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Row and column removal theorems for homomorphisms between Specht modules

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Abstract

We prove analogues of (Donkin's generalisations of) James's row and column removal theorems, in the context of homomorphisms between Specht modules for symmetric groups.

1 Introduction

In [4], James proves two theorems regarding the decomposition numbers for the symmetric groups; these concern the removal of a row or a column from a partition diagram. These results are generalised by Donkin in [2]. In trying to understand the structure of Specht modules, the homomorphism spaces $\mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, S^\mu)$ are of similar interest to the decomposition numbers, and here we prove analogues of Donkin's results in this context.

For the remainder of this section, we recall some basic results and establish notation. In Section 2 we state our main results, and examine their interdependence; it turns out that we need only prove the generalised row removal theorem, which we do in Section 3. Finally, in Section 4, we consider the representation of homomorphisms in terms of polytabloids, and attempt to describe our results in this way.

Similar results ought to be true for homomorphisms between Weyl modules for general linear groups. Certainly a simple column removal theorem is true (even in characteristic two), since adding a column to the Young diagram corresponds to tensoring with the determinant representation. Such a result would not immediately imply our results, however, since spaces of homomorphisms are not preserved under the Schur functor.

It is also unclear whether corresponding results hold for representations of finite general linear groups in non-defining characteristic; partial results are obtained in the second author's doctoral thesis [5].

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1.1 Definitions and notation

Throughout this paper, we assume that k is a field *of characteristic not two*, and we let $\mathfrak{S}_n$ denote the symmetric group on n letters. We are interested in the space of $k\mathfrak{S}_n$-homomorphisms between the Specht modules S^λ and S^μ, for λ and μ partitions of n.

We take our notation from James's book [3], from which we also recall some useful results. Recall that a *composition* of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers summing to n, and that a composition λ is a *partition* if $\lambda_1 \geqslant \lambda_2 \geqslant \dots$. Write $|\lambda| = n$. We define the Young diagram

$$[\lambda] = \{(i, j) \mid j \leqslant \lambda_i\} \subset \mathbb{N}^2.$$

Note that the Young diagram is conventionally drawn with i increasing down the page. For example, the Young diagram of the partition $(4, 2, 1)$ is

$$\begin{matrix}\times & \times & \times & \times \\ \times & \times & & \\ \times & & & \end{matrix}\,.$$

Accordingly, we shall refer to *lower* rows of $[\lambda]$ to mean those corresponding to higher values of i.

Let λ be a composition of n. A *λ-tableau* is a bijection from $[\lambda]$ to $\{1, \dots, n\}$ (or occasionally some other specified set of size n), which we shall regard as '$[\lambda]$ with the nodes replaced by the elements of $\{1, \dots, n\}$'. The row equivalence relation $\sim_{\mathrm{row}}$ is defined on the set of λ-tableaux in an obvious way, and column equivalence $\sim_{\mathrm{col}}$ is defined similarly. An equivalence class of λ-tableaux under $\sim_{\mathrm{row}}$ is called a *λ-tabloid*, and we denote by $\{t\}$ the tabloid containing t. The set of tableaux and the set of tabloids are acted upon in a natural way by $\mathfrak{S}_n$, and we let M^λ be the $k\mathfrak{S}_n$-module which has the set of λ-tabloids as a basis. If $x \in M^\lambda$ and $I \subseteq \{1, \dots, n\}$, we say that x is *alternating in the elements of I* if $x(a\ b) = -x$ for all $a, b \in I$ with $a < b$. If λ is a partition, then for a tableau t we define the *polytabloid*

$$e_t = \sum_{s \sim_{\mathrm{col}} t} (-1)^{st}\{s\} \in M^\lambda,$$

where $(-1)^{st}$ is the signature of the permutation sending s to t. We define the Specht module $S^\lambda \leqslant M^\lambda$ to be the k-span of all λ-polytabloids.

We are interested in homomorphisms between such modules. In order to define some of these, we extend our definition of tableaux, allowing them to have repeated entries. Specifically, if λ and μ are compositions of n, a *λ-tableau of type μ* is a diagram obtained by replacing the nodes of $[\lambda]$ with μ_1 1s, μ_2 2s, and so on. We let $\mathcal{T}(\lambda, \mu)$ denote the set of λ-tableaux of type μ. The relations $\sim_{\mathrm{row}}$ and $\sim_{\mathrm{col}}$ are defined in the obvious way on $\mathcal{T}(\lambda, \mu)$.

Let t be a fixed λ-tableau (of type (1^n)). Given a λ-tableau S of type μ, we define $d_t(S)$ to be any μ-tableau in which $i \in \{1, \dots, n\}$ appears in row j, where i and j occupy corresponding positions in t and S. This defines $d_t(S)$ up to $\sim_{\mathrm{row}}$ equivalence, and thus defines $\{d_t(S)\}$ uniquely.

Now, for each $T \in \mathcal{T}(\lambda, \mu)$, we define a homomorphism

$$\Theta_T : M^\lambda \longrightarrow M^\mu$$

by stipulating that

$$\{t\} \longmapsto \sum_{S \sim_{\mathrm{row}} T} \{d_t(S)\}$$

and extending linearly. Note that Θ_T only depends on the $\sim_{\mathrm{row}}$ equivalence class of T.

A particular set of such homomorphisms gives us a new characterisation of the Specht module S^λ, which will prove very useful. Let d be a positive integer, and choose t such that $0 \leqslant t < \lambda_{d+1}$. Define ν by

$$\nu_i = \begin{cases} \lambda_i + \lambda_{i+1} - t & (i = d) \\ t & (i = d+1) \\ \lambda_i & (\text{otherwise}). \end{cases}$$

Let T be the λ-tableau with all entries in row i equal to i, except for $i = d+1$, when there are $\lambda_{d+1} - t$ entries equal to d and t entries equal to $d+1$. T is then a λ-tableau of type ν, and we write $\psi_{d,t}$ for the homomorphism $\Theta_T : M^\lambda \to M^\nu$. (We may omit to mention ν, since it is determined by λ, d and t.) We then have the following.

Theorem 1.1 ([3], Corollary 17.18)**.** *If λ is a partition of n, then*

$$S^\lambda = \bigcap_{d \geqslant 1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker(\psi_{d,t}).$$

We are also interested in homomorphisms from S^λ, when λ is a partition. Accordingly, we let $\hat{\Theta}_T : S^\lambda \to M^\mu$ be the restriction of Θ_T to S^λ. We say that a λ-tableau T of type μ is *semistandard* if its entries are non-decreasing along each row and strictly increasing down each column. We let $\mathcal{T}_0(\lambda, \mu)$ denote the set of semistandard λ-tableaux of type μ. For example, we have

$$\mathcal{T}_0((5,2),(3,2,2)) = \left\{ \begin{smallmatrix} 1&1&1&2&2 \\ 3&3 \end{smallmatrix}, \begin{smallmatrix} 1&1&1&2&3 \\ 2&3 \end{smallmatrix}, \begin{smallmatrix} 1&1&1&3&3 \\ 2&2 \end{smallmatrix} \right\}.$$

The following result is originally due to Carter and Lusztig [1].

Theorem 1.2 ([3], Theorem 13.13)**.** *If λ is a partition of n and μ a composition of n, and if k is a field of characteristic not two, then the set $\{\hat{\Theta}_T \mid T \in \mathcal{T}_0(\lambda,\mu)\}$ is a basis for* $\mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, M^\mu)$.

This has the following corollary, which we shall use. Recall that, for compositions λ and μ, we say that λ *dominates* μ (and write $\lambda \trianglerighteq \mu$) if

$$\sum_{i=1}^{j} \lambda_i \geqslant \sum_{i=1}^{j} \mu_i$$

for all j.

Corollary 1.3. *If λ is a partition of n, and μ a composition of n with $\lambda \not\trianglerighteq \mu$, then* $\mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, M^\mu) = 0$.

Proof. It is easily seen that there are no semistandard λ-tableaux of type μ unless $\lambda \trianglerighteq \mu$. □

We end this introduction with a few items of notation. For a partition λ of n, λ' will denote the conjugate partition, that is,

$$\lambda'_i = \max\{j \mid \lambda_j \geqslant i\}.$$

M^* will denote the dual of any module M, and sgn will denote the one-dimensional signature representation of $k\mathfrak{S}_n$. All tensor products will be taken over k.

2 The main results

Our main theorem is as follows.

Theorem 2.1. *Let k be any field with characteristic not equal to two. Suppose λ^{R} and μ^{R} are partitions of l, each with at most s non-zero parts, and λ^{B} and μ^{B} are partitions of m with $\lambda^{\mathrm{B}}_1, \mu^{\mathrm{B}}_1 \leqslant r$. Let $n = rs + l + m$, and define the partitions*

$$\lambda = (r + \lambda^{\mathrm{R}}_1, \dots, r + \lambda^{\mathrm{R}}_s, \lambda^{\mathrm{B}}_1, \lambda^{\mathrm{B}}_2, \dots),$$
$$\mu = (r + \mu^{\mathrm{R}}_1, \dots, r + \mu^{\mathrm{R}}_s, \mu^{\mathrm{B}}_1, \mu^{\mathrm{B}}_2, \dots).$$

Then, as k-vector spaces,

$$\mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, S^\mu) \cong \mathrm{Hom}_{k\mathfrak{S}_l}(S^{\lambda^{\mathrm{R}}}, S^{\mu^{\mathrm{R}}}) \otimes \mathrm{Hom}_{k\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}}).$$

In fact, we shall prove the following two theorems, which are direct analogues of Donkin's results.

Theorem 2.2. *Let k be any field with characteristic not equal to two. Let λ and μ be partitions of n, and suppose that for some s we have*

$$\lambda_1 + \dots + \lambda_s = \mu_1 + \dots + \mu_s.$$

Define

$$\lambda^{\mathrm{T}} = (\lambda_1, \dots, \lambda_s),$$
$$\lambda^{\mathrm{B}} = (\lambda_{s+1}, \lambda_{s+2}, \dots),$$
$$\mu^{\mathrm{T}} = (\mu_1, \dots, \mu_s),$$
$$\mu^{\mathrm{B}} = (\mu_{s+1}, \mu_{s+2}, \dots).$$

Then, putting $m = |\lambda^{\mathrm{B}}| = |\mu^{\mathrm{B}}|$, we have

$$\mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, S^\mu) \cong \mathrm{Hom}_{k\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}}) \otimes \mathrm{Hom}_{k\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$$

as k-vector spaces.

Theorem 2.3. *Let k be any field with characteristic not equal to two. Let λ and μ be partitions of n, and suppose that for some r we have*

$$\lambda'_1 + \dots + \lambda'_r = \mu'_1 + \dots + \mu'_r.$$

Define

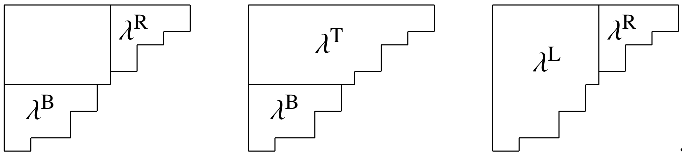
$$\lambda^{\mathrm{L}} = (\min(\lambda_1, r), \min(\lambda_2, r), \dots),$$
$$\lambda^{\mathrm{R}} = (\max(\lambda_1 - r, 0), \max(\lambda_2 - r, 0), \dots),$$
$$\mu^{\mathrm{L}} = (\min(\mu_1, r), \min(\mu_2, r), \dots),$$
$$\mu^{\mathrm{R}} = (\max(\mu_1 - r, 0), \max(\mu_2 - r, 0), \dots).$$

Then, putting $l = |\lambda^{\mathrm{R}}| = |\mu^{\mathrm{R}}|$, we have

$$\mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, S^\mu) \cong \mathrm{Hom}_{k\mathfrak{S}_{n-l}}(S^{\lambda^{\mathrm{L}}}, S^{\mu^{\mathrm{L}}}) \otimes \mathrm{Hom}_{k\mathfrak{S}_l}(S^{\lambda^{\mathrm{R}}}, S^{\mu^{\mathrm{R}}})$$

as k-vector spaces.

The partitions $\lambda^{\mathrm{R}}, \lambda^{\mathrm{L}}, \lambda^{\mathrm{T}}, \lambda^{\mathrm{B}}$ appearing in these theorems may be viewed pictorially as follows:



Proposition 2.4. *Theorems 2.2 and 2.3 together are equivalent to Theorem 2.1.*

Proof. First of all, assume Theorem 2.1, and suppose that λ^{T}, μ^{T}, λ^{B} and μ^{B} are as in Theorem 2.2. If either $\lambda^{\mathrm{T}}_s > \mu^{\mathrm{T}}_s$ or $\lambda^{\mathrm{B}}_1 < \mu^{\mathrm{B}}_1$, then $\lambda \ntrianglerighteq \mu$ and either $\lambda^{\mathrm{T}} \ntrianglerighteq \mu^{\mathrm{T}}$ or $\lambda^{\mathrm{B}} \ntrianglerighteq \mu^{\mathrm{B}}$, so that both vector spaces in Theorem 2.2 are zero. So we may assume $\lambda^{\mathrm{T}}_s \leqslant \mu^{\mathrm{T}}_s$ and $\lambda^{\mathrm{B}}_1 \geqslant \mu^{\mathrm{B}}_1$. Put $r = \lambda^{\mathrm{T}}_s$, $l = n - m - rs$, and define

$$\lambda^{\mathrm{R}} = (\lambda^{\mathrm{T}}_1 - r, \dots, \lambda^{\mathrm{T}}_s - r), \qquad \mu^{\mathrm{R}} = (\mu^{\mathrm{T}}_1 - r, \dots, \mu^{\mathrm{T}}_s - r);$$

then $\lambda^{\mathrm{R}}, \mu^{\mathrm{R}}, \lambda^{\mathrm{B}}, \mu^{\mathrm{B}}$ satisfy the conditions of Theorem 2.1. So we have

$$\mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, S^\mu) \cong \mathrm{Hom}_{k\mathfrak{S}_l}(S^{\lambda^{\mathrm{R}}}, S^{\mu^{\mathrm{R}}}) \otimes \mathrm{Hom}_{k\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$$

by Theorem 2.1, and we need only show that

$$\mathrm{Hom}_{k\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}}) \cong \mathrm{Hom}_{k\mathfrak{S}_l}(S^{\lambda^{\mathrm{R}}}, S^{\mu^{\mathrm{R}}}).$$

But Theorem 2.1 implies (replacing λ^{B} and μ^{B} with the partition of zero) that

$$\begin{aligned}\mathrm{Hom}_{k\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}}) &\cong \mathrm{Hom}_{k\mathfrak{S}_l}(S^{\lambda^{\mathrm{R}}}, S^{\mu^{\mathrm{R}}}) \otimes \mathrm{Hom}_{k\mathfrak{S}_0}(S^{(0)}, S^{(0)})\\ &\cong \mathrm{Hom}_{k\mathfrak{S}_l}(S^{\lambda^{\mathrm{R}}}, S^{\mu^{\mathrm{R}}}) \otimes k\\ &\cong \mathrm{Hom}_{k\mathfrak{S}_l}(S^{\lambda^{\mathrm{R}}}, S^{\mu^{\mathrm{R}}}).\end{aligned}$$

Theorem 2.3 is deduced from Theorem 2.1 similarly.

Conversely, assume Theorems 2.2 and 2.3 and the conditions of Theorem 2.1. The conditions of Theorem 2.2 hold, with

$$\lambda^{\mathrm{T}} = (\lambda^{\mathrm{R}}_1 + r, \dots, \lambda^{\mathrm{R}}_s + r), \qquad \mu^{\mathrm{T}} = (\mu^{\mathrm{R}}_1 + r, \dots, \mu^{\mathrm{R}}_s + r);$$

by Theorem 2.3 (with $\lambda^{\mathrm{L}} = \mu^{\mathrm{L}} = (r^s)$) we have

$$\mathrm{Hom}_{k\mathfrak{S}_l}(S^{\lambda^{\mathrm{R}}}, S^{\mu^{\mathrm{R}}}) \cong \mathrm{Hom}_{k\mathfrak{S}_{l+rs}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}});$$

now Theorem 2.2 gives the conclusion of Theorem 2.1. □

Fortunately (and unlike the corresponding results for decomposition numbers), we only need to prove one of Theorems 2.2 and 2.3.

Proposition 2.5. *Theorems 2.2 and 2.3 are equivalent.*

We need the following result from [3].

Proposition 2.6 ([3], Theorem 8.15)**.** *Let* sgn *denote the alternating representation of the symmetric group. Then, for any* λ *and any field k,*

$$S^{\lambda'} \otimes \mathrm{sgn} \cong (S^{\lambda})^*.$$

From this we prove the following.

Corollary 2.7. *For any partitions* λ *and* μ *we have*

$$\mathrm{Hom}_{k\mathfrak{S}_n}(S^{\lambda'}, S^{\mu'}) \cong \mathrm{Hom}_{k\mathfrak{S}_n}(S^{\mu}, S^{\lambda}).$$

Proof. Since sgn is one-dimensional, we have

$$\mathrm{Hom}_{k\mathfrak{S}_n}(M \otimes \mathrm{sgn}, N \otimes \mathrm{sgn}) \cong \mathrm{Hom}_{k\mathfrak{S}_n}(M, N)$$

for any modules M, N. So we have

$$\begin{aligned}\mathrm{Hom}_{k\mathfrak{S}_n}(S^{\lambda'}, S^{\mu'}) &\cong \mathrm{Hom}_{k\mathfrak{S}_n}(S^{\lambda'} \otimes \mathrm{sgn}, S^{\mu'} \otimes \mathrm{sgn})\\ &\cong \mathrm{Hom}_{k\mathfrak{S}_n}((S^{\lambda})^*, (S^{\mu})^*)\\ &\cong \mathrm{Hom}_{k\mathfrak{S}_n}(S^{\mu}, S^{\lambda}).\end{aligned}$$

□

Proof of Proposition 2.5. If λ and μ satisfy the conditions of Theorem 2.2, then λ' and μ' satisfy the conditions of Theorem 2.3, with $r = s$, $l = m$ and with

$$\begin{aligned}(\lambda')^{\mathrm{L}} &= (\lambda^{\mathrm{T}})',\\ (\lambda')^{\mathrm{R}} &= (\lambda^{\mathrm{B}})',\\ (\mu')^{\mathrm{L}} &= (\mu^{\mathrm{T}})',\\ (\mu')^{\mathrm{R}} &= (\mu^{\mathrm{B}})'.\end{aligned}$$

So we have

$$\mathrm{Hom}_{k\mathfrak{S}_n}(S^{\mu'}, S^{\lambda'}) \cong \mathrm{Hom}_{k\mathfrak{S}_n}(S^{\lambda}, S^{\mu})$$

and

$$\mathrm{Hom}_{k\mathfrak{S}_{n-m}}(S^{\mu'^{\mathrm{L}}}, S^{\lambda'^{\mathrm{L}}}) \otimes \mathrm{Hom}_{k\mathfrak{S}_m}(S^{\mu'^{\mathrm{R}}}, S^{\lambda'^{\mathrm{R}}}) \cong \mathrm{Hom}_{k\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}}) \otimes \mathrm{Hom}_{k\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$$

by Corollary 2.7. □

Remark. It is not clear to what extent our results hold in characteristic two. The crucial point is the failure of Theorem 1.2, but in fact some our later arguments (in the proofs of Propositions 3.2 and 3.4) rely on the characteristic not being two. An unmodified version of Theorem 2.1 in characteristic two is false: put $r = 1, s = 2$ and

$$\lambda^{\mathrm{R}} = (1, 1), \qquad \mu^{\mathrm{R}} = (2), \qquad \lambda^{\mathrm{B}} = \mu^{\mathrm{B}} = (0).$$

Then

$$\mathrm{Hom}_{\mathbb{F}_2\mathfrak{S}_4}(S^{\lambda}, S^{\mu}) = \mathrm{Hom}_{\mathbb{F}_2\mathfrak{S}_4}(S^{(2,2)}, S^{(3,1)}) = 0,$$

while

$$\mathrm{Hom}_{\mathbb{F}_2\mathfrak{S}_2}(S^{(1,1)}, S^{(2)}) \cong \mathrm{Hom}_{\mathbb{F}_2\mathfrak{S}_0}(S^{(0)}, S^{(0)}) \cong \mathbb{F}_2.$$

However, the authors suspect that Theorem 2.1 holds in characteristic two when λ is 2-regular.

3 The proof of theorem 2.2

We shall prove Theorem 2.2 using Theorem 1.1: given homomorphisms $V : S^{\lambda^{\mathrm{T}}} \to S^{\mu^{\mathrm{T}}}$ and $W : S^{\lambda^{\mathrm{B}}} \to S^{\mu^{\mathrm{B}}}$, we shall define a homomorphism $(V, W) : S^{\lambda} \to M^{\mu}$ and show that $(V, W)\psi_{d,t} = 0$ for all d and t, so that the image of (V, W) lies in S^{μ}; we proceed similarly in the other direction.

Let λ be a partition of n and ν a composition of n with $\lambda_1 + \cdots + \lambda_s = \nu_1 + \cdots + \nu_s = n - m$. Define λ^{T} and λ^{B} as in Theorem 2.2, and ν^{T} and ν^{B} analogously.

Definition. Suppose that $R \in \mathcal{T}(\lambda^{\mathrm{T}}, \nu^{\mathrm{T}})$ has entry x^i_j in position i of row j and $S \in \mathcal{T}(\lambda^{\mathrm{B}}, \nu^{\mathrm{B}})$ has entry y^i_j in position i of row j. Define (R, S) to be the λ-tableau of type ν with entry z^i_j in position i of row j, where

$$z^i_j = \begin{cases} x^i_j & (j \leqslant s) \\ y^i_j + s & (j > s). \end{cases}$$

For example, if

$$R = \begin{matrix} 1&1&1&1&1&2 \\ 2&2&2&3&& \\ 3&3&3&&& \end{matrix}, \qquad S = \begin{matrix} 1&1&2 \\ 2&3& \\ 3&& \end{matrix},$$

then

$$(R, S) = \begin{matrix} 1&1&1&1&1&2 \\ 2&2&2&3&& \\ 3&3&3&&& \\ 4&4&5&&& \\ 5&6&&&& \\ 6&&&&& \end{matrix}.$$

Lemma 3.1. *If R and S are semistandard, then so is (R, S), and $\{(R, S) \mid R \in \mathcal{T}_0(\lambda^{\mathrm{T}}, \nu^{\mathrm{T}}), S \in \mathcal{T}_0(\lambda^{\mathrm{B}}, \nu^{\mathrm{B}})\}$ is precisely the set of semistandard λ-tableaux of type ν.*

Proof. The first statement is clear. Suppose T is a semistandard λ-tableau of type ν with entry z^i_j in position i of row j. Then the entries of T strictly increase down the columns, so that for $j > s$ we have that $z^i_j > s$. Since the number $|\nu^{\mathrm{T}}|$ of entries less than or equal to s equals the number $|\lambda^{\mathrm{T}}|$ of positions in rows $1, \ldots, s$, it also is also true that $z^i_j \leqslant s$ for $j \leqslant s$. So the top s rows of T constitute a λ^{T}-tableau of type ν^{T} (which is clearly semistandard), and a similar statement holds for the lower rows. □

Definition. Suppose

$$V = \sum_{R \in \mathcal{T}_0(\lambda^{\mathrm{T}}, \nu^{\mathrm{T}})} a_R \hat{\Theta}_R \in \mathrm{Hom}_{k\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, M^{\nu^{\mathrm{T}}})$$

and

$$W = \sum_{S \in \mathcal{T}_0(\lambda^{\mathrm{B}}, \nu^{\mathrm{B}})} b_S \hat{\Theta}_S \in \mathrm{Hom}_{k\mathfrak{S}_{m}}(S^{\lambda^{\mathrm{B}}}, M^{\nu^{\mathrm{B}}}).$$

Define $(V, W) \in \mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, M^\nu)$ by

$$(V, W) = \sum_R \sum_S a_R b_S \hat{\Theta}_{(R,S)}.$$

Proposition 3.2. *Suppose $U \in \mathcal{T}(\lambda^{\mathrm{T}}, \nu^{\mathrm{T}})$ and $S \in \mathcal{T}_0(\lambda^{\mathrm{B}}, \nu^{\mathrm{B}})$. Then*

$$\hat{\Theta}_{(U,S)} = (\hat{\Theta}_U, \hat{\Theta}_S).$$

In order to prove this, we need a lemma.

Lemma 3.3. *Let t be a λ-tableau, $\pi \in \mathfrak{S}_n$ an element of the column stabiliser of t, and $\theta : S^\lambda \to M$ any $k\mathfrak{S}_n$-homomorphism. Then*

$$e_t\theta\pi = \epsilon(\pi)e_t\theta.$$

Proof. We have $e_{t\pi} = \epsilon(\pi)e_t$, so that

$$\begin{aligned}\epsilon(\pi)e_t\theta &= (e_{t\pi})\theta\\ &= (e_t\pi)\theta\\ &= e_t\theta\pi\end{aligned}$$

since θ is a $k\mathfrak{S}_n$-homomorphism. □

Proof of Proposition 3.2. By Theorem 1.2, we may write

$$\hat{\Theta}_U = \sum_{i=1}^{a} c_i\hat{\Theta}_{R_i}$$

with $R_1, \dots, R_a$ semistandard. We then need to prove that

$$\hat{\Theta}_{(U,S)} = \sum_{i=1}^{a} c_i\hat{\Theta}_{(R_i,S)}.$$

We apply both homomorphisms to an arbitrary λ-polytabloid e_t, and compare coefficients of an arbitrary ν-tabloid $\{u\}$. Note that if π is in the column stabiliser of t, then

$$(e_t\hat{\Theta}_{(U,S)})\pi = (\epsilon(\pi))e_t\hat{\Theta}_{(U,S)}$$

and

$$(e_t(\hat{\Theta}_U, \hat{\Theta}_S))\pi = (\epsilon(\pi))e_t(\hat{\Theta}_U, \hat{\Theta}_S),$$

by Lemma 3.3.

Suppose first that there are distinct numbers a and b in the same row of u and the same column of t. Then the coefficient of $\{u\}$ in the image of e_t under both maps is equal to the coefficient of $\{u\}$ in the image of $e_{t(a\,b)}$, and hence must be zero. We may therefore assume that the entries of any column of t occur in distinct rows of u.

For each $c \geqslant 1$, let $t_{c,1}, t_{c,2}, \dots$ be the numbers in column c of t, with $t_{c,1} < t_{c,2} < \dots$. Then there is a unique element π of the column stabiliser of t such that, in $u\pi$, the entry $t_{c,i+1}$ occurs in a lower row than $t_{c,i}$ for any i and any c. This means that none of the entries $t_{c,s+1}, t_{c,s+2}, \dots$ can occur in rows $1, \dots, s$. So rows $1, \dots, s$ of $u\pi$ contain only entries $t_{c,i}$ for $i \leqslant s$; but the number of such entries is $|\lambda^{\mathrm{T}}|$, and so all $t_{c,i}$ with $i \leqslant s$ occur in rows $1, \dots, s$ of $u\pi$. By replacing u with $u\pi$, we may simply assume that u has this property.

Let t^{T} be the λ^{T}-tableau consisting of the first s rows of t, and t^{B} the λ^{B}-tableau consisting of the remaining rows, and define u^{T} and u^{B} similarly. Then $\{u^{\mathrm{T}}\}$ is a tabloid with entries in the same set as t^{T}, and $\{u^{\mathrm{B}}\}$ is a tabloid with entries in the same set as t^{B}. So the coefficient of $\{u\}$ in $e_t\hat{\Theta}_{(U,S)}$ equals

$$\begin{aligned}
&(\text{coefficient of } \{u^{\mathrm{T}}\} \text{ in } e_{t^{\mathrm{T}}}\hat{\Theta}_U) \times (\text{coefficient of } \{u^{\mathrm{B}}\} \text{ in } e_{t^{\mathrm{B}}}\hat{\Theta}_S)\\
&= \sum_{i=1}^{a} c_i(\text{coefficient of } \{u^{\mathrm{T}}\} \text{ in } e_{t^{\mathrm{T}}}\hat{\Theta}_{R_i}) \times (\text{coefficient of } \{u^{\mathrm{B}}\} \text{ in } e_{t^{\mathrm{B}}}\hat{\Theta}_S)\\
&= \text{coefficient of } \{u\} \text{ in } e_t\left(\sum_{i=1}^{a} c_i\hat{\Theta}_{(R_i,S)}\right).
\end{aligned}$$

□

A similar argument proves the following.

Proposition 3.4. *Suppose $U \in \mathcal{T}_0(\lambda^{\mathrm{T}}, \nu^{\mathrm{T}})$ and $S \in \mathcal{T}(\lambda^{\mathrm{B}}, \nu^{\mathrm{B}})$. Then*

$$\hat{\Theta}_{(U,S)} = (\hat{\Theta}_U, \hat{\Theta}_S).$$

Now we suppose that λ and μ are as in Theorem 2.2.

Lemma 3.5. *If $V \in \mathrm{Hom}_{k\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, M^{\mu^{\mathrm{T}}})$ and $W \in \mathrm{Hom}_{k\mathfrak{S}_n}(S^{\lambda^{\mathrm{B}}}, M^{\mu^{\mathrm{B}}})$, then*

$$(V, W)\psi_{d,t} = \begin{cases} (V\psi_{d,t}, W) & (d < s)\\ (V, W\psi_{d-s,t}) & (d > s).\end{cases}$$

Proof. We deal with the case where $d < s$; the other case follows similarly. By linearity, we need only show that

$$(\hat{\Theta}_R, \hat{\Theta}_S)\psi_{d,t} = (\hat{\Theta}_R\psi_{d,t}, \hat{\Theta}_S)$$

for $R \in \mathcal{T}_0(\lambda^{\mathrm{T}}, \mu^{\mathrm{T}})$, $S \in \mathcal{T}_0(\lambda^{\mathrm{B}}, \mu^{\mathrm{B}})$. Let $U_{d,t}$ be the set of row standard λ^{T}-tableaux which may be obtained from R by replacing $\lambda_{d+1} - t$ entries equal to $d+1$ with ds. Let $U \in U_{d,t}$ and suppose that U contains a_{Ui} entries equal to d in row i while R contains b_i entries equal to d in row i. Then

$$(\hat{\Theta}_R\psi_{d,t}, \hat{\Theta}_S) = \left(\sum_{U \in U_{d,t}} \prod_i \binom{a_{Ui}}{b_i}\hat{\Theta}_U, \hat{\Theta}_S\right) = \sum_{U \in U_{d,t}} \prod_i \binom{a_{Ui}}{b_i}\hat{\Theta}_{(U,S)}$$

by Proposition 3.2 and

$$(\hat{\Theta}_R, \hat{\Theta}_S)\psi_{d,t} = \hat{\Theta}_{(R,S)}\psi_{d,t} = \sum_{U \in U_{d,t}} \prod_i \binom{a_{Ui}}{b_i}\hat{\Theta}_{(U,S)}$$

as required. □

Lemma 3.6. *Suppose* $V \in \mathrm{Hom}_{k\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}})$ *and* $W \in \mathrm{Hom}_{k\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$. *Then* $(V, W) \in \mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, S^\mu)$.

Proof. We have that $(V, W) \in \mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, S^\mu)$ if $(V, W)\psi_{d,t} = 0$ for all d and t. Let ν be the composition such that $\psi_{d,t} : M^\mu \to M^\nu$.

If $d = s$ then (since $\lambda_1 + \cdots + \lambda_s = \mu_1 + \cdots + \mu_s$) we have $\lambda \ntrianglerighteq \nu$ and so $\mathrm{Hom}(S^\lambda, M^\nu) = 0$. Hence $(V, W)\psi_{d,t} = 0$.

If $d < s$ then by Lemma 3.5 we have

$$(V, W)\psi_{d,t} = (V\psi_{d,t}, W),$$

which is zero, since the image of V lies in $S^{\mu^{\mathrm{T}}}$. Similarly for $d > s$. □

Definition. Let

$$\phi : \mathrm{Hom}_{k\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, M^{\mu^{\mathrm{T}}}) \otimes \mathrm{Hom}_{k\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, M^{\mu^{\mathrm{B}}}) \longrightarrow \mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, M^\mu)$$

be given by

$$V \otimes W \longmapsto (V, W).$$

Note that ϕ is a linear bijection, by Lemma 3.1 and Proposition 3.2.

Theorem 3.7. *Let* $\hat{\phi}$ *denote the restriction of* ϕ *to* $\mathrm{Hom}_{k\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}}) \otimes \mathrm{Hom}_{k\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$. *Then* $\hat{\phi}$ *is a bijection between* $\mathrm{Hom}_{k\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}}) \otimes \mathrm{Hom}_{k\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$ *and* $\mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, S^\mu)$.

Proof. Lemma 3.6 shows that if $V \otimes W \in \mathrm{Hom}_{k\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}}) \otimes \mathrm{Hom}_{k\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$ then $(V \otimes W)\hat{\phi} \in \mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, S^\mu)$. Also since ϕ is injective, we have that $\hat{\phi}$ is injective. It remains only to show that $\hat{\phi}$ is surjective.

Suppose that $\{A_i \mid i \in I\}$ is a basis of $\mathrm{Hom}_{k\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}})$ and that $\{B_j \mid j \in J\}$ is a basis of $\mathrm{Hom}_{k\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$.

Let

$$Z = \sum_{R \in \mathcal{T}_0(\lambda^{\mathrm{T}}, \mu^{\mathrm{T}})} \sum_{S \in \mathcal{T}_0(\lambda^{\mathrm{B}}, \mu^{\mathrm{B}})} c_{R,S}(\hat{\Theta}_R, \hat{\Theta}_S) \in \mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, S^\mu).$$

Then $\left(\sum_R \sum_S c_{R,S}(\hat{\Theta}_R, \hat{\Theta}_S)\right)\psi_{d,t} = 0$ for all d. When $d > s$, this gives

$$\sum_R \sum_S (\hat{\Theta}_R, c_{R,S}\hat{\Theta}_S\psi_{d-s,t}) = 0.$$

Since ϕ is injective, we get

$$\sum_S c_{R,S}\hat{\Theta}_S\psi_{d-s,t} = 0$$

for every R, so that $\sum_S c_{R,S}\hat{\Theta}_S \in \mathrm{Hom}_{k\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$ for every R. Writing

$$\sum_S c_{R,S}\hat{\Theta}_S = \sum_{j \in J} \alpha_{R,j} B_j$$

for $\alpha_{R,j} \in k$, we have

$$\left(\sum_{j\in J}\sum_{R} \alpha_{R,j}(\hat{\Theta}_R, B_j)\right)\psi_{d,t} = 0$$

for all $d < s$. The injectivity of ϕ gives

$$\sum_{R} \alpha_{R,j}\hat{\Theta}_R\psi_{d,t} = 0$$

for every j when $d < s$, so that

$$\sum_{R} \alpha_{R,j}\hat{\Theta}_R \in \mathrm{Hom}_{k\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}}).$$

Writing $\sum_R \alpha_{R,j}\hat{\Theta}_R = \sum_{i\in I} \beta_{i,j}A_i$ for each j, we have

$$Z = \sum_{i\in I}\sum_{j\in J} \beta_{i,j}(A_i, B_j) = Z'\hat{\phi},$$

where

$$Z' = \sum_{i,j} \beta_{i,j}(A_i \otimes B_j).$$

This completes the proof of Theorem 2.2, and hence of Theorem 2.1. □

4 Expressing homomorphisms in terms of polytabloids

Given a homomorphism $\theta : S^{\lambda} \to S^{\mu}$, it is often useful to have a description of the action of θ in terms of polytabloids, i.e.

$$e_t \longmapsto \sum_{s} c_s e_s$$

for a λ-tableau t, and μ-tableaux s. We address this for Theorem 2.1; in particular, given such descriptions of homomorphisms $S^{\lambda^{\mathrm{R}}} \to S^{\mu^{\mathrm{R}}}$ and $S^{\lambda^{\mathrm{B}}} \to S^{\mu^{\mathrm{B}}}$, we provide one for the corresponding homomorphism $S^{\lambda} \to S^{\mu}$. We do this in two stages: first for Theorem 2.3 and then for the following special case of Theorem 2.2, which may be regarded as an ordinary row removal theorem for homomorphisms.

Proposition 4.1. *Let ν and ξ be partitions of m with $\nu_1, \xi_1 \leqslant r$, and define*

$$\overline{\nu} = (r, \nu_1, \nu_2, \dots), \qquad \overline{\xi} = (r, \xi_1, \xi_2, \dots).$$

Then

$$\mathrm{Hom}_{k\mathfrak{S}_{m+r}}(S^{\overline{\nu}}, S^{\overline{\xi}}) \cong \mathrm{Hom}_{k\mathfrak{S}_m}(S^{\nu}, S^{\xi}).$$

It is easily shown (by modifying the proof of Proposition 2.4) that Theorem 2.3 and Proposition 4.1 are together equivalent to Theorem 2.1.

We begin by addressing Theorem 2.3, which is fairly easily dealt with. Let λ^{L}, λ^{R}, μ^{L} and μ^{R} be as in Theorem 2.3. Given a λ^{L}-tableau T^{L} and a λ^{R}-tableau T^{R} (with any entries), define the λ-tableau $T^{\mathrm{L}}|T^{\mathrm{R}}$ simply by juxtaposing T^{L} and T^{R}. For λ^{L}- and λ^{R}-tabloids u^{L} and u^{R}, define $u^{\mathrm{L}}|u^{\mathrm{R}}$ similarly, and extend this to the whole of M^λ, so that if $x^{\mathrm{L}} = \sum_{u^{\mathrm{L}}} c_{u^{\mathrm{L}}} u^{\mathrm{L}}$ is a linear combination of λ^{L}-tabloids on $\{1,\dots,n-l\}$, and $x^{\mathrm{R}} = \sum_{u^{\mathrm{R}}} c_{u^{\mathrm{R}}} u^{\mathrm{R}}$ is a linear combination of λ^{R}-tabloids on $\{n-l+1,\dots,n\}$, then we define

$$x^{\mathrm{L}}|x^{\mathrm{R}} = \sum_{u^{\mathrm{L}},u^{\mathrm{R}}} c_{u^{\mathrm{L}}} c_{u^{\mathrm{R}}} u^{\mathrm{L}}|u^{\mathrm{R}};$$

we make similar definitions for μ^{L}- and μ^{R}-tableaux and tabloids. It is then clear that for a μ^{L}-tableau t^{L} on $\{1,\dots,n-l\}$, and a μ^{R}-tableau t^{R} on $\{n-l+1,\dots,n\}$, we have

$$e_{t^{\mathrm{L}}|t^{\mathrm{R}}} = e_{t^{\mathrm{L}}}|e_{t^{\mathrm{R}}}.$$

We now assume that $\lambda^{\mathrm{L}} \trianglerighteq \mu^{\mathrm{L}}$ and $\lambda^{\mathrm{R}} \trianglerighteq \mu^{\mathrm{R}}$; there is no loss in doing this, since otherwise all the homomorphism spaces involved are zero. We require the following lemma.

Lemma 4.2. *Suppose T^{L} is a semistandard λ^{L}-tableau of type μ^{L} and that T^{R} is a semistandard λ^{R}-tableau of type μ^{R}. Suppose also that S is a row permutation of $T^{\mathrm{L}}|T^{\mathrm{R}}$ in which each column has distinct entries. Then $S = U^{\mathrm{L}}|U^{\mathrm{R}}$ for some row permutations U^{L} and U^{R} of T^{L} and T^{R} respectively.*

Proof. Let s be the number of non-zero parts of μ^{R}. Then, since we are assuming $\lambda^{\mathrm{L}} \trianglerighteq \mu^{\mathrm{L}}$ and $\lambda^{\mathrm{R}} \trianglerighteq \mu^{\mathrm{R}}$, we have

$$\lambda^{\mathrm{L}}_1 = \dots = \lambda^{\mathrm{L}}_s = \mu^{\mathrm{L}}_1 = \dots = \mu^{\mathrm{L}}_s = r,$$

and we know that the number of non-zero parts of λ^{R} is at most s.

So, for $1 \leqslant i \leqslant s$, every entry of the ith row of T^{L} equals i. We must therefore show that the ith row of S begins with r entries equal to i, for all such i. Suppose we have shown this for $i = s, s-1, \dots, j+1$, and consider row j. The entries of T^{R} are all at most s, while the entries in row j of S are all at least j (since T^{L} and T^{R} are semistandard). So the first r entries of row j of S all lie between j and s. But none of them can be greater than j, since then S would have equal entries in some column. The result follows. □

Now suppose T^{L} is a semistandard λ^{L}-tableau of type μ^{L}, and T^{R} a semistandard λ^{R}-tableau of type μ^{R}. In order to calculate the image of a λ-polytabloid under $\hat{\Theta}_{T^{\mathrm{L}}|T^{\mathrm{R}}}$, we need only consider row permutations of $T^{\mathrm{L}}|T^{\mathrm{R}}$ in which each column has distinct entries. By Lemma 4.2, any such row permutation is of the form $U^{\mathrm{L}}|U^{\mathrm{R}}$, where U^{L} and U^{R} are row permutations of T^{L} and T^{R} respectively. For a λ^{L}-tableau t^{L} on $\{1,\dots,n-l\}$ and a λ^{R}-tableau t^{R} on $\{n-l+1,\dots,n\}$, this gives us

$$(e_{t^{\mathrm{L}}|t^{\mathrm{R}}})\hat{\Theta}_{T^{\mathrm{L}}|T^{\mathrm{R}}} = \big((e_{t^{\mathrm{L}}})\hat{\Theta}_{T^{\mathrm{L}}}\big)\Big|\big((e_{t^{\mathrm{R}}})\hat{\Theta}_{T^{\mathrm{R}}}\big).$$

Hence we have the following.

Theorem 4.3. *Let t^{L} be a λ^{L}-tableau on $\{1,\dots,n-l\}$, and t^{R} a λ^{R}-tableau on $\{n-l+1,\dots,n\}$. Suppose that*

$$\theta^{\mathrm{L}} = \sum_{T^{\mathrm{L}} \in \mathcal{T}_0(\lambda^{\mathrm{L}},\mu^{\mathrm{L}})} c_{T^{\mathrm{L}}} \hat{\Theta}_{T^{\mathrm{L}}}$$

is a homomorphism from $S^{\lambda^{\mathrm{L}}}$ to $M^{\mu^{\mathrm{L}}}$ with

$$(e_{t^{\mathrm{L}}})\theta^{\mathrm{L}} = \sum_{s^{\mathrm{L}}} d_{s^{\mathrm{L}}} e_{s^{\mathrm{L}}},$$

for some μ^{L}-tableaux s^{L} on $\{1,\dots,n-l\}$, and some $c_{T^{\mathrm{L}}}, d_{s^{\mathrm{L}}} \in k$, and similarly suppose that

$$\theta^{\mathrm{R}} = \sum_{T^{\mathrm{R}} \in \mathcal{T}_0(\lambda^{\mathrm{R}},\mu^{\mathrm{R}})} c_{T^{\mathrm{R}}} \hat{\Theta}_{T^{\mathrm{R}}}$$

is a homomorphism from $S^{\lambda^{\mathrm{R}}}$ to $M^{\mu^{\mathrm{R}}}$ with

$$(e_{t^{\mathrm{R}}})\theta^{\mathrm{R}} = \sum_{s^{\mathrm{R}}} d_{s^{\mathrm{R}}} e_{s^{\mathrm{R}}},$$

for μ^{R}-tableaux s^{R} on $\{n-l+1,\dots,n\}$.

Then the homomorphism

$$\theta = \sum_{T^{\mathrm{L}} \in \mathcal{T}_0(\lambda^{\mathrm{L}},\mu^{\mathrm{L}})} \sum_{T^{\mathrm{R}} \in \mathcal{T}_0(\lambda^{\mathrm{R}},\mu^{\mathrm{R}})} c_{T^{\mathrm{L}}} c_{T^{\mathrm{R}}} \hat{\Theta}_{T^{\mathrm{L}}|T^{\mathrm{R}}}$$

satisfies

$$(e_{t^{\mathrm{L}}|t^{\mathrm{R}}})\theta = \sum_{s^{\mathrm{L}}} \sum_{s^{\mathrm{R}}} d_{s^{\mathrm{L}}} d_{s^{\mathrm{R}}} e_{s^{\mathrm{L}}|s^{\mathrm{R}}};$$

in particular, the image of θ lies in the Specht module S^{μ}.

Now we turn our attention to Proposition 4.1; this is rather more complicated, and requires additional notation. Given a ν- or ξ-tableau t on $\{1,\dots,m\}$, recall the definition of $\bar{\nu}$ and $\bar{\xi}$ from Proposition 4.1, and define the $\bar{\nu}$- or $\bar{\xi}$-tableau $\bar{t}$ by adding a row with entries $m+1,\dots,m+r$ in order. Given a homomorphism $\theta : S^{\nu} \to S^{\xi}$ with

$$(e_t)\theta = \sum_s c_s e_s,$$

we would like to find a homomorphism $\bar{\theta} : S^{\bar{\nu}} \to S^{\bar{\xi}}$ with

$$(e_{\bar{t}})\bar{\theta} = \sum_s c_s e_{\bar{s}}.$$

Unfortunately, this does not work in general. We must express the image $(e_t)\theta$ in a particular way.

Suppose that $I = \{I_1,\dots,I_r\}$ is a partition of the set $\{1,\dots,m\}$. We define an equivalence relation $\sim_I$ on ξ-tableaux by putting $s \sim_I u$ if and only if for each i, the elements of I_i between them occupy the same positions in s as in u. (We shall be considering the case where t is a ν-tableau, and I_i the set of entries of the ith column of t. Then we shall have $s \sim_I u$ if and only if $u = s\pi$ for some $\pi \in C(t)$.)

Given a ξ-tableau s, define

$$C_I(s) = \{u \mid s \sim_I u\},$$

and let $\{u_1,\dots,u_a\}$ be a transversal of the $\sim_{\mathrm{col}}$ equivalence classes in $C_I(s)$. Now define the *multipolytabloid*

$$e^I_s = \sum_{j=1}^{a} (-1)^{su_j} e_{u_j}.$$

Remarks.

1. The multipolytabloid is constructed in order to be alternating in the elements of each I_i; in particular, $e^I_s = 0$ if any two elements of some I_i lie in the same row of s.

2. Since $e_u = (-1)^{uv} e_v$ whenever $u \sim_{\mathrm{col}} v$, the multipolytabloid e^I_s is independent of the choice of $u_1, \dots, u_a$.

We shall use multipolytabloids to describe the images of homomorphisms between Specht modules, but in fact we shall need to be slightly more stringent. We say that the multipolytabloid e^I_s is *rectified* if the elements of I_i occur in the first i columns of s, for each i.

Proposition 4.4. *Suppose that an element x of S^ξ is alternating in the elements of each I_i. Then x can be expressed as a linear combination of rectified multipolytabloids e^I_s.*

Proof. By permuting the elements of $\{1, \dots, m\}$, we may assume that the sets I_i are intervals in increasing order, i.e. that if $a \in I_i$, $b \in I_j$ with $i < j$, then $a < b$.

Recall the lexicographic order on ξ-tabloids: $\{s\} \leqslant \{u\}$ if the least entry which does not lie in the same row of s as of u lies in a higher row of s. Write $\{s\} < \{u\}$ if $\{s\} \leqslant \{u\}$ and $\{s\} \neq \{u\}$. This gives a total order on ξ-tabloids. By putting $s \leqslant u$ if $\{s\} \leqslant \{u\}$, we obtain a partial order on the set of ξ-tableaux, and a total order on the set of standard ξ-tableaux. We shall need the following crucial fact (which is weak form of [3, Lemma 8.3]): for any tableau t and any standard tableau s with $t < s$, the tabloid $\{t\}$ does not occur in the polytabloid e_s.

Express x as a linear combination of standard polytabloids, and suppose that e_s is the first standard polytabloid (with respect to the order $\leqslant$) appearing with a non-zero coefficient c. We claim that e^I_s is rectified, and that the first standard polytabloid occurring in $x - ce^I_s$ is later with respect to the order $\leqslant$ than s. This is then sufficient, since $x - ce^I_s$ is alternating in the elements of each I_i, and so by induction can expressed as a linear combination of rectified multipolytabloids.

To prove the claim, we assert that for each i, the entries of I_i occur in increasing rows of s, that is, if $a, b \in I_i$ with $a < b$, then a occurs in a strictly higher row of s than b does. By the fact stated above, the tabloid $\{s\}$ occurs in x with coefficient c, and no tabloid $\{t\}$ with $t < s$ occurs. For $a, b \in I_i$, the tabloid $\{s\}(a\ b)$ occurs in x with coefficient $-c$, and so we must have $\{s\} < \{s\}(a\ b)$, i.e. a must appear in a strictly higher row of s than b does. So the elements of I_i occur in distinct rows of s, and this (together with the fact that s is standard) tells us that e^I_s is rectified: the elements of I_1 occur as the first $|I_1|$ entries of column 1 of s; the elements of I_2 each occur either in the first column of s or immediately to the right of an element of I_1, and so on.

Now consider the effect of subtracting the multipolytabloid e^I_s from x. The above description of s shows us that we may choose $s = u_1, \dots, u_a$ to be standard; since the elements of I_i lie in increasing rows of s, we have $s < u_2, \dots, u_a$, so that the first standard polytabloid occurring in $x - ce^I_s$ is later than s. □

Given this proposition, we may find the expression we seek for $\overline{\theta} : S^{\overline{\nu}} \to S^{\overline{\xi}}$ in terms of polytabloids. For a semistandard ν-tableau T of type ξ, define the semistandard $\overline{\nu}$-tableau $\overline{T}$ of type $\overline{\xi}$ by adding 1 to each entry of T, and then adding a row of r 1s at the top.

Given a tableau t, let $I(t)_i$ be the set of entries in the ith column of t. Let $I(t) = \{I(t)_1, I(t)_2, \dots\}$.

Theorem 4.5. *Let ν and ξ be as in Proposition 4.1, and suppose that*

$$\theta = \sum_{T \in \mathcal{T}_0(\nu,\xi)} c_T \hat{\Theta}_T$$

is a homomorphism with image lying inside the Specht module S^ξ. Then there exist rectified ξ-multipolytabloids $e^{I(t)}_s$ and coefficients d_s such that

$$(e_t)\theta = \sum_s d_s e^{I(t)}_s$$

and such that the homomorphism

$$\overline{\theta} = \sum_{T \in \mathcal{T}_0(\nu,\xi)} c_T \hat{\Theta}_{\overline{T}}$$

satisfies

$$(e_{\bar{t}})\overline{\theta} = \sum_s d_s e_{\overline{s}}^{I(\bar{t})};$$

in particular, the image of $\overline{\theta}$ *lies in the Specht module* $S^{\overline{\xi}}$.

Proof. The polytabloid e_t, and hence its image $(e_t)\theta$, are alternating in the entries of each $I(t)_i$ by definition, so by Proposition 4.4 $(e_t)\theta$ can be described as a linear combination of rectified multipolytabloids. So we need only prove the last equation, by comparing the coefficient of an arbitrary $\overline{\xi}$-tabloid $\{r\}$ on each side.

Both sides of the equation are alternating in the entries of each $I(\bar{t})_i$ by construction, so we may assume that the entries of $I(\bar{t})_i$ occur in distinct rows of r, for each i. This means that the first row of r must contain exactly one element of each $I(\bar{t})_i$; by the alternating property, we may assume these elements are $m+1,\dots,m+r$, i.e. that $r = \overline{q}$ for some ξ-tableau q. It is clear that the coefficient of $\{\overline{q}\}$ in $(e_{\bar{t}})\hat{\Theta}_{\overline{T}}$ equals the coefficient of $\{q\}$ in $(e_t)\hat{\Theta}_T$ for every T, so it suffices to show that the coefficient of $\{\overline{q}\}$ in $e_{\overline{s}}^{I(\bar{t})}$ equals the coefficient of $\{q\}$ in $e_s^{I(t)}$. If we let $u_1,\dots,u_a$ be the tableaux used in the definition of $e_{\overline{s}}^{I(\bar{t})}$, then (since $e_{\overline{s}}^{I(\bar{t})}$ is rectified) the entry $m+i$ occurs in the first i columns of u_j for each i, j. So the only way $\{r\}$ can occur in e_{u_j} is if $m+i$ lies in column i of u_j for each i; let $u_{j_1},\dots,u_{j_b}$ be the set of such u_j. We may assume that the entries $m+1,\dots,m+r$ lie in the first row of u_{j_l}, i.e. that $u_{j_l} = \overline{v_l}$ for some ξ-tableau v_l. Furthermore, we may choose $v_1,\dots,v_b$ to be a possible set of tableaux used in the definition of $e_s^{I(t)}$, i.e.

$$e_s^{I(t)} = \sum_{l=1}^{b} (-1)^{s v_l} e_{v_l}.$$

Then we see that the coefficient of $\{r\}$ in u_{i_l} equals the coefficient of $\{q\}$ in v_l, which gives the result. □

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