}$). Make corresponding definitions for $\breve{\lambda}$ and μ°, with $\mathfrak{r}_{\breve{\mathtt{S}}}$ the μ°-tableau corresponding to $\breve{\mathtt{S}}$. The following lemma will be important.

Lemma 3.2. *Suppose* $\mathtt{S}, \mathtt{U} \in \mathcal{T}_{\mathrm{c}}(\lambda, \mu)$. *Then*

$$(-1)^{\mathfrak{r}_{\mathtt{S}}\mathfrak{r}_{\mathtt{U}}} = (-1)^{\mathfrak{r}_{\breve{\mathtt{S}}}\mathfrak{r}_{\breve{\mathtt{U}}}}.$$

Proof. We continue to use the notation $\mathtt{S}[i]$ for the set of entries in the ith column of $\mathtt{S}$. We say that $\mathtt{S}$ and $\mathtt{U}$ are *close* if for some i there are $a \in \mathtt{S}[i] \setminus \mathtt{S}[i+1]$ and $b \in \mathtt{S}[i+1] \setminus \mathtt{S}[i]$ such that:

$$\begin{aligned} \mathtt{U}[i] &= \mathtt{S}[i] \cup \{b\} \setminus \{a\}; \\ \mathtt{U}[i+1] &= \mathtt{S}[i+1] \cup \{a\} \setminus \{b\}; \\ \mathtt{U}[k] &= \mathtt{S}[k] \text{ whenever } k \neq i, i+1. \end{aligned}$$

Claim. There exists a chain $\mathtt{S} = \mathtt{S}_0, \mathtt{S}_1, \dots, \mathtt{S}_N = \mathtt{U}$ such that $\mathtt{S}_{j-1}$ and $\mathtt{S}_j$ are close for all j.

Proof. Assuming $\mathtt{S} \neq \mathtt{U}$, let i be minimal such that $\mathtt{S}[i] \neq \mathtt{U}[i]$, and take $a \in \mathtt{S}[i] \setminus \mathtt{U}[i]$ and $b \in \mathtt{U}[i] \setminus \mathtt{S}[i]$. Since $\mathtt{S}, \mathtt{U}$ are tableaux of the same type, there must be an entry equal to b in column l of $\mathtt{S}$, for some $l > i$. By swapping entries between columns l and $l-1$, then between columns $l-1$ and $l-2$ and so on, we may find a chain $\mathtt{S} = \mathtt{S}_0, \mathtt{S}_1, \dots, \mathtt{S}_M$ where

- $\mathtt{S}_{j-1}$ and $\mathtt{S}_j$ are close for all j,
- $\mathtt{S}_M[k] = \mathtt{S}[k]$ for all $k < i$, and
- $\mathtt{S}_M[i] = \mathtt{S}[i] \cup \{b\} \setminus \{a\}$.

Now replace $\mathtt{S}$ with $\mathtt{S}_M$ and proceed by (downwards) induction on i and $|\mathtt{S}[i] \cap \mathtt{U}[i]|$. □

In view of this claim, and since we have $(-1)^{\mathfrak{r}_\mathtt{S}\mathfrak{r}_\mathtt{T}}(-1)^{\mathfrak{r}_\mathtt{T}\mathfrak{r}_\mathtt{U}} = (-1)^{\mathfrak{r}_\mathtt{S}\mathfrak{r}_\mathtt{U}}$ for any $\mathtt{S}, \mathtt{T}, \mathtt{U}$, we may assume that $\mathtt{S}$ and $\mathtt{U}$ are close. Let a and b be as above; by interchanging $\mathtt{S}$ and $\mathtt{U}$ if necessary, we may assume that $a < b$.

Let $a = a_0 < a_1 < \dots < a_p$ be the elements of $\mathtt{S}[i]$ which lie between a and b, and let $x, \dots, x+p$ be the numbers which occupy the corresponding positions in column i of $\mathtt{t}^\lambda$. Let $b_0 < b_1 < \dots < b_q = b$ be the elements of $\mathtt{S}[i+1]$ which lie between a and b, and let $y, \dots, y+q$ be the numbers which occupy the corresponding positions in column $i+1$ of $\mathtt{t}^\lambda$. Then the permutation which takes $\mathfrak{r}_\mathtt{S}$ to $\mathfrak{r}_\mathtt{U}$ is simply the cycle

$$(x+p \;\; x+p-1 \;\; \dots \;\; x+1 \;\; x \;\; y \;\; y+1 \;\; \dots \;\; y+q-1 \;\; y+q),$$

which has signature $(-1)^{p+q+1}$.

Now consider $\check{\mathtt{S}}$ and $\check{\mathtt{U}}$. We have

$$\begin{aligned} \check{\mathtt{U}}[s-i] &= \check{\mathtt{S}} \cup \{b\} \setminus \{a\}, \\ \check{\mathtt{U}}[s-i+1] &= \check{\mathtt{S}}[s-i+1] \cup \{a\} \setminus \{b\}, \\ \check{\mathtt{U}}[k] &= \check{\mathtt{S}}[k] \text{ whenever } k \neq s-i, s-i+1. \end{aligned}$$

As above we find that the sign of the permutation taking $\mathfrak{r}_{\check{\mathtt{S}}}$ to $\mathfrak{r}_{\check{\mathtt{U}}}$ is $(-1)^{\tilde{p}+\tilde{q}+1}$, where

$$\tilde{p} = \left|\check{\mathtt{S}}[s-i] \cap \{a, \dots, b\}\right|, \qquad \tilde{q} = \left|\check{\mathtt{S}}[s-i+1] \cap \{a, \dots, b\}\right|.$$

But $\{1, \dots, r\}$ is the disjoint union of $\mathtt{S}[i]$ and $\check{\mathtt{S}}[s-i+1]$, so that $p+\tilde{q} = b-a+1$. Similarly, $q+\tilde{p} = b-a+1$, so that we have

$$\begin{aligned} (-1)^{\tilde{p}+\tilde{q}+1} &= (-1)^{2b-2a+2-p-q+1} \\ &= (-1)^{p+q+1}. \end{aligned}$$

□

4 Comparing the images of semi-standard homomorphisms

The usual way to obtain information about homomorphisms between Specht modules is to combine Theorem 1.3 with the Kernel Intersection Theorem [6, Corollary 17.18], which expresses S^μ as the intersection of the kernels of certain homomorphisms $\psi_{d,t} : M^\mu \to M^\nu$. This does not seem to be particularly easy to apply to our result, and so we adopt a slightly different approach. Define

$$\Theta_{\lambda\mu} = \langle \theta(e_{\mathrm{t}^\lambda}) \mid \theta \in \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, M^\mu)\rangle.$$

Since S^λ is generated by e_{t^λ} [6, 4.5], we have

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, S^\mu) = \dim_{\mathbb{F}}(\Theta_{\lambda\mu} \cap S^\mu).$$

In this section, we shall compare $\Theta_{\lambda\mu}$ with $\Theta_{\check{\lambda}\mu^\circ}$, which is defined analogously.

Let $\mathcal{T}(\lambda,\mu)$ be a basis for M^μ as in 1.2.2, using the λ-tableau $\mathrm{t} = \mathrm{t}^\lambda$. Now define the subspace

$$A_{\lambda\mu} = \{x \in M^\mu \mid x \text{ is alternating in the elements of each column of } \mathrm{t}^\lambda\}$$

of M^μ, and define $A_{\check{\lambda}\mu^\circ} \leqslant M^{\mu^\circ}$ similarly. Then $\Theta_{\lambda\mu} \leqslant A_{\lambda\mu}$, since for $\theta \in \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, M^\mu)$ and a, b in the same column of t^λ we have

$$(a\ b)\theta(e_{\mathrm{t}^\lambda}) = \theta((a\ b)e_{\mathrm{t}^\lambda}) = \theta(-e_{\mathrm{t}^\lambda}) = -\theta(e_{\mathrm{t}^\lambda}).$$

Similarly, $\Theta_{\check{\lambda}\mu^\circ} \leqslant A_{\check{\lambda}\mu^\circ}$. We aim to construct a vector space isomorphism $A_{\lambda\mu} \to A_{\check{\lambda}\check{\mu}}$ which restricts to an isomorphism $\Theta_{\lambda\mu} \to \Theta_{\check{\lambda}\mu^\circ}$.

4.1 A basis for $A_{\lambda\mu}$

Suppose $a \in A_{\lambda\mu}$, and write $a = \sum_{\mathtt{T}\in\mathcal{T}(\lambda,\mu)} a_{\mathtt{T}}\mathtt{T}$, with $a_{\mathtt{T}} \in \mathbb{F}$. The condition that a is alternating in the elements of each column of t^λ is equivalent to

$$a_{\mathtt{T}} = (-1)^{\mathtt{ST}} a_{\mathtt{S}} \text{ whenever } \mathtt{S} \sim_{\mathrm{col}} \mathtt{T},$$

where $(-1)^{\mathtt{ST}}$ is the sign of *any* column permutation taking $\mathtt{S}$ to $\mathtt{T}$. Hence we may easily find a basis for $A_{\lambda\mu}$; we assume henceforth that $\mathrm{char}(\mathbb{F}) \neq 2$.

Lemma 4.1. *For* $\mathtt{T} \in \mathcal{T}_{\mathrm{c}}(\lambda,\mu)$*, define*

$$E_{\mathtt{T}} = \sum_{\mathtt{S}\sim_{\mathrm{col}}\mathtt{T}} (-1)^{\mathtt{ST}}\mathtt{S}.$$

Then the set

$$\{E_{\mathtt{T}} \mid \mathtt{T} \in \mathcal{T}_{\mathrm{c}}(\lambda,\mu)\}$$

is a basis for $A_{\lambda\mu}$.

We may therefore construct a vector space isomorphism $A_{\lambda\mu} \to A_{\check{\lambda}\mu^\circ}$ by means of the bijection $\mathcal{T}_{\mathrm{c}}(\lambda,\mu) \to \mathcal{T}_{\mathrm{c}}(\check{\lambda},\mu^\circ)$ given by $\mathtt{T} \mapsto \check{\mathtt{T}}$. Define the linear map $\phi : A_{\lambda\mu} \to A_{\check{\lambda}\mu^\circ}$ by

$$\phi : E_{\mathtt{T}} \longmapsto E_{\check{\mathtt{T}}}.$$

Given the second statement of Lemma 3.1(3), we might hope that the vector space isomorphism

$$\Theta_{\lambda\mu} \longrightarrow \Theta_{\check{\lambda}\mu^\circ}$$
$$\hat{\Theta}_{\mathtt{T}}(e_{\mathrm{t}^\lambda}) \longmapsto \hat{\Theta}_{\check{\mathtt{T}}}(e_{\mathrm{t}^{\check{\lambda}}})$$

is a restriction of ϕ. This is not the case in general, and we have to find the appropriate restriction of ϕ differently.

Recall the dominance order [6, Definition 13.8] on $\mathcal{T}_{\mathrm{c}}(\lambda,\mu)$: we write $\mathtt{S} \rhd \mathtt{T}$ if $\mathtt{T}$ can be obtained from $\mathtt{S}$ by interchanging entries w and x, where $w < x$ and w belongs to an earlier column of $\mathtt{S}$ than x; then we extend $\trianglerighteq$ to a partial order on $\mathcal{T}_{\mathrm{c}}(\lambda,\mu)$. It is easily seen that $\mathtt{S} \trianglerighteq \mathtt{T}$ if and only if $\check{\mathtt{S}} \trianglerighteq \check{\mathtt{T}}$.

Proposition 4.2. *Suppose that* $\mathtt{T} \in \mathcal{T}_{\mathrm{c}}(\lambda,\mu)$ *is not semi-standard. Then there exist* $\mathtt{T}_1,\dots,\mathtt{T}_l \in \mathcal{T}_{\mathrm{c}}(\lambda,\mu)$ *and* $c_1,\dots,c_l \in \mathbb{F}$ *such that:*

- $\mathtt{T}_i \rhd \mathtt{T}$ *for all* i*;*
- *if* $a = \sum_{\mathtt{S}\in\mathcal{T}_{\mathrm{c}}(\lambda,\mu)} a_{\mathtt{S}} E_{\mathtt{S}}$ *is any element of* $\Theta_{\lambda\mu}$*, then*

$$a_{\mathtt{T}} + c_1 a_{\mathtt{T}_1} + \cdots + c_l a_{\mathtt{T}_l} = 0;$$

- *if* $b = \sum_{\mathtt{U}\in\mathcal{T}_{\mathrm{c}}(\check{\lambda},\mu^\circ)} b_{\mathtt{U}} E_{\mathtt{U}}$ *is any element of* $\Theta_{\check{\lambda}\mu^\circ}$*, then*

$$b_{\check{\mathtt{T}}} + c_1 b_{\check{\mathtt{T}}_1} + \cdots + c_l b_{\check{\mathtt{T}}_l} = 0.$$

Proof. We use an idea from the proof of [6, Lemma 13.12], which uses the Garnir relations to show that the semi-standard homomorphisms span $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, M^\mu)$. We must find Garnir relations which are 'the same' for $\mathtt{T}$ and $\check{\mathtt{T}}$.

Write $\mathtt{T}_{ji}$ for the jth entry in the ith column of $\mathtt{T}$. Because $\mathtt{T}$ is not semi-standard, we have $\mathtt{T}_{ji} > \mathtt{T}_{j(i+1)}$ for some i, j; let i, j be such that this holds but $\mathtt{T}_{mi} \leqslant \mathtt{T}_{m(i+1)}$ for $m < j$. Suppose $\mathtt{T}_{j(i+1)} = d$.

Now, using the proof of Lemma 3.1(3), we find that d appears in the $(s+1-i)$th column of $\check{\mathtt{T}}$, say $\check{\mathtt{T}}_{k(s+1-i)} = d$, and that $\check{\mathtt{T}}_{k(s-i)} > \check{\mathtt{T}}_{k(s+1-i)}$. Let X be the set of entries in positions $j,\dots,\lambda'_i$ of column i of t^λ, and Y the set of entries in positions $1,\dots,j$ of column $i+1$. Let $\check{X}$ be the set of entries in positions $k,\dots,r-\lambda'_{i+1}$ of column $s-i$ of $\mathrm{t}^{\check{\lambda}}$, and $\check{Y}$ the set of entries in positions $1,\dots,k$ of column $s+1-i$. We are going to use the Garnir elements $G_{X,Y}$ and $G_{\check{X},\check{Y}}$, but we need to choose our coset representatives carefully. Given a right coset $\sigma(\mathfrak{S}_X \times \mathfrak{S}_Y)$ of $\mathfrak{S}_X \times \mathfrak{S}_Y$ in $\mathfrak{S}_{X\cup Y}$, let $x_1 < \cdots < x_b$ be the elements of X mapped to elements of Y by σ^{-1}, and $y_1 < \cdots < y_b$ the elements of Y mapped to elements of X by σ^{-1}. Clearly these determine $\sigma(\mathfrak{S}_X \times \mathfrak{S}_Y)$ but are independent of σ. Choose the involution $(x_1\ b_1)(x_2\ b_2)\dots(x_b\ y_b)$ as the coset representative for $\sigma(\mathfrak{S}_X \times \mathfrak{S}_Y)$. Because the coset representatives $\sigma_1,\dots,\sigma_a$ so chosen are involutions, they also form a set of *left* coset representatives for $\mathfrak{S}_X \times \mathfrak{S}_Y$ in $\mathfrak{S}_{X\cup Y}$.

Now define $G_{X,Y}$ as in 1.2.1. We have $|X \cup Y| = \lambda'_i + 1$, and so $G_{X,Y} e_{\mathrm{t}^\lambda} = 0$. If $a = \theta(e_{\mathrm{t}^\lambda})$ with θ a homomorphism from S^λ to M^μ, then we have

$$0 = \theta(G_{X,Y} e_{\mathrm{t}^\lambda}) = G_{X,Y} a;$$

in particular, the coefficient of $\mathtt{T}$ in $G_{X,Y} a$ is zero. If we write a in terms of the basis $\mathcal{T}(\lambda,\mu)$ for M^μ, say $a = \sum_{\mathtt{S}\in\mathcal{T}(\lambda,\mu)} \alpha_{\mathtt{S}} \mathtt{S}$, then we have

$$(-1)^{\sigma_1} \alpha_{\sigma_1 \mathtt{T}} + \cdots + (-1)^{\sigma_a} \alpha_{\sigma_a \mathtt{T}} = 0$$

(since $\sigma_m = \sigma_m^{-1}$ for each m). Now if the tableau $\sigma_m\mathtt{T}$ has two identical entries in some column, then $\alpha_{\sigma_m\mathtt{T}} = 0$, and so we let $\mathtt{S}_0, \dots, \mathtt{S}_l$ be the tableaux among $\sigma_1\mathtt{T}, \dots, \sigma_a\mathtt{T}$ which have distinct entries in each column. Let $\mathtt{T}_0, \dots, \mathtt{T}_l$ be the tableaux in $\mathcal{T}_{\mathrm{c}}(\lambda, \mu)$ such that $\mathtt{T}_m \sim_{\mathrm{col}} \mathtt{S}_m$ for each m. Then, assuming $\mathtt{S}_0 = \mathtt{T}_0 = \mathtt{T}$, we have

$$0 = a_{\mathtt{T}} + (-1)^{\mathtt{TS}_1}(-1)^{\mathtt{S}_1\mathtt{T}_1}a_{\mathtt{T}_1} + \dots + (-1)^{\mathtt{TS}_l}(-1)^{\mathtt{S}_l\mathtt{T}_l}a_{\mathtt{T}_l},$$

where $\mathtt{TS}_m$ is the number of entries moved from column i to column $i+1$ to get from $\mathtt{T}$ to $\mathtt{S}_m$, and $(-1)^{\mathtt{S}_m\mathtt{T}_m}$ is the sign of the column permutation taking $\mathtt{S}_m$ to $\mathtt{T}_m$.

In fact, we may easily describe $\mathtt{T}_1, \dots, \mathtt{T}_l$. A permutation σ_m simply interchanges some subset of X with some subset of Y of the same size. And so (recalling that we write $\mathtt{T}[i]$ for the set of entries in the ith column of $\mathtt{T}$) a tableau $\sigma_m\mathtt{T}$ is obtained by interchanging A and B, where A is a subset of $\mathtt{T}[i]$ which consists of integers greater than d, B is a subset of $\mathtt{T}[i+1]$ consisting of integers which are at most d, and $|A| = |B|$. The tableau $\sigma_m\mathtt{T}$ then has distinct entries in each column if and only if A is disjoint from $\mathtt{T}[i+1]$ and B is disjoint from $\mathtt{T}[i]$.

Now consider $\check{\mathtt{T}}$. Notice that a set A of integers greater than d is a subset of $\mathtt{T}[i]$ disjoint from $\mathtt{T}[i+1]$ if and only if it is a subset of $\check{\mathtt{T}}[s-i]$ disjoint from $\check{\mathtt{T}}[s+1-i]$, while a set B of integers equal to or less than d is a subset of $\mathtt{T}[i+1]$ disjoint from $\mathtt{T}[i]$ if and only if it is a subset of $\check{\mathtt{T}}[s+1-i]$ disjoint from $\check{\mathtt{T}}[s-i]$. So if we repeat the above procedure with $\check{\lambda}, \mu^\circ$ in place of λ, μ and let $\check{\mathtt{S}}_1, \dots, \check{\mathtt{S}}_l$ and $\check{\mathtt{T}}_1, \dots, \check{\mathtt{T}}_m$ be the tableaux obtained, then $\check{\mathtt{T}}_1, \dots, \check{\mathtt{T}}_l$ are in one-to-one correspondence with $\mathtt{T}_1, \dots, \mathtt{T}_l$ via the pairs (A, B); without loss, assume that $\mathtt{T}_m$ and $\check{\mathtt{T}}_m$ correspond in this way, for each m. Moreover, notice that we have $(-1)^{\mathtt{TS}_m} = (-1)^{|A|} = (-1)^{\check{\mathtt{T}}\check{\mathtt{S}}_m}$ for each m. So it suffices to prove that $(-1)^{\mathtt{S}_m\mathtt{T}_m} = (-1)^{\check{\mathtt{S}}_m\check{\mathtt{T}}_m}$ for each m. Now $(-1)^{\mathtt{S}_m\mathtt{T}_m}$ is the sign of the permutation required to put columns i and $i+1$ of $\mathtt{S}_m$ in ascending order; column i of $\mathtt{S}_m$ is the same as column i of $\mathtt{T}$, except that the elements $a_1 < \dots < a_x$ of A are replaced with the elements $b_1 < \dots < b_x$ of B in order. So the number of transpositions required to put this column in ascending order is

$$\sum_{y=1}^{x} \left|(\mathtt{T}[i] \setminus A) \cap \{b_y + 1, \dots, a_y - 1\}\right|.$$

Similarly, the signature of the permutation required to put the entries of column $i+1$ of $\mathtt{S}_m$ in order is

$$\sum_{y=1}^{x} \left|(\mathtt{T}[i+1] \setminus B) \cap \{b_y + 1, \dots, a_y - 1\}\right|.$$

And so, letting $C_y = \{b_y + 1, \dots, a_y - 1\} \setminus (A \cup B)$, we have

$$(-1)^{\mathtt{S}_m\mathtt{T}_m} = (-1)^{\sum_{y=1}^{x}\left|\mathtt{T}[i]\cap C_y\right|+\left|\mathtt{T}[i+1]\cap C_y\right|}.$$

Similarly we have

$$(-1)^{\check{\mathtt{S}}_m\check{\mathtt{T}}_m} = (-1)^{\sum_{y=1}^{x}\left|\check{\mathtt{T}}[s-i]\cap C_y\right|+\left|\check{\mathtt{T}}[s+1-i]\cap C_y\right|};$$

but

$$\begin{aligned}\left|\check{\mathtt{T}}[s-i] \cap C_y\right| + \left|\check{\mathtt{T}}[s+1-i] \cap C_y\right| &= \left|(\{1, \dots, r\} \setminus \mathtt{T}[i]) \cap C_y\right| + \left|(\{1, \dots, r\} \setminus \mathtt{T}[i+1]) \cap C_y\right| \\ &= 2|C_y| - |\mathtt{T}[i] \cap C_y| - |\mathtt{T}[i+1] \cap C_y| \\ &\equiv |\mathtt{T}[i] \cap C_y| + |\mathtt{T}[i+1] \cap C_y| \pmod 2,\end{aligned}$$

and so $(-1)^{\mathtt{S}_m\mathtt{T}_m} = (-1)^{\check{\mathtt{S}}_m\check{\mathtt{T}}_m}$. □

Corollary 4.3. *ϕ restricts to a vector space isomorphism $\Theta_{\lambda\mu} \to \Theta_{\check{\lambda}\mu^\circ}$.*

Proof. The dominance order guarantees that the relations

$$a_{\mathtt{T}} = c_1 a_{\mathtt{T}_1} + \cdots + c_l a_{\mathtt{T}_l}$$

of Proposition 4.2 are linearly independent; since the number of such relations is the number of non-semi-standard tableaux in $\mathcal{T}_{\mathrm{c}}(\lambda, \mu)$, and the dimension of $\Theta_{\lambda\mu}$ is the number of semi-standard tableaux, it follows that $\Theta_{\lambda\mu}$ is precisely the subspace of $A_{\lambda\mu}$ satisfying these relations. A similar statement holds for $\Theta_{\check{\lambda}\mu^\circ}$. But by Proposition 4.2, ϕ maps the relations for $\Theta_{\lambda\mu}$ to those for $\Theta_{\check{\lambda}\mu^\circ}$, and hence $\phi(\Theta_{\lambda\mu}) = \Theta_{\check{\lambda}\mu^\circ}$. □

For later use, we wish to express Corollary 4.3 as a statement purely about column standard tableaux. The next lemma follows immediately from the definitions.

Lemma 4.4. *If $\mathtt{T} \in \mathcal{T}_0(\lambda, \mu)$ and we write*

$$\hat{\Theta}_{\mathtt{T}}(e_{\mathrm{t}^\lambda}) = \sum_{\mathtt{S} \in \mathcal{T}_{\mathrm{c}}(\lambda,\mu)} a_{\mathtt{S}} E_{\mathtt{S}},$$

then we have

$$a_{\mathtt{S}} = \sum_{\substack{\mathtt{R} \in \mathcal{T}(\lambda,\mu) \\ \mathtt{T} \sim_{\mathrm{row}} \mathtt{R} \sim_{\mathrm{col}} \mathtt{S}}} (-1)^{\mathtt{RS}}.$$

Now the following is simply a re-casting of Corollary 4.3.

Proposition 4.5. *Let $\boldsymbol{M}$ be the matrix over $\mathbb{F}$ with rows indexed by the set $\mathcal{T}_{\mathrm{c}}(\lambda, \mu)$, and columns indexed by the set $\mathcal{T}(\lambda, \mu)$, and in which the $(\mathtt{S}, \mathtt{T})$-entry is*

$$\sum_{\substack{\mathtt{R} \in \mathcal{T}(\lambda,\mu) \\ \mathtt{T} \sim_{\mathrm{row}} \mathtt{R} \sim_{\mathrm{col}} \mathtt{S}}} (-1)^{\mathtt{RS}}.$$

Let $\check{\boldsymbol{M}}$ be the matrix over $\mathbb{F}$ with rows indexed by the set $\mathcal{T}_{\mathrm{c}}(\check{\lambda}, \mu^\circ)$, and columns indexed by the set $\mathcal{T}(\check{\lambda}, \mu^\circ)$, and in which the $(\mathtt{S}, \mathtt{T})$-entry is

$$\sum_{\substack{\mathtt{R} \in \mathcal{T}(\check{\lambda},\mu^\circ) \\ \mathtt{T} \sim_{\mathrm{row}} \mathtt{R} \sim_{\mathrm{col}} \mathtt{S}}} (-1)^{\mathtt{RS}}.$$

Then, under the correspondence $\mathtt{S} \leftrightarrow \check{\mathtt{S}}$ between rows of $\boldsymbol{M}$ and rows of $\check{\boldsymbol{M}}$, the span of the columns of $\boldsymbol{M}$ equals the span of the columns of $\check{\boldsymbol{M}}$.

5 The intersection of $A_{\lambda\mu}$ with S^μ

In this section and the next, we complete the proof of Theorem 1.1 by showing that the map $\phi : A_{\lambda\mu} \to A_{\check{\lambda}\mu^\circ}$ also restricts to a linear isomorphism $(A_{\lambda\mu} \cap S^\mu) \to (A_{\check{\lambda}\mu^\circ} \cap R^{\mu^\circ})$, by using multipolytabloids to describe these intersections.

Recall the notation and definitions of 1.2.3. Notice that if $e_{\mathfrak{s}}^I$ is a rectified multipolytabloid, then the μ-tableau $\mathtt{T}_{\mathfrak{s}}$ is transpose semi-standard; every $\mathtt{T} \in \mathcal{T}_1(\mu, \lambda')$ arises in this way, and if two different

μ-tableaux $\mathfrak{s}_1$ and $\mathfrak{s}_2$ give the same element $\mathtt{T}_{\mathfrak{s}_1} = \mathtt{T}_{\mathfrak{s}_2}$ of $\mathcal{T}_1(\mu, \lambda')$, then $e^I_{\mathfrak{s}_1} = \pm e^I_{\mathfrak{s}_2}$. Given $\mathtt{T} \in \mathcal{T}_1(\mu, \lambda')$, we choose such an $\mathfrak{s}$, and write $e^{\mathtt{T}} = e^I_{\mathfrak{s}}$.

We would like to know the coefficient of each basis element $E_{\mathtt{S}}$ in a given multipolytabloid $e^{\mathtt{T}}$. Recall the tableau $\mathfrak{r}_{\mathtt{S}}$ from Section 3; the coefficient of $E_{\mathtt{S}}$ in any element a of $A_{\lambda\mu}$ equals the coefficient of $\{\mathfrak{r}_{\mathtt{S}}\}$ in a (when expressed as a linear combination of tabloids), and similarly for $E_{\check{\mathtt{S}}}$ and $\{\mathfrak{r}_{\check{\mathtt{S}}}\}$. So we examine the coefficients of the tabloids $\{\mathfrak{r}_{\mathtt{S}}\}$ in the multipolytabloids $e^{\mathtt{T}}$. The following lemma follows from the definitions.

Lemma 5.1. *Let $e^I_{\mathfrak{s}}$ be a multipolytabloid, with $\mathfrak{u}_1, \dots, \mathfrak{u}_a$ representatives of $\sim_{\mathrm{col}}$ classes as in 1.2.3. Then the coefficient of a μ-tabloid $\{\mathfrak{r}\}$ in $e^I_{\mathfrak{s}}$ equals*

$$\sum_{i=1}^{a} (-1)^{\mathfrak{s}\mathfrak{u}_i} \sum_{\mathfrak{u}_i \sim_{\mathrm{col}} \mathfrak{u} \sim_{\mathrm{row}} \mathfrak{r}} (-1)^{\mathfrak{u}\mathfrak{u}_i}.$$

We would like to express Lemma 5.1 using μ-tableaux of type λ'. Take $\mathtt{S} \in \mathcal{T}_{\mathrm{c}}(\lambda, \mu)$ and let $\mathfrak{r}_{\mathtt{S}}$ be as above; suppose also that $e^I_{\mathfrak{s}} = e^{\mathtt{T}}$ for $\mathtt{T} \in \mathcal{T}_1(\mu, \lambda')$. Notice that if $\mathfrak{u}_i \sim_{\mathrm{col}} \mathfrak{u} \sim_{\mathrm{row}} \mathfrak{r}_{\mathtt{S}}$, then $\mathtt{T} \sim_{\mathrm{col}} \mathtt{T}_{\mathfrak{u}} \sim_{\mathrm{row}} \mathtt{T}_{\mathfrak{r}_{\mathtt{S}}} = \acute{\mathtt{S}}$; moreover, if $\mathtt{T} \sim_{\mathrm{col}} \mathtt{U} \sim_{\mathrm{row}} \acute{\mathtt{S}}$, then there is a unique i and a unique $\mathfrak{u}$ such that $\mathfrak{u}_i \sim_{\mathrm{col}} \mathfrak{u} \sim_{\mathrm{row}} \mathfrak{r}_{\mathtt{S}}$ and $\mathtt{U} = \mathtt{T}_{\mathfrak{u}}$. In this case, we have $(-1)^{\mathtt{U}\acute{\mathtt{S}}} = (-1)^{\mathfrak{u}\mathfrak{r}_{\mathtt{S}}}$.

So we find that the coefficient of $\{\mathfrak{r}_{\mathtt{S}}\}$ in $e^{\mathtt{T}} = e^I_{\mathfrak{s}}$ is

$$(-1)^{\mathfrak{s}\mathfrak{r}_{\mathtt{S}}} \sum_{\substack{\mathtt{U} \in \mathcal{T}(\mu, \lambda') \\ \mathtt{T} \sim_{\mathrm{col}} \mathtt{U} \sim_{\mathrm{row}} \acute{\mathtt{S}}}} (-1)^{\mathtt{U}\acute{\mathtt{S}}}. \tag{1}$$

We perform the same calculation with $\check{\lambda}, \mu^\circ$ in place of λ, μ: for $\mathtt{S} \in \mathcal{T}_{\mathrm{c}}(\lambda, \mu)$, let $\mathfrak{r}_{\check{\mathtt{S}}}$ be the row standard μ°-tableau corresponding to $\check{\mathtt{S}}$, and for $\mathtt{T} \in \mathcal{T}_1(\mu, \lambda')$ let $e^{\check{\mathtt{T}}} = e^I_{\hat{\mathfrak{s}}}$ be the μ°-multipolytabloid corresponding to $\check{\mathtt{T}}$, with $\hat{\mathfrak{s}}$ some standard μ°-tableau. Then the coefficient of $\{\mathfrak{r}_{\check{\mathtt{S}}}\}$ in $e^{\check{\mathtt{T}}}$ is

$$(-1)^{\hat{\mathfrak{s}}\mathfrak{r}_{\check{\mathtt{S}}}} \sum_{\substack{\mathtt{U} \in \mathcal{T}(\mu^\circ, (\check{\lambda})') \\ \check{\mathtt{T}} \sim_{\mathrm{col}} \mathtt{U} \sim_{\mathrm{row}} \grave{\check{\mathtt{S}}}}} (-1)^{\mathtt{U}\grave{\check{\mathtt{S}}}}. \tag{2}$$

Now we are able to prove the main result of this section.

Proposition 5.2. *ϕ restricts to a linear isomorphism $(A_{\lambda\mu} \cap S^\mu) \to (A_{\check{\lambda}\mu^\circ} \cap R^{\mu^\circ})$.*

Proof. Let $\boldsymbol{N}$ be the matrix with columns indexed by transpose semi-standard μ-tableaux of type λ', and rows indexed by column standard λ-tableaux of type μ, and in which the entry in column $\mathtt{T}$ and row $\mathtt{S}$ is the coefficient of $E_{\mathtt{S}}$ in $e^{\mathtt{T}}$. Let $\check{\boldsymbol{N}}$ be the matrix with columns indexed by transpose semi-standard μ°-tableaux of type $(\check{\lambda})'$ and rows indexed by column standard $\check{\lambda}$-tableaux of type μ°, and in which the entry in column $\check{\mathtt{T}}$ and row $\check{\mathtt{S}}$ is the coefficient of $E_{\check{\mathtt{S}}}$ in $e^{\check{\mathtt{T}}}$. We need to show that, under the correspondence $\mathtt{S} \leftrightarrow \check{\mathtt{S}}$ between rows of $\boldsymbol{N}$ and rows of $\check{\boldsymbol{N}}$, the column spaces of $\boldsymbol{N}$ and $\check{\boldsymbol{N}}$ are equal.

We re-label the rows of $\boldsymbol{N}$, replacing $\mathtt{S} \in \mathcal{T}_{\mathrm{c}}(\lambda, \mu)$ with $\acute{\mathtt{S}} \in \mathcal{T}_{\mathrm{r}}(\mu, \lambda')$. Similarly, re-label the rows of $\check{\boldsymbol{N}}$, replacing $\mathtt{S}$ with $\grave{\mathtt{S}}$. By Lemma 3.1(5), the rows of $\boldsymbol{N}$ and $\check{\boldsymbol{N}}$ now correspond via $\mathtt{S} \leftrightarrow \check{\mathtt{S}}$.

Now choose and fix some $\mathtt{V} \in \mathcal{T}_{\mathrm{r}}(\mu, \lambda')$, and then for each $\mathtt{S} \in \mathcal{T}_{\mathrm{r}}(\mu, \lambda')$ multiply row $\mathtt{S}$ of $\boldsymbol{N}$ by $(-1)^{\mathfrak{r}_{\mathtt{V}}\mathfrak{r}_{\mathtt{S}}}$, and multiply row $\check{\mathtt{S}}$ of $\check{\boldsymbol{N}}$ by $(-1)^{\mathfrak{r}_{\check{\mathtt{V}}}\mathfrak{r}_{\check{\mathtt{S}}}}$. By Lemma 3.2, a row of $\boldsymbol{N}$ gets multiplied by the same factor as the corresponding row of $\check{\boldsymbol{N}}$, and so this operation does not affect whether the column spaces of $\boldsymbol{N}$ and $\check{\boldsymbol{N}}$ coincide. We also multiply the columns of $\boldsymbol{N}$ and $\check{\boldsymbol{N}}$ by ± 1 (which will not affect the column space): multiply column $\mathtt{T}$ of $\boldsymbol{N}$ by $(-1)^{\mathfrak{s}\mathfrak{r}_{\mathtt{V}}}$, where $e^{\mathtt{T}} = e^I_{\mathfrak{s}}$, and similarly for $\check{\boldsymbol{N}}$.

Using expressions (1) and (2), we find that after these operations the entry in column T and row S of $\boldsymbol{N}$ is

$$\sum_{\substack{\mathtt{U}\in\mathcal{T}(\mu,\lambda')\\ \mathtt{T}\sim_{\mathrm{col}}\mathtt{U}\sim_{\mathrm{row}}\mathtt{S}}} (-1)^{\mathtt{US}}$$

and the entry in column T and row S of $\check{\boldsymbol{N}}$ is

$$\sum_{\substack{\mathtt{U}\in\mathcal{T}(\mu^\circ,(\check{\lambda})')\\ \mathtt{T}\sim_{\mathrm{col}}\mathtt{U}\sim_{\mathrm{row}}\mathtt{S}}} (-1)^{\mathtt{US}}.$$

Now the fact that $\boldsymbol{N}$ and $\check{\boldsymbol{N}}$ have the same column space follows from Proposition 4.5 by taking the transposes of all tableaux. □

6 The proof of Theorem 1.1

Proof of Theorem 1.1. ϕ is a linear isomorphism $A_{\lambda\mu} \to A_{\check{\lambda}\mu^\circ}$ which restricts to linear isomorphisms

$$\Theta_{\lambda\mu} \longrightarrow \Theta_{\check{\lambda}\mu^\circ}$$

and

$$(A_{\lambda\mu} \cap S^\mu) \longrightarrow (A_{\check{\lambda}\mu^\circ} \cap R^{\mu^\circ}),$$

and hence to a linear isomorphism

$$(\Theta_{\lambda\mu} \cap S^\mu) \longrightarrow (\Theta_{\check{\lambda}\mu^\circ} \cap R^{\mu^\circ}).$$

But, as stated at the start of Section 4, we have

$$\dim_{\mathbb{F}}(\Theta_{\lambda\mu} \cap S^\mu) = \dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, S^\mu).$$

Similarly,

$$\dim_{\mathbb{F}}(\Theta_{\check{\lambda}\mu^\circ} \cap R^{\mu^\circ}) = \dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_{rs-n}}(S^{\check{\lambda}}, R^{\mu^\circ}).$$

But R^{μ° and $S^{\check{\mu}}$ are isomorphic $\mathbb{F}\mathfrak{S}_{rs-n}$-modules, are so we are done. □

We now deduce the row removal theorem in [4].

Corollary 6.1. [4, Proposition 4.1] *Suppose $\mathbb{F}$ is a field of odd characteristic and that λ and μ are partitions of n with $\lambda_1 = \mu_1$. Define $\bar{\lambda} = (\lambda_2, \lambda_3, \dots)$ and $\bar{\mu} = (\mu_2, \mu_3, \dots)$. Then*

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, S^\mu) = \dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_{n-\lambda_1}}(S^{\bar{\lambda}}, S^{\bar{\mu}}).$$

Proof. Take $r \geqslant \lambda'_1, \mu'_1$ and $s \geqslant \lambda_1$, and define $\check{\lambda}$ and $\check{\mu}$ as in Theorem 1.1. If we $\check{\bar{\lambda}}$ and $\check{\bar{\mu}}$ similarly, but using $r-1$ and s, then in fact we have $\check{\lambda} = \check{\bar{\lambda}}$ and $\check{\mu} = \check{\bar{\mu}}$. The result now follows from Theorem 1.1. □

We may deduce the column removal theorem similarly, but it seems unlikely that one can deduce the generalised version [4, Theorem 2.1].

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