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Homomorphisms between Specht modules

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Abstract

In positive characteristic, the Specht modules S^λ corresponding to partitions λ are not necessarily irreducible, and understanding their structure is vital to understanding the representation theory of the symmetric group. In this paper, we address the related problem of finding the spaces of homomorphisms between Specht modules. Indeed in [2], Carter and Payne showed that the space of homomorphisms from S^λ to S^μ is non-zero for certain pairs of partitions λ and μ where the Young diagram for μ is obtained from that for λ by moving several nodes from one row to another. We also consider partitions of this type, and, by explicitly examining certain combinations of semi-standard homomorphisms, we are able to give a constructive proof of the Carter–Payne theorem and to generalise.

1 Introduction

Let k be a field, and let $\mathfrak{S}_n$ denote the symmetric group on n letters. If the characteristic of k is infinite, the irreducible representations of $\mathfrak{S}_n$ over k are afforded by the Specht modules S^λ, as λ ranges over the set of partitions of n. If the characteristic of k is prime, the S^λ are no longer necessarily simple, and understanding their structure is vital to understanding the representation theory of the symmetric group over field of prime characteristic; for example, determining the decomposition numbers – the simple composition factors of the Specht modules – is widely regarded as the central problem in the subject. In this paper, we address the related issue of finding the spaces of homomorphisms between Specht modules. In [2], Carter and Payne showed, using corresponding results from the theory of algebraic groups based on the seminal work of Carter and Lusztig [1], that $\mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, S^\mu)$ is non-zero for certain pairs of partitions λ and μ where the Young diagram for μ is obtained from that for λ by moving several nodes from row i to row j, for some $i < j$. We address partitions of the same type; by explicitly examining certain combinations of semi-standard homomorphisms, we are able to extend their results.

Künzer [5, 6] addresses the same problem in the case where the number of nodes moved is at most two.

We now indicate the layout of this paper. For the remainder of this section, we summarise some generalities concerning the representation theory of the symmetric groups. In Section 2, we introduce semi-standard homomorphisms and prove some simple results which will be key to our calculations. In Section 3, we specialise to the particular type of partitions considered by Carter and Payne, and we define a particular semi-standard homomorphism f over the integers which will be used to construct homomorphisms between Specht modules. In Section 4, we perform long calculations to evaluate $\psi^t_d f$ explicitly, and in Section 5 we use these results first to give a new proof of the Carter-Payne theorem, and then (via further manipulations of tableaux) to try to generalise this result.

1.1 Background and notation

The standard reference to characteristic-free representation theory of the symmetric groups is James's book [4], from which we take our notation and several important results. Recall that a *composition* of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers summing to n, and that λ is a *partition* if $\lambda_1 \geqslant \lambda_2 \geqslant \dots$. The *Young diagram* for λ is the subset

$$[\lambda] = \{(i, j) \mid j \leqslant \lambda_i\}$$

of $\mathbb{N}^2$, whose elements we call *nodes*. A λ*-tableau* consists of $[\lambda]$ with the nodes replaced by integers, usually the integers $1, \dots, n$ in some order. There are natural equivalence relations $\sim_{\mathrm{row}}$ and $\sim_{\mathrm{col}}$ on the set of λ-tableaux ($s \sim_{\mathrm{row}} t$ if we can obtain t by permuting the entries within each row of s, and similarly for columns). An equivalence class under $\sim_{\mathrm{row}}$ is called a *tabloid*, and we write $\{t\}$ for the tabloid containing t. The k-vector space with the set of λ-tabloids as a basis and the natural action of $\mathfrak{S}_n$ is the permutation module M^λ_k. If λ is a partition and t a λ-tableau, we define the *polytabloid*

$$e_t = \sum_{s \sim_{\mathrm{col}} t} (-1)^{st} \{s\} \in M^\lambda_k,$$

where $(-1)^{st}$ is the signature of the permutation required to change s into t. The k-span of the polytabloids is called the *Specht module* S^λ_k. We often drop the field subscript when the field is understood and write M^λ and S^λ for these modules respectively.

2 Homomorphisms between Specht modules and permutation modules

In order to construct homomorphisms between Specht modules, we shall use two important results from [4]. The first concerns homomorphisms from S^λ to the permutation module M^μ.

Let λ be a partition of n and μ a composition of n. Recall that a λ*-tableau of type* μ is a tableau of shape λ with μ_i entries equal to i, for each i. We say that a λ-tableau of type μ is *row standard* if the entries are increasing along rows, *column standard* if the entries are strictly increasing down columns, and *semi-standard* if both row standard and column standard. We let $\mathcal{T}(\lambda, \mu)$ denote the set of λ-tableaux of type μ, and $\mathcal{T}_0(\lambda, \mu)$ the set of semi-standard tableaux of type μ.

Let t be a fixed λ-tableau (of type (1^n)), and for a composition μ define a bijection between $\mathcal{T}(\lambda, \mu)$ and the set of μ-tabloids by letting $T \in \mathcal{T}(\lambda, \mu)$ correspond to $\{r\}$, where, if corresponding positions of t and T are occupied by the numbers i and j, then i appears in row j of r. We shall frequently use $\mathcal{T}(\lambda, \mu)$ as a basis for M^μ via this bijection.

Now, for each $T \in \mathcal{T}(\lambda, \mu)$, we define a homomorphism

$$\Theta_T : M^\lambda \longrightarrow M^\mu$$

over any field, by stipulating that

$$\{t\} \longmapsto \sum_{S \sim_{\mathrm{row}} T} S$$

and extending linearly. We then define the homomorphism $\hat{\Theta}_T : S^\lambda \to M^\mu$ to be the restriction of Θ_T to the Specht module S^λ. If $T \in \mathcal{T}_0(\lambda, \mu)$, we call this a semi-standard homomorphism. We have the following, due originally to Carter and Lusztig [1].

Theorem 1. [4, Theorem 13.13] *Let λ and μ be partitions of n. Then the set $\{\hat{\Theta}_T \mid T \in \mathcal{T}_0(\lambda,\mu)\}$ is linearly independent over k. Moreover, it is a k-basis for $\mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda_k, M^\mu_k)$, except possibly when* $\mathrm{char}(k) = 2$ *and λ is 2-singular.*

Later, we shall need the following lemma, which helps us to cope with the failure of Theorem 1 when $p = 2$ and λ is 2-singular. The authors are indebted to Gordon James for providing a proof of this.

Lemma 2. *Let λ and μ be partitions of n. Then the set $\{\hat{\Theta}_T \mid T \in \mathcal{T}_0(\lambda,\mu)\}$ of semi-standard homomorphisms spans the same subspace of $\mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda_k, M^\mu_k)$ as the set $\{\hat{\Theta}_T \mid T \in \mathcal{T}(\lambda,\mu)\}$.*

Proof. We must show that if $T \in \mathcal{T}(\lambda,\mu)$, then $\hat{\Theta}_T$ may be expressed as a linear combination of semi-standard homomorphisms. This depends on the following claim, which is analogous to [4, Lemma 13.12]: if we write $\hat{\Theta}_T(e_t)$ as a linear combination $\sum c_S S$ of basis elements for M^μ, then

1. $c_S = 0$ whenever S has two equal entries in some column, and

2. $c_S \neq 0$ for some semi-standard tableau S.

To prove 1, observe that when we apply Θ_T to a polytabloid, we need only consider row permutations of T in which any two entries in the same column are distinct. The proof of 2 is identical to the proof of [4, Lemma 12.12(ii)]. The proof of Lemma 2 is now completed using an identical argument to that in [4, Lemma 13.13]. □

In order to use homomorphisms $S^\lambda \to M^\mu$ to construct homomorphisms $S^\lambda \to S^\mu$, we need the following characterization of the Specht module. Let d be a positive integer, and choose t such that $0 \leqslant t < \lambda_{d+1}$. Define the parts of ν by the following prescription:

$$\nu_i = \begin{cases} \lambda_i + t & (i = d) \\ \lambda_i - t & (i = d+1) \\ \lambda_i & (\text{otherwise}). \end{cases}$$

Let T be any λ-tableau with all entries in row i equal to i, except for $i = d+1$, when there are t entries equal to d and $\lambda_{d+1} - t$ entries equal to $d+1$. T is then a λ-tableau of type ν, and we write ψ^t_d for the homomorphism $\Theta_T : M^\lambda \to M^\nu$. (We may omit to mention ν, since it is determined by λ, d and t.) We then have the following.

Theorem 3. [4, Corollary 17.18] *If λ is a partition of n, then*

$$S^\lambda = \bigcap_{d \geqslant 1} \bigcap_{t=1}^{\lambda_{d+1}} \ker(\psi^t_d).$$

It should be noted that ψ^t_d coincides with $\psi_{d,\lambda_{d+1}-t}$ as defined in [4]. Our strategy for constructing homomorphisms $S^\lambda \to S^\mu$ is as follows: we take a particular linear combination of semi-standard homomorphisms, and show that the image lies in the Specht module by composing with the maps ψ^t_d.

2.1 Some useful results concerning semi-standard homomorphisms

Lemma 4. *Let $1 \leqslant i_1 < i_2 < \cdots < i_s$, and suppose $T_1, \dots, T_a$ are λ-tableaux of type μ which are identical except in rows $i_1, i_2, \dots, i_s$. Let $\tilde{T}_j$ be the tableau obtained from T_j by deleting all rows apart from $i_1, \dots, i_s$. If $\sum_{j=1}^{a} c_j \hat{\Theta}_{\tilde{T}_j} = 0$ for some $c_j \in \mathbb{Z}$, then $\sum_{j=1}^{a} c_j \hat{\Theta}_{T_j} = 0$.*

Proof. Let $\tilde{\lambda}$ be the partition $(\lambda_{i_1}, \dots, \lambda_{i_s})$, and let $\tilde{\mu}$ be the composition in which μ_i is the number of entries equal to i in each $\tilde{T}_j$, so that $\tilde{T}_1, \dots, \tilde{T}_a$ are $\tilde{\lambda}$-tableaux of type $\tilde{\mu}$; suppose $\tilde{\lambda}$ and $\tilde{\mu}$ are partitions of m. Let $\tilde{t}$ be a $\tilde{\lambda}$-tableau, and let t be a λ-tableau which agrees with $\tilde{t}$ outside rows $i_1, \dots, i_s$.

We need to define some $\mathfrak{S}_m$-homomorphisms from $M^{\tilde{\mu}}$ to M^{μ} (viewing M^{μ} as a $\mathbb{Z}\mathfrak{S}_m$-module by restriction): if $S \in \mathcal{T}(\lambda, \mu)$ is such that when rows $i_1, \dots, i_s$ are deleted we get a $\tilde{\lambda}$-tableau of type $\tilde{\mu}$, define

$$\pi_S : M^{\tilde{\mu}} \longrightarrow M^{\mu}$$

by

$$R \longmapsto U,$$

where R is a $\tilde{\lambda}$-tableau of type $\tilde{\mu}$, and U is a λ-tableau which agrees with S in rows $i_1, \dots, i_s$ and agrees with R elsewhere. This clearly defines a $\mathbb{Z}\mathfrak{S}_m$-homomorphism.

Now suppose that $\sum_{j=1}^{a} c_j \hat{\Theta}_{\tilde{T}_j} = 0$. Let C_t be the column stabiliser of t, and as in [4] let $\kappa_t = \sum_{\sigma \in C_t} (-1)^{\sigma} \sigma$. Make similar definitions for $\tilde{t}$. The agreement between t and $\tilde{t}$ guarantees that $C_{\tilde{t}} \leqslant C_t$; if we let $\sigma_1, \dots, \sigma_d$ be coset representatives, then we have

$$\kappa_t = \sum_{i=1}^{d} (-1)^{\sigma_i} \sigma_i \kappa_{\tilde{t}}.$$

So we have

$$\begin{aligned} \Theta_{T_j}(e_t) &= \Theta_{T_j}(\kappa_t \{t\}) \\ &= \sum_{i=1}^{d} (-1)^{\sigma_i} \sigma_i \kappa_{\tilde{t}} \sum_{S \sim_{\mathrm{row}} T_j} S. \end{aligned}$$

We define new equivalence relations on $\mathcal{T}(\lambda, \mu)$: say $S \approx T$ if S and T differ by a row permutation on rows $i_1, \dots, i_s$ and agree elsewhere, or $S \approxeq T$ if S and T agree in rows $i_1, \dots, i_s$ and differ by a row permutation elsewhere. Then we have

$$\begin{aligned} \sum_{S \sim_{\mathrm{row}} T_j} S &= \sum_{U \approx T_j} \sum_{S \approxeq U} S \\ &= \sum_{U \approx T_j} \pi_U(\Theta_{\tilde{T}_j}(\{\tilde{t}\})). \end{aligned}$$

Hence

$$\sum_{j=1}^{a} c_j \hat{\Theta}_{T_j}(e_t) = \sum_{j=1}^{a} c_j \sum_{i=1}^{d} (-1)^{\sigma_i} \sigma_i \kappa_{\tilde{t}} \sum_{U \approx T_j} \pi_U(\Theta_{\tilde{T}_j}(\{\tilde{t}\}))$$

$$= \sum_{i=1}^{d} (-1)^{\sigma_i} \sigma_i \sum_{U \approx T_1} \pi_U \left(\sum_{j=1}^{a} c_j \Theta_{\tilde{T}_j}(\kappa_{\tilde{t}}\{\tilde{t}\}) \right) \quad \square$$

$$= 0. \quad \square$$

Given a λ-tableau T of type μ, we write T^i_j for the number of entries equal to i in row j. Note that the row equivalence class of T (and hence Θ_T) are determined by the T^i_j.

Lemma 5. *Let T be a λ-tableau of type μ, and let*

$$\mathcal{T}(T, d, t) = \left\{ (t_1, t_2, \dots) \;\middle|\; 0 \leqslant t_j \leqslant T^{d+1}_j, \sum_j t_j = t \right\}.$$

For $\mathbf{t} = (t_1, t_2, \dots) \in \mathcal{T}(T, d, t)$, let $T(\mathbf{t})$ be a tableau obtained by changing t_j of the entries $d + 1$ into entries d in row j, for each j. Then

$$\psi^t_d \Theta_T = \sum_{\mathbf{t} \in \mathcal{T}(T,d,t)} \left(\prod_j \binom{T^d_j + t_j}{t_j} \right) \Theta_{T(\mathbf{t})}.$$

Proof. Choosing a λ-tableau t and using the bases $\mathcal{T}(\lambda, \mu)$ and $\mathcal{T}(\lambda, \nu)$ for M^μ and M^ν, the map ψ^t_d is given by

$$S \longmapsto \sum_{R \in (S)^t_d} R,$$

where $(S)^t_d$ is the set of λ-tableaux which may be obtained from S by changing t of the entries $d + 1$ to the value d.

To prove the last equation of Lemma 5, it suffices to apply both sides to $\{t\}$. Thus we are required to prove

$$\psi^t_d \sum_{S \sim_{\text{row}} T} S = \sum_{\mathbf{t} \in \mathcal{T}(T,d,t)} \left(\prod_j \binom{T^d_j + t_j}{t_j} \right) \sum_{R \sim_{\text{row}} T(\mathbf{t})} R.$$

Now the condition $R \sim_{\text{row}} T(\mathbf{t})$ defines $\mathbf{t}$ uniquely; moreover, the number of S such that $S \sim_{\text{row}} T$ and $R \in (S)^t_d$ is $\prod_j \binom{T^d_j + t_j}{t_j}$, and this is enough to complete the proof. $\square$

2.1.1 Multisets

We now need a little notation concerning multisets: if P is a multiset of integers, we write P^i for the number of elements P equal to i. We also write $P \subseteq Q$ to mean that $P^i \leqslant Q^i$ for all i. The *union* of P and Q is the multiset R with $R^i = P^i + Q^i$ for all i. We say that P and Q are *disjoint subsets* of R if $P \cup Q \subseteq R$.

If P and Q are multisets, we write

$$\binom{P}{Q} = \prod_i \binom{P^i}{Q^i}.$$

The following is then immediate.

Lemma 6.

$$\sum_{Q\subseteq P\ \mid\ |Q|=q} \binom{P}{Q} = \binom{|P|}{q}.$$

Our next lemma tells us how to express $\hat{\Theta}_T$ in terms of other homomorphisms $\hat{\Theta}_S$ by moving entries between rows.

Lemma 7. *Suppose λ is a partition, and j and l are integers with $\lambda_j \geqslant \lambda_l$. Suppose T is a λ-tableau of type ν in which row j contains e entries equal to i and a multiset C of other entries and row l contains f entries equal to i and a multiset D of other entries. For every submultiset F of C with $|F| = f$, let T_F be any tableau obtained by replacing the elements of F with entries equal to i, and replacing the entries equal to i in row l with the elements of F. Then*

$$\hat{\Theta}_T = (-1)^f \sum_{F\subseteq C\ \mid\ |F|=f} \binom{D\cup F}{F} \hat{\Theta}_{T_F}.$$

Proof. By Lemma 4, we may assume that λ has only two parts, with $j = 1$, $l = 2$. We begin by considering the special case where the elements of $C \cup D$ are distinct. Then for each F we have $\binom{D\cup F}{F} = 1$. Choose a λ-tableau t and $S \in \mathcal{T}(\lambda, \mu)$, and compare the coefficients of S in $\hat{\Theta}_T(e_t)$ and $\hat{\Theta}_{T_F}(e_t)$. The coefficient of S in $\hat{\Theta}_T(e_t)$ is zero if any of the following occurs:

- two elements of C occur in the same column of S;
- two elements of D occur in the same column of S;
- two entries equal to i occur in the same column of S;
- an element of D occurs in a column of S of length 1.

Suppose none of these happens, and that S has

- u columns of the form $\begin{array}{|c|}\hline i\\\hline c\\\hline\end{array}$ with $c \in C$,
- v columns of the form $\begin{array}{|c|}\hline c\\\hline i\\\hline\end{array}$ with $c \in C$,
- w columns of the form $\begin{array}{|c|}\hline i\\\hline d\\\hline\end{array}$ with $d \in D$,
- x columns of the form $\begin{array}{|c|}\hline d\\\hline i\\\hline\end{array}$ with $d \in D$,
- y columns of the form $\begin{array}{|c|}\hline c\\\hline d\\\hline\end{array}$ with $c \in C$, $d \in D$,
- z columns of the form $\begin{array}{|c|}\hline d\\\hline c\\\hline\end{array}$ with $c \in C$, $d \in D$.

Then we must have $u + v = f$, and the coefficient of S in $\hat{\Theta}_T(e_t)$ is $(-1)^{u+x+z}$.

Now look at the coefficient of S in $\hat{\Theta}_{T_F}$. This is zero if two elements of D occur in the same column, two entries equal to i occur in the same column or an element of D occurs in a column of length 1. If two elements c_1 and c_2 of C occur a column of S and the coefficient of S in $\hat{\Theta}_{T_F}$ is non-zero, then F must contain exactly one of c_1 and c_2. If we construct F' by interchanging c_1 and c_2, then the coefficient of S in $\hat{\Theta}_{T_{F'}}$ is minus the coefficient of S in $\hat{\Theta}_{T_F}$. So the coefficient of S in $\sum_F \hat{\Theta}_{T_F}$ is zero. So we assume that no two elements of C occur together in a column of S, and that u, v, w, x, y, z are as above. Then the coefficient of S in $\hat{\Theta}_{T_F}$ is zero unless F is precisely the set of elements of C which occur in columns of the form $\begin{array}{|c|}\hline i\\\hline c\\\hline\end{array}$ or $\begin{array}{|c|}\hline c\\\hline i\\\hline\end{array}$ with $c \in C$, in which case it is $(-1)^{v+x+z}$. Hence the coefficient of S in $(-1)^f \sum_F \hat{\Theta}_{T_F}$ is $(-1)^{u+x+z}$. This completes the case where the elements of $C \cup D$ are distinct.

Now we proceed by downwards induction on the number of distinct elements of $C \cup D$, with the initial case being that considered above. Suppose that the number a appears more than once in $C \cup D$, and let T' be a tableau obtained from T by replacing one occurrence of a with a number α distinct from any other element of $C \cup D$. By induction Lemma 7 holds for T'; assuming without loss of generality that $\alpha = a + 1$, we take the equation for $\hat{\Theta}_{T'}$ and compose with ψ^1_a, using Lemma 5.

First suppose that α occurs in row l of T'. Recalling the definition of D^a for an element a of a multiset D, we have

$$\psi^1_a \hat{\Theta}_{T'} = D^a \hat{\Theta}_T.$$

Given F, we define T'_F in an obvious way, and we find

$$\psi^1_a \hat{\Theta}_{T'_F} = (D^a + F^a)\hat{\Theta}_{T_F}.$$

Thus we have

$$D^a \hat{\Theta}_T = (-1)^f \sum_{F \subseteq C \mid |F|=f} \left(\prod_{b \neq a} \binom{D^b + F^b}{F^b} \right) \binom{D^a - 1 + F^a}{F^a} (F^a + D^a) \hat{\Theta}_{T_F},$$

and the result follows; we may divide by D^a since it must be positive and since we may assume we are working over $\mathbb{Z}$.

Now suppose that α occurs in row j of T'. Given F with $F^a > 0$, we define a submultiset F^+ of the set of entries in row j of T' by replacing an entry equal to a with α. We then define T'_{F^+} in an obvious way, so that our inductive assumption is

$$\hat{\Theta}_{T'} = (-1)^f \left(\sum_{F \subseteq C \mid |F|=f} \binom{D \cup F}{F} \hat{\Theta}_{T'_F} + \sum_{F \subseteq C \mid |F|=f, F^a > 0} \binom{D \cup F^+}{F^+} \hat{\Theta}_{T'_{F^+}} \right).$$

Composing with ψ^1_a using Lemma 5 again gives the result. □

3 Carter–Payne partitions

The pairs of partitions (λ, μ) for which we shall construct homomorphisms are those for which

$$\mu_i = \begin{cases} \lambda_i - s & (i = a) \\ \lambda_i + s & (i = b) \\ \lambda_i & (i \neq a, b) \end{cases}$$

for some a, b, s. These are the partitions considered by Carter and Payne, whose main result is as follows.

Theorem 8. [2, p. 425, Theorem] *Let λ and μ be as above, and let S^λ and S^μ be the Specht modules defined over a field k of characteristic p. Suppose also that for some e we have $s < p^e$ and $\lambda_a - \lambda_b + b - a - s \equiv 0 \pmod{p^e}$. Then $\mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, S^\mu) \neq 0$.*

In order to simplify matters, we use the following results, due to the first author and Lyle.

Theorem 9. [3, Theorems 2.2, 2.3] *Suppose that k is a field of characteristic not two, and that ξ and ν are partitions of n with $\xi_1 = \nu_1 = r$ (or with $\xi'_1 = \nu'_1 = r$). Denote by $\bar{\xi}$, $\bar{\nu}$ the partitions of $n - r$ obtained by removing the first row (respectively, the first column) from $[\xi], [\nu]$. Then*

$$\dim_k \mathrm{Hom}_{k\mathfrak{S}_n}(S^\xi, S^\nu) = \dim_k \mathrm{Hom}_{k\mathfrak{S}_{n-r}}(S^{\bar{\xi}}, S^{\bar{\nu}}).$$

In view of this result, we assume that, with λ and μ as above, we have $a = 1$ and $\lambda_b = 0$.

Remark. The failure of Theorem 9 in characteristic two need not worry us, since the homomorphisms we define are all linear combinations of semi-standard homomorphisms. With $\bar{\xi}$ and $\bar{\nu}$ as in Theorem 9, let $\underline{\mathrm{Hom}}_{k\mathfrak{S}_n}(S^\xi, S^\nu)$ denote the space of homomorphisms from S^ξ to S^ν which are linear combinations of semi-standard homomorphisms. In the case where $\bar{\xi}$ and $\bar{\nu}$ are obtained by removing the first column, Theorem 4.3 of [3] provides a linear injection from $\underline{\mathrm{Hom}}(S^{\bar{\xi}}, S^{\bar{\nu}})$ to $\underline{\mathrm{Hom}}(S^\xi, S^\nu)$, which works even when the characteristic is two. Using [4, Theorem 8.15] we deduce a similar result for row removal. So, since we are only interested in showing the existence of homomorphisms, we may remove rows and columns from λ and μ.

We are now in a position to define the linear combination of semi-standard homomorphisms from which we shall construct a homomorphism $S^\lambda \to S^\mu$. We write

$$\lambda = (l_0 + s, l_1^{m_1 - 1}, l_2^{m_2 - m_1}, \dots, l_r^{m_r - m_{r-1}})$$

where $l_0 \geqslant l_1 > l_2 > \dots > l_r \geqslant s$, so that

$$\mu = (l_0, l_1^{m_1 - 1}, l_2^{m_2 - m_1}, \dots, l_r^{m_r - m_{r-1}}, s),$$

and we assume henceforth that $r \geqslant 1$; the case where $r = 0$ is easily dealt with.

We shall not in fact use semi-standard λ-tableaux of type μ, but *pseudo-standard* tableaux. We say that T is *pseudo-standard* if:

- the entries in each row of T are weakly increasing;
- T^i_j is non-zero only if $i = j$ or $\lambda_i < \lambda_j$.

Let $\mathcal{T}^0(\lambda, \mu)$ denote the set of pseudo-standard λ-tableaux of type μ.

For example, if $\lambda = (5, 3^2)$ and $\mu = (3^3, 2)$, the pseudo-standard λ-tableaux of type μ are

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&1&4&4\\\hline 2&2&2\\\cline{1-3} 3&3&3\\\cline{1-3}\end{array}\ ,\qquad \begin{array}{|c|c|c|c|c|}\hline 1&1&1&3&4\\\hline 2&2&2\\\cline{1-3} 3&3&4\\\cline{1-3}\end{array}\ ,\qquad \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&4\\\hline 2&2&4\\\cline{1-3} 3&3&3\\\cline{1-3}\end{array}$$

$$\begin{array}{ccccc}1&1&1&3&3\\2&2&2&&\\3&4&4&&\end{array},\qquad \begin{array}{ccccc}1&1&1&2&3\\2&2&4&&\\3&3&4&&\end{array},\qquad \begin{array}{ccccc}1&1&1&2&2\\2&4&4&&\\3&3&3&&\end{array}$$

while the semi-standard λ-tableaux of type μ are

$$\begin{array}{ccccc}1&1&1&4&4\\2&2&2&&\\3&3&3&&\end{array},\qquad \begin{array}{ccccc}1&1&1&3&4\\2&2&2&&\\3&3&4&&\end{array},\qquad \begin{array}{ccccc}1&1&1&2&4\\2&2&3&&\\3&3&4&&\end{array}$$

$$\begin{array}{ccccc}1&1&1&3&3\\2&2&2&&\\3&4&4&&\end{array},\qquad \begin{array}{ccccc}1&1&1&2&3\\2&2&3&&\\3&4&4&&\end{array},\qquad \begin{array}{ccccc}1&1&1&2&2\\2&3&3&&\\3&4&4&&\end{array}.$$

Remark. The homomorphisms corresponding to pseudo-standard tableaux are not linearly independent in general. We shall discuss the consequences of this in Section 5.

Before defining our homomorphism $f : S^\lambda \to M^\mu$, we introduce a small item of notation. Given integers a and b with $b \geqslant 0$, we write

$$a^{b\downarrow} = \prod_{i=0}^{b-1}(a-i)$$

and

$$a^{b\uparrow} = \prod_{i=0}^{b-1}(a+i).$$

Let λ be as above, with ξ any composition of n, and let $\gamma_1, \dots, \gamma_r$ be integers. Given $T \in \mathcal{T}^0(\lambda, \xi)$, define $f(T) \in \mathbb{Z}$ as follows. For each h, let $T_h = \sum_{j>h} T_h^j$, and for $1 \geqslant i \geqslant r$, define

$$n_i(T) = \sum_{h=m_{i-1}+1}^{m_i} T_h$$

and

$$c_i(T) = \gamma_i^{(s-n_i(T))\downarrow} \prod_{h=m_{i-1}+1}^{m_i} T_h!.$$

Now define

$$f(T) = \prod_{i=1}^{r} c_i(T).$$

For example, the values of $f(T)$ for $T \in \mathcal{T}^0((5,3^2),(3^3,2))$ as above are

$$f\left(\begin{array}{ccccc}1&1&1&4&4\\2&2&2&&\\3&3&3&&\end{array}\right) = \gamma_1(\gamma_1-1),\qquad f\left(\begin{array}{ccccc}1&1&1&3&4\\2&2&2&&\\3&3&3&&\end{array}\right) = \gamma_1,\qquad f\left(\begin{array}{ccccc}1&1&1&2&4\\2&2&4&&\\3&3&3&&\end{array}\right) = \gamma_1,$$

$$f\left(\begin{array}{ccccc}1&1&1&3&3\\2&2&2&&\\3&4&4&&\end{array}\right) = 2,\qquad f\left(\begin{array}{ccccc}1&1&1&2&3\\2&2&4&&\\3&3&4&&\end{array}\right) = 1,\qquad f\left(\begin{array}{ccccc}1&1&1&2&2\\2&4&4&&\\3&3&3&&\end{array}\right) = 2.$$

Now define $f : S^\lambda_{\mathbb{Z}} \to M^\mu_{\mathbb{Z}}$ by

$$f = \sum_{T \in \mathcal{T}^0(\lambda,\mu)} f(T)\hat{\Theta}_T.$$

We shall use f to construct homomorphisms over $\mathbb{F}_p$ by dividing by a suitable power of p, and then reducing modulo p. The precise way in which we do this will depend upon the results of the next section.

4 Composing f with ψ^t_d

In this section we discover the important properties of our map $f : S^\lambda_{\mathbb{Z}} \to M^\mu_{\mathbb{Z}}$. These are concerned with composing with the maps $\psi^t_d : M^\mu \to M^\nu$. We shall show that $\psi^t_d f$ is divisible by certain integers; our technique will be to evaluate $\psi^t_d f$ explicitly, and to express it in terms of pseudo-standard λ-tableaux of type ν. We begin with the case where $\lambda_d = \lambda_{d+1}$.

Proposition 10. *Suppose $\lambda_d = \lambda_{d+1}$, i.e. $m_l < d \leqslant m_{l+1}$ for some i. Then for any $0 < t \leqslant \lambda_{d+1}$ we have*

$$\psi^t_d f = 0.$$

We prove this proposition in two stages: given a pseudo-standard λ-tableau T of type μ, we express $\psi^t_d\hat{\Theta}_T$ as a linear combination of homomorphisms $\hat{\Theta}_S$ for S pseudostandard. Then we combine these expressions in order to express $\psi^t_d f$ in the same way.

Lemma 11. *Choose t and d such that $\lambda_d = \lambda_{d+1}$. For a pseudo-standard λ-tableau R of type μ or ν, define $R[d]$ to be the multiset of entries in row d which are greater than d, and define $R(d)$ to be the multiset of rows other than d containing entries equal to d. That is:*

$$R[d]^i = \begin{cases} R^i_d & (i > d) \\ 0 & (i \leqslant d); \end{cases}$$

$$R(d)^i = \begin{cases} R^d_i & (i < d) \\ 0 & (i \geqslant d). \end{cases}$$

Define $R[d+1]$ and $R(d+1)$ analogously. Now fix $T \in \mathcal{T}^0(\lambda,\mu)$, and let

$$\begin{aligned} \mathcal{S}(T) = \{S \in \mathcal{T}^0(\lambda,\nu) \mid\ & T[d] \subseteq S[d], \\ & S[d] \cup S[d+1] = T[d] \cup T[d+1], \\ & S(d) \subseteq T(d), \\ & S(d) \cup S(d+1) = T(d) \cup T(d+1)\}. \end{aligned}$$

Then

$$\psi^t_d\hat{\Theta}_T = \sum_{S \in \mathcal{S}(T)} (-1)^{|S[d]\setminus T[d]|} \binom{S[d+1]}{T[d+1]} \binom{S(d)}{T(d)}.$$

Proof. We compose $\hat{\Theta}_T$ with ψ^t_d using Lemma 5. Given a submultiset I of $T(d+1)$ of size i, we change those entries $d+1$ into entries d in rows corresponding to elements of I, and we change $t-i$ entries $d+1$ into entries d in row $d+1$ (maintaining the row standard property). By Lemma 5, the homomorphism corresponding to the resulting tableau occurs with coefficient $\binom{T(d)\cup I}{I}$.

Now we use Lemma 7 to move the entries equal to d in row $d+1$ into row d. Given a submultiset Z of $T[d]$ of size $t-i$, we move the elements of Z from row d to row $d+1$, and move those entries equal to d in row $d+1$ into row d, maintaining the row standard property. We get an additional coefficient $(-1)^{t-i}\binom{T[d+1]\cup Z}{Z}$, by Lemma 7, and the resulting tableau lies in $\mathcal{S}(T)$. In fact, each $S \in \mathcal{S}(T)$ corresponds to a unique choice of I and Z, and the result follows. □

Proof of Proposition 10. Let S be a pseudo-standard λ-tableau S of type ν, and assume the notation of Lemma 11. By that lemma, we find that the coefficient of $\hat{\Theta}_S$ in $\psi^t_d f$ is

$$\sum_{T\in\mathcal{T}^0(\lambda,\mu)|S\in\mathcal{S}(T)} f(T)(-1)^{|S[d]\setminus T[d]|}\binom{S[d+1]}{T[d+1]}\binom{S(d)}{T(d)}.$$

Moreover, a tableau T such that $S\in\mathcal{S}(T)$ corresponds to multisets $I\subseteq S[d]$ and $Z\subseteq S(d+1)$ of sizes i and $t-i$ for some i, as in the proof of Lemma 11. Also, the tableau T corresponding to I and Z has

$$c_j(T)=\begin{cases}\dfrac{|S(d)\cup Z|!|S(d+1)\setminus Z|!c_j(S)}{|S(d)|!|S(d+1)|!} & (j=l)\\ c_j(S) & (j\neq l).\end{cases}$$

Thus the coefficient of $\hat{\Theta}_S$ in $\psi^t_d f$ is

$$\frac{\prod_{j=1}^r c_j(S)}{|S(d)|!|S(d+1)|!}\sum_{i=0}^t(-1)^{t-i}\sum_{\substack{Z\subseteq S(d+1)\\|Z|=t-i}}\sum_{\substack{I\subseteq S[d]\\|I|=i}}\binom{S(d+1)}{Z}\binom{S[d]}{I}|S(d)\cup Z|!|S(d+1)\setminus Z|!$$
$$=\frac{\prod_{j=1}^r c_j(S)}{|S(d)|!|S(d+1)|!}\sum_{i=0}^t(-1)^{t-i}(|S[d]|-i)!(|S(d+1)|-t+i)!\binom{|S(d+1)|}{t-i}\binom{|S[d]|}{i}$$

by Lemma 6

$$=\frac{\prod_{j=1}^r c_j(S)}{|S(d)|!|S(d+1)|!}\frac{|S[d]|!|S(d+1)|!}{t!}(-1)^t\sum_{i=0}^t(-1)^i\binom{t}{i}.$$

The alternating sum of binomial coefficients is zero, since $t>0$. □

When rows d and $d+1$ of λ are of different lengths, things become a little more complex; in particular, $\psi^t_d f$ need not be zero. However, we can calculate an exact expression for $\psi^t_d f$. Suppose that $d=m_b$ for some $0\leqslant b\leqslant r$, and let ν be the composition such that $\psi^t_d : M^\mu\to M^\nu$. Given a pseudo-standard λ-tableau S of type ν, define

$$g(S)=\begin{cases}\left(\prod_{i=1}^r c_i(S)\right)\frac{(l_0+s-l_1+m_1-1-\gamma_1)^{t\uparrow}}{t!} & (b=0)\\ \left(\prod_{i\neq b} c_i(S)\right)\left(\gamma_b^{(s-n_b(S)-t)\downarrow}(S_d+t)!\prod_{j=m_{b-1}+1}^{d-1}S_j!\right)\frac{(l_b-l_{b+1}+m_{b+1}-m_b+\gamma_b-\gamma_{b+1})^{t\uparrow}}{t!} & (0<b<r)\\ \left(\prod_{i=1}^{r-1} c_i(S)\right)\left(\gamma_r^{(s-n_r(S)-t)\downarrow}(S_d+t)!\prod_{j=m_{r-1}+1}^{d-1}S_j!\right)\frac{(l_r-s+1+\gamma_r)^{t\uparrow}}{t!} & (b=r).\end{cases}$$

Theorem 12. *Suppose $\lambda_d > \lambda_{d+1}$, i.e. $d = m_b$ for some $0 \leqslant b \leqslant r$. Then*

$$\psi^t_d f = \sum_{S \in \mathcal{T}^0(\lambda,\nu)} g(S)\hat{\Theta}_S.$$

To assist with our calculations, we need some simple results concerning factorials and binomial coefficients.

Lemma 13. *Suppose that $a_1, \dots, a_m, b_1, \dots, b_n, c$ are non-negative integers such that*

$$a_1 + \cdots + a_m = b_1 + \cdots + b_n = c.$$

Let C be the set of all arrays (c_{ij}), $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ of non-negative integers such that

$$\sum_{i=1}^{m} c_{ij} = b_j$$

for all j, and

$$\sum_{j=1}^{n} c_{ij} = a_i$$

for all i. Then

$$\sum_{(c_{ij}) \in C} \frac{\prod_i a_i! \prod_j b_j!}{\prod_{i,j} c_{ij}!} = c!.$$

Proof. It suffices to prove

$$\frac{c!}{\prod_i a_i!} = \sum_{(c_{ij}) \in C} \frac{\prod_j b_j!}{\prod_{i,j} c_{ij}!};$$

the left-hand side is the number of ways of partitioning a set of size c into sets of sizes $b_1, \dots, b_n$, while the right-hand side is the number of ways of partitioning the disjoint union of sets of sizes $a_1, \dots, a_m$ into sets of sizes $b_1, \dots, b_n$. Clearly, these are the same. □

4.0.2 Binomial coefficients

In the course of proving Theorem 12, we shall allow ourselves to use binomial coefficients $\binom{a}{b}$ in which a is negative (and b non-negative). Such a binomial coefficient is simply defined as

$$\binom{a}{b} = \frac{a^{b\downarrow}}{b!}.$$

There are some standard properties of binomial coefficients which we cannot use here; for example, we cannot write $\binom{a}{b} = \binom{a}{a-b}$, although we can now write $\binom{a}{b} = (-1)^b\binom{-a+b-1}{b}$. The only standard properties of binomial coefficients we shall use are the following, which still hold with our extended definition.

Lemma 14. *For any integers a, b, c with $c \leqslant 0$,*

$$\sum_{i=0}^{c} \binom{a}{i}\binom{b}{c-i} = \binom{a+b}{c}.$$

Proof. This is a standard result when a and b are non-negative. On the other hand, it may be regarded as a polynomial identity in a and b, and so holds for all a and b. □

Lemma 15. *Let c be a positive integer, and let $x_1, \dots, x_c, n$ be integers, with $n \geqslant 0$. Let A be the set of all c-tuples $(a_1, \dots, a_c)$ of non-negative integers summing to n. Then*

$$\sum_{(a_1,\dots,a_c)\in A} \prod_{j=1}^{c} \binom{x_j+a_j}{a_j} = \binom{x_1+\dots+x_c+n+c-1}{n}.$$

Proof. This is readily proved by induction on n. □

4.1 The proof of Theorem 12

The proof of Theorem 12 is a long calculation in which we use Lemma 5 and Lemma 7 repeatedly. Essentially, we proceed as in the proof of Proposition 10: we take a pseudo-standard λ-tableau T of type μ and express $\psi^t_d\hat{\Theta}_T$ as a linear combination of homomorphisms $\hat{\Theta}_S$ with S pseudo-standard. Then we combine these expressions to get an expression for $\psi^t_d f$, and compare coefficients of an arbitrary $\hat{\Theta}_S$.

We assume that $0 < b < r$ throughout; the proofs of the cases $b = 0$ and $b = r$ are simpler, and may be left to the reader to reconstruct; as we proceed, we shall indicate the places where they differ from the general case.

We write $a = m_{b-1}, c = m_{b+1}$, and we shall use the word 'set' to mean 'multiset', with the conventions for 'subset', 'union' and 'disjoint' described in 2.1.1.

4.1.1 Sets associated with T

Let T be a pseudo-standard λ-tableau of type μ. We shall need to refer to the following sets:

- E is the set of rows other than d which contain entries equal to d (with multiplicity), i.e. $E^i = T^d_i$ for $i < d$;
- G is the set of rows higher than row $a + 1$ which contain entries equal to $d + 1$, i.e. $G^i = T^{d+1}_i$ for $i \leqslant a$;
- B is the set of rows between $a + 1$ and $d - 1$ which contain entries equal to $d + 1$, i.e. $B^i = T^{d+1}_i$ for $a < i < d$;
- for $a + 1 \leqslant i \leqslant d - 1$, C_i is the set of entries between $d + 2$ and c in row i, and D_i is the set of entries greater than c in row i;
- C is the set of entries between $d + 2$ and c in row d, and D is the set of entries greater than c in row d;
- Q is the set of entries greater than $d + 1$ in row $d + 1$;
- for $d + 2 \leqslant j \leqslant c$, F_j is the set of entries not equal to j in row j.

Note that we have $T^{d+1}_d + |C| + |D| = |E|$ and $|Q| = T^{d+1}_d + |G| + |B|$.

4.1.2 Composing $\hat{\Theta}_T$ with ψ^t_d

In order to express $\psi^t_d\hat{\Theta}_T$ as a linear combination of pseudo-standard homomorphisms, we need to consider all possible choices of the following:

- sets $H \subseteq B$ and $I \subseteq G$ and non-negative integers $h \leqslant T^{d+1}_d$ and $k \leqslant l_{b+1} - |Q|$ such that $|H| + h + k + |I| = t$;
- disjoint subsets $J_{a+1}, \dots, J_{d-1}$ of C, disjoint subsets $K_{a+1}, \dots, K_{d-1}$ of D and non-negative integers $P^{a+1}, \dots, P^{d-1}$ such that $P^i + |J_i| + |K_i| = H^i$ for each i;
- sets $L \subseteq C \setminus \bigcup_i J_i$ and $M \subseteq D \setminus \bigcup_i K_i$ and a non-negative integer m such that $m + |L| + |M| = k$;
- a set $N_j \subseteq F_j$ with $|N_j| = L^j$, for $j = d+2, \dots, c$.

We let P be the set defined by the P^i, and we write $\mathbf{J}$ for $(J_{a+1}, \dots, J_{d-1})$ and similarly for $\mathbf{K}$ and $\mathbf{N}$.

Now we compose $\hat{\Theta}_T$ with ψ^t_d. Let $T(H, I, h, k)$ be the row standard tableau obtained from T by replacing entries equal to $d+1$ with entries equal to d; the number of entries replaced in row i is I^i if $i \leqslant a$, H^i if $a < i < d$, h if $i = d$ or k if $i = d+1$. By Lemma 5, we have

$$\psi^t_d\hat{\Theta}_T = \sum_{H,I,h,k} \binom{E \cup I}{I}\binom{l_b - |E| + h}{h}\hat{\Theta}_{T(H,I,h,k)}.$$

(In the case $b = 0$, we have no sets I, H.)

We seek to manipulate $T(H, I, h, k)$ using Lemma 7 to make it pseudo-standard. Let $T(H, I, h, k, \mathbf{J}, \mathbf{K}, P)$ be the row standard tableau obtained from $T(H, I, h, k)$ by replacing the entries equal to d in row $a+i$ with P^i entries equal to $d+1$ and the elements of J_i and K_i for $a < i < d$, replacing the elements of $\bigcup_i J_i \cup \bigcup_i K_i$ in row d together with $|P|$ of the entries equal to $d+1$ with entries equal to d, and performing a row permutation if necessary. By Lemma 7 we have

$$\hat{\Theta}_{T(H,I,h,k)} = \sum_{\mathbf{J},\mathbf{K},P} (-1)^{|H|}\binom{(B \setminus H) \cup P}{P}\prod_i \binom{C_i \cup J_i}{J_i}\binom{D_i \cup K_i}{K_i}\hat{\Theta}_{T(H,I,h,k,\mathbf{J},\mathbf{K},P)}.$$

(In the case $b = 0$, this step is unnecessary, and the corresponding coefficients may be neglected.)

The next step in expressing $\psi^t_d\hat{\Theta}_T$ in terms of pseudo-standard homomorphisms is to eliminate all the entries equal to d in rows $d+1$ and below. Let $T(H, I, h, k, \mathbf{J}, \mathbf{K}, P, L, M, m)$ be the row standard tableau obtained from $T(H, I, h, k, \mathbf{J}, \mathbf{K}, P)$ by replacing the elements of $L \cup M$ in row d, together with m of the entries equal to $d+1$ in row d, with entries equal to d, and replacing the entries equal to d in row $d+1$ with the elements of $L \cup M$ and m entries $d+1$. By Lemma 7 we get

$$\hat{\Theta}_{T(H,I,h,k,\mathbf{J},\mathbf{K},P)} = \sum_{L,M,m} (-1)^k\binom{l_{b+1} - |Q| - k + m}{m}\binom{Q \cup M}{M}\hat{\Theta}_{T(H,I,h,k,\mathbf{J},\mathbf{K},P,L,M,m)}.$$

Finally, we need to move the entries of L in row $d+1$ to their correct rows. Let $T(H, I, h, k, \mathbf{J}, \mathbf{K}, P, L, M, m, \mathbf{N})$ be the row standard tableau obtained from $T(H, I, h, k, \mathbf{J}, \mathbf{K}, P, L, M, m)$ by replacing the entries equal to j in row $d+1$ with the elements of N_j, and replacing the elements of N_j in row j with entries equal to j, for $j = d+2, \dots, c$. Applying Lemma 7 for each j in turn, we find that

$$\hat{\Theta}_{T(H,I,h,k,\mathbf{J},\mathbf{K},P,L,M,m)} = \sum_{\mathbf{N}} (-1)^{|L|}\binom{Q \cup M \cup N_{d+2}}{N_{d+2}}\binom{Q \cup M \cup N_{d+2} \cup N_{d+3}}{N_{d+3}} \dots \binom{Q \cup M \cup N_{d+2} \cup N_{d+3} \cup \dots \cup N_c}{N_c}\hat{\Theta}_{T(H,I,h,k,\mathbf{J},\mathbf{K},P,L,M,m,\mathbf{N})}.$$

(This step may be ignored when $b = r$, or indeed when $m_{b+1} = m_b + 1$.)

By combining these equations, we can find an expression for $\psi^t_d \hat{\Theta}_T$ as a linear combination of pseudo-standard homomorphisms.

4.1.3 Comparing coefficients of S

Now we take a pseudo-standard λ-tableau S of type ν, and examine how arises in the expressions for $\psi^t_d \hat{\Theta}_T$ for various T, Again, we shall label several sets. Suppose that, in S:

- X is the set of rows other than d that contain elements equal to d, i.e. $X^i = T^d_i$ for $i \leqslant a$;
- for $a+1 \leqslant i \leqslant d-1$, Γ_i is the set of entries between $d+2$ and c in row i, and Δ_i is the set of entries greater than c in row i;
- Γ is the set of entries between $d+2$ and c in row d, and Δ is the set of entries greater than c in row d;
- Ψ is the set of entries in row $d+1$ not equal to $d+1$;
- for $d+2 \leqslant j \leqslant c$, Φ_j is the set of entries not equal to j in row j;
- Π is the set of rows between $a+1$ and $d-1$ which contain entries equal to $d+1$ (i.e. $\Pi^i = S^{d+1}_i$ for $a < i < d$), Y is the set of rows higher than row $a+1$ containing entries equal to $d+1$, i.e. $Y^i = S^{d+1}_i$ for $i \leqslant a$);

We wish to write S as $T(H, I, h, k, \mathbf{J}, \mathbf{K}, P, L, M, m, \mathbf{N})$, and to this end we choose:

- $J_i \subseteq \Gamma_i$ and $K_i \subseteq \Delta_i$, for $a+1 \leqslant i \leqslant d-1$;
- $P \subseteq \Pi$, $I \subseteq X$ and disjoint subsets $M, N_{d+2}, \dots, N_c$ of Ψ.

We then let

- $C_i = \Gamma_i \setminus J_i$,
- $D_i = \Delta_i \setminus K_i$,
- $H^i = P^i + |J_i| + |K_i|$,
- $B = (\Pi \setminus P) \cup H$,
- $E = X \setminus I$,
- $G = Y \cup I$,
- $Q = \Psi \setminus (M \cup N_{d+2} \cup \cdots \cup N_c)$,
- $F_j = \Phi_j \cup N_j$,
- $L^j = |N_j|$,
- $C = \Gamma \cup \bigcup_i J_i \cup L$,
- $D = \Delta \cup \bigcup_i K_i \cup M$ and

Finally, we choose k such that $|L|+|M| \leqslant k \leqslant l_{b+1}-|Q|$, and put $m = k-|L|-|M|$ and $h = t-k-|H|-|I|$. There is then a unique pseudo-standard λ-tableau T of type μ such that

$$S = T(H, I, h, k, \mathbf{J}, \mathbf{K}, P, L, M, m, \mathbf{N}),$$

and we seek the sum, over all T obtained in this way, of the product of the coefficients determined above and $f(T)$.

We have $c_i(T) = c_i(S)$ whenever $i \neq b, b+1$, while

$$c_b(T)c_{b+1}(T) = \\ |Q|!\prod_j |F_j|!\prod_i (B^i + |C_i| + |D_i|)!(T_d^{d+1} + |C| + |D|)!\gamma_b^{(s-|B|-\sum|C_i|-\sum|D_i|-|E|)\downarrow}\gamma_{b+1}^{(s-|Q|-\sum|F_j|)\downarrow}.$$

By combining all the expressions we have found so far, the proof of Theorem 12 reduces to the following combinatorial manipulation.

Proposition 16. *The sum of*

$$|Q|!\prod_j |F_j|!\prod_i (B^i + |C_i| + |D_i|)!(T_d^{d+1} + |C| + |D|)!$$
$$\times \gamma_b^{(s-|B|-\sum|C_i|-\sum|D_i|-|E|)\downarrow}\gamma_{b+1}^{(s-|Q|-\sum|F_j|)\downarrow}$$
$$\times \binom{E\cup I}{I}\binom{l_b-|E|+h}{h}(-1)^{|H|}\prod_i \binom{C_i\cup J_i}{J_i}\binom{D_i\cup K_i}{K_i}\binom{(B\setminus H)\cup P}{P}$$
$$\times \binom{l_c-|Q|-k+m}{m}\binom{Q\cup M}{M}(-1)^{k+|L|}\binom{Q\cup M\cup N_{d+2}}{N_{d+2}}\binom{Q\cup M\cup N_{d+2}\cup N_{d+3}}{N_{d+3}}\cdots\binom{Q\cup M\cup N_{d+2}\cup N_{d+3}\cup\cdots\cup N_c}{N_c}$$

over all possible choices of $\mathbf{J}, \mathbf{K}, P, I, M, \mathbf{N}, k$ *as above, equals*

$$\left(\gamma_b^{(s-n_b(S)-t)\downarrow}(S_d + t)!\prod_{j=m_{b-1}+1}^{d-1} S_j!\right)\frac{(l_b - l_{b+1} + m_{b+1} - m_b + \gamma_b - \gamma_{b+1})^{t\uparrow}}{t!}.$$

Proof. We eliminate the ranges of summation one by one.

Summing over N

We eliminate $N_{d+2}, \ldots, N_c$ in stages: to begin with, we sum over the possible choices of $N_{d+2}, \ldots, N_c$ with fixed union N. The only term which depends on the N_j rather than just their union is the product

$$\prod_{j=d+2}^{c} |F_j|!\binom{Q\cup M\cup N_{d+2}}{N_{d+2}}\binom{Q\cup M\cup N_{d+2}\cup N_{d+3}}{N_{d+3}}\cdots\binom{Q\cup M\cup N_{d+2}\cup N_{d+3}\cup\cdots\cup N_c}{N_c}$$
$$= \prod_{j=d+2}^{c} |\Phi_j + N_j|!\prod_{i=c+1}^{m_r}\frac{\Psi^i!}{(Q\cup M)^i!N_{d+2}^i!\ldots N_c^i!}.$$

We first sum over $N_{d+2}, \ldots, N_c$ of fixed sizes $|N_{d+2}|, \ldots, |N_c|$; by Lemma 13, we get

$$|N|!\prod_{i=c+1}^{m_r}\frac{\Psi^i!}{N^i!}\prod_{j=d+2}^{c}\frac{|\Phi_j + N_j|!}{|N_j|!}.$$

Now we sum over the possible $|N_j|$ (but still fixing N): we have

$$\prod_{j=d+2}^{c} \frac{|\Phi_j + N_j|!}{|N_j|!} = \prod_{j=d+2}^{c} |\Phi_j|!\binom{|\Phi_j+N_j|}{|N_j|},$$

which yields

$$\prod_{j=d+2}^{c} |\Phi_j|!\binom{\sum_j |\Phi_j|+c-d+|N|-2}{|N|}$$

by Lemma 15. Next, we sum over possible sets N while fixing the cardinality $|N|$; the only terms which depend on N rather than just $|N|$ give

$$\binom{\Psi\setminus N}{N}\binom{\Psi}{N} = \binom{\Psi}{M}\binom{\Psi\setminus M}{N},$$

which gives

$$\binom{\Psi}{M}\binom{|\Psi\setminus M|}{|N|}$$

by Lemma 6.

Summing over $M, I, \mathbf{J}, \mathbf{K}, P$ of given cardinalities

Using Lemma 6, we can also sum over the possible choices of M, I, $J_{a+1}, \ldots, J_{d-1}$, $K_{a+1}, \ldots, K_{d-1}$, P of given sizes. Re-assembling the terms, we find that the summand in Proposition 16 is equal to

$$\begin{aligned}
&(S_{d+1} - |M| - |N|)! \prod_{i=a+1}^{d-1} S_i!|X \setminus I|! \prod_{j=d+2}^{c} S_j! \\
&\times \gamma_b^{(s-\sum_{i=a+1}^{d} S_d - t+|I|)\downarrow} \gamma_{b+1}^{(s-\sum_{j=d+1}^{c} S_j+|M|)\downarrow} \\
&\times \binom{|X|}{|I|}\binom{l_b-|X|+t-k-|P|-\sum_i |J_i|-\sum_i |K_i|}{l_b-|X|+|I|}\left(\prod_i \binom{|\Gamma_i|}{|J_i|}\binom{|\Delta_i|}{|K_i|}\right)\binom{|\Pi|}{|P|} \\
&\times \binom{l_{b+1}-S_{d+1}}{k-|N|-|M|}\binom{\sum_{j=d+2}^{c} S_j+c-d+|N|-2}{|N|}|N|!\binom{S_{d+1}}{|M|}\binom{S_{d+1}-|M|}{|N|} \\
&\times (-1)^{|P|+\sum_i |J_i|+\sum_i |K_i|+k+|N|}
\end{aligned}$$

(since $|\Psi| = S_{d+1}$, $|\Phi_j| = S_j$ and $B^i + |C_i| + |D_i| = \Pi^i + |\Gamma_i| + |\Delta_i| = S_i$).

Summing over $|M|, |N|, |P|$

Next, we sum over $|N|$; the product

$$(S_{d+1} - |M| - |N|)!|N|!\binom{S_{d+1}-|M|}{|N|}$$

equals $(S_{d+1} - |M|)!$, which does not depend on $|N|$, so we need only sum

$$\binom{l_{b+1}-S_{d+1}}{k-|N|-|M|}\binom{\sum_{j=d+2}^{c} S_j+c-d+|N|-2}{|N|}(-1)^{|N|}$$

which equals

$$\binom{l_{b+1}-c+d-S_{d+1}-\sum_{j=d+2}^{c} S_j+1}{k-|M|}$$

by Lemma 14. To sum over $|M|$, we write

$$\begin{aligned}\gamma_{b+1}^{(s-\sum_{j=d+1}^{c} S_j+|M|)\downarrow} &= \gamma_{b+1}^{(s-n_{b+1}(S)+|M|)\downarrow}\\ &= \gamma_{b+1}^{(s-n_{b+1}(S))\downarrow}(\gamma_{b+1}-s+n_{b+1}(S))^{|M|\downarrow},\end{aligned}$$

observe that $(S_{d+1}-|M|)!\binom{S_{d+1}}{|M|} = \frac{S_{d+1}!}{|M|!}$, and sum

$$\binom{l_{b+1}-c+d-n_{b+1}(S)+1}{k-|M|}\binom{\gamma_{b+1}-s+n_{b+1}(S)}{|M|}$$

to get

$$\binom{l_{b+1}-c+d+1+\gamma_{b+1}-s}{k}.$$

Next we sum over $|P|$; the terms involving $|P|$ give

$$\begin{aligned}&(-1)^{|P|}\binom{l_b-|X|+t-k-|P|-\sum_i|J_i|-\sum_i|K_i|}{t-k-|P|-\sum_i|J_i|-\sum_i|K_i|-|I|}\binom{|\Pi|}{|P|}\\ &= (-1)^{t+k+\sum_i|J_i|+\sum_i|K_i|+|I|}\binom{-l_b+|X|-|I|-1}{t-k-|P|-\sum_i|J_i|-\sum_i|K_i|-|I|}\binom{|\Pi|}{|P|},\end{aligned}$$

which in turn gives

$$(-1)^{t+k+\sum_i|J_i|+\sum_i|K_i|+|I|}\binom{|\Pi|-l_b+|X|-|I|-1}{t-k-\sum_i|J_i|-\sum_i|K_i|-|I|}.$$

Summing over $|J_i|, |K_i|, |I|$

Next we sum over all possible $|J_i|$ and $|K_i|$ with fixed sums $J=\sum_i|J_i|$ and $K=\sum_i|K_i|$; the summand becomes

$$\begin{aligned}&\prod_{i=a+1}^{d-1} S_i!(|X|-|I|)!c_{b+1}(S)\\ &\times\gamma_b^{(s-n_b(S)-t+|I|)\downarrow}\\ &\times\binom{|X|}{|I|}\binom{|\Pi|-l_b+|X|-|I|-1}{t-k-\sum_i|J_i|-\sum_i|K_i|-|I|}\binom{\sum_i|\Gamma_i|}{J}\binom{\sum_i|\Delta_i|}{K}\\ &\times(-1)^{t+|I|}\frac{l_{b+1}-c+d+1+\gamma_{b+1}-s}{k}.\end{aligned}$$

Summing over J, K and k so that the last two lines of this product give

$$\binom{|X|}{|I|}\binom{|\Pi|-l_b+|X|-|I|+\sum_i|\Gamma_i|+\sum_i|\Delta_i|+l_{b+1}+d-c-s+\gamma_{b+1}}{t-|I|}(-1)^{t+|I|}.$$

Finally, we put

$$\gamma_b^{(s-n_b(S)-t+|I|)\downarrow} = \gamma_b^{(s-n_b(S)-t)\downarrow}(\gamma_b-s+n_b(S)+t)^{|I|\downarrow},$$

and then sum over $|I|$. We end up with

$$\begin{aligned}&c_{b+1}(S)\prod_{i=a+1}^{d-1} S_i!|X|!\gamma_b^{(s-|\Pi|-\sum_i|\Gamma_i|-\sum_i|\Delta_i|-|X|)\downarrow}\\ &\times(-1)^t\frac{(l_b-l_{b+1}+m_{b+1}-m_b+\gamma_b-\gamma_{b+1})^{t\uparrow}}{t!},\end{aligned}$$

as required. □

5 Consequences for $\mathbb{F}_p\mathfrak{S}_n$-homomorphisms

We now try to use our map f defined above to construct homomorphisms between Specht modules in characteristic p. The way in which we do this depends upon the following obvious lemma.

Lemma 17. *Suppose there exists a homomorphism $f : S^\lambda_{\mathbb{Z}} \to M^\mu_{\mathbb{Z}}$ such that, for some $e > 0$,*

- *for all d and t, $\psi^t_d f \in p^e \operatorname{Hom}_{\mathbb{Z}\mathfrak{S}_n}(S^\lambda, M^\nu)$, and*
- *$f \not\subseteq p^e \operatorname{Hom}_{\mathbb{Z}\mathfrak{S}_n}(S^\lambda_{\mathbb{Z}}, M^\mu_{\mathbb{Z}})$.*

Then $\operatorname{Hom}_{\mathbb{F}_p\mathfrak{S}_n}(S^\lambda_{\mathbb{F}_p}, S^\mu_{\mathbb{F}_p}) \neq 0$.

5.1 A proof of Theorem 8

Our first application of our results is a new proof of the Carter–Payne theorem. We are able to avoid problems with linear dependence of pseudo-standard homomorphisms, chiefly by using the following lemma. We maintain the definitions of λ, μ and ν.

Lemma 18. *Let ξ equal either μ or ν. Given $\mathbf{e} = (e_1 \leqslant \ldots \leqslant e_s)$, let $\mathcal{T}_{\mathbf{e}}(\lambda, \xi)$ be the set of pseudo-standard λ-tableaux of type ξ whose first row is*

$$1 \quad 1 \quad \ldots \quad 1 \quad e_1 \quad \ldots \quad e_s.$$

Let $V_{\mathbf{e}}$ be the $\mathbb{Z}$-span of the set

$$\{\hat{\Theta}_T \mid T \in \mathcal{T}_{\mathbf{e}}(\lambda, \xi)\}.$$

Then the sum

$$\sum_{\mathbf{e}} V_{\mathbf{e}}$$

is direct.

Proof. It is enough to show that, given a pseudo-standard λ-tableau T of type ξ, we can write

$$\hat{\Theta}_T = \sum_R x_R \hat{\Theta}_R,$$

where each R is a semi-standard tableau with the same first row as T.

Let $\bar{T}$ be the tableau obtained by removing the first row of T; note that each entry of $\bar{T}$ is at least 2. By Lemma 2 we may write $\hat{\Theta}_{\bar{T}} = \sum_{\bar{R}} x_{\bar{R}} \hat{\Theta}_{\bar{R}}$, where each $\bar{R}$ is semi-standard. For each $\bar{R}$, define a tableau R by adding a row at the top equal to the first row of T. Then the above formula holds (with $x_R = x_{\bar{R}}$) by Lemma 4, and each R is semi-standard, because each entry of R is at least 2 and the first row of T has weakly increasing entries, the first l_0 of which are equal to 1. □

Proof of Theorem 8. We define $f : S^\lambda_{\mathbb{Z}} \to M^\mu_{\mathbb{Z}}$ as in Section 3, taking

$$\gamma_i = -l_i + s - 1 + m_i - m_r$$

for all i. By Proposition 10 and Theorem 12, we have

$$\psi^t_d f = 0$$

except possibly when $d = 1$. So we assume $d = 1$ from now on, and we have

$$\psi^t_1 f = \left(\sum_{S \in \mathcal{T}^0(\lambda,\nu)} \left(\prod_{i=1}^{r} c_i(S) \right) \right) \frac{D^{t\uparrow}}{t!},$$

where D is the distance through which nodes are moved to get from λ to μ. We may also assume that $t \leqslant s$, since for $t > s$ there are no pseudo-standard λ-tableaux of type ν, and so we have $\psi^t_d f = 0$. Write

$$a = \frac{D^{t\uparrow}}{t!};$$

the fact that D is divisible by some power of p greater than s (and hence greater than t) implies that a is divisible by p.

Given a pseudo-standard λ-tableau of type ν whose first row is

$$1 \quad 1 \quad \dots \quad 1 \quad e_1 \quad \dots \quad e_s,$$

we must have $e_1 = \cdots = e_t = 1$; define $\mathbf{e} = (e_1, \dots, e_s)$, and define $\hat{\mathbf{e}}$ by changing $e_1, \dots, e_t$ from 1 to 2. Now if S is a pseudo-standard λ-tableau of type ν, define $\hat{S}$ to be the row standard tableau obtained by changing t of the entries equal to 1 in row 1 to 2s. Then ˆ induces a bijection

$$\mathcal{T}_{\mathbf{e}}(\lambda,\nu) \leftrightarrow \mathcal{T}_{\hat{\mathbf{e}}}(\lambda,\mu)$$

whenever the left-hand side is defined. In fact, ˆ induces a linear isomorphism

$$V_{\mathbf{e}} \cong V_{\hat{\mathbf{e}}};$$

although the set

$$\{\hat{\Theta}_S \mid S \in \mathcal{T}_{\mathbf{e}}(\lambda,\nu)\}$$

might not be linearly independent, we have

$$\left(\sum_{S \in \mathcal{T}_{\mathbf{e}}(\lambda,\nu)} x_S \hat{\Theta}_S = 0 \right) \Leftrightarrow \left(\sum_{S \in \mathcal{T}_{\mathbf{e}}(\lambda,\nu)} x_S \hat{\Theta}_{\hat{S}} = 0 \right)$$

by Lemma 4. Notice also that if $S \in \mathcal{T}^0(\lambda,\nu)$, then

$$\prod_{i=1}^{r} c_i(S) = f(\hat{S}).$$

The result of this is as follows. Let $\mathbf{E}$ be the set of all $\mathbf{e}$ such that $\mathcal{T}_{\mathbf{e}}(\lambda,\nu)$ is defined, and let $\mathbf{G}$ be the set of all $\mathbf{g}$ such that $\mathcal{T}_{\mathbf{g}}(\lambda,\mu)$ is defined. Then we can write

$$f = \sum_{\mathbf{g} \in \mathbf{G}} f_{\mathbf{g}}$$

where $f_{\mathbf{g}} \in V_{\mathbf{g}}$, and we can write

$$\psi_1^t f = \sum_{\mathbf{e}\in\mathbf{E}} v_{\mathbf{e}}$$

with $v_{\mathbf{e}} \in V_{\mathbf{g}}$. By Lemma 18, f is divisible by p^i (i.e. f lies in $p^i \operatorname{Hom}_{\mathbb{Z}\mathfrak{S}_n}(S^\lambda, M^\mu)$) if and only if each $f_{\mathbf{g}}$ is divisible by p^i, and $\psi_d^t f$ is divisible by p^j if and only if each $v_{\mathbf{e}}$ is divisible by p^j. By the above discussion, we have

$$v_{\mathbf{e}} = a f_{\hat{\mathbf{e}}}$$

for all $\mathbf{e} \in \mathbf{E}$; since a is divisible by p, we find that ψ_d^t is divisible by a greater power of p than f is (the fact that f is non-zero follows from Proposition 21 below), and this is sufficient. □

5.2 Quasi-standard homomorphisms

Now we try to find more general applications of Proposition 10 and Theorem 12. As remarked earlier, the fact that the pseudo-standard homomorphisms are linearly dependent presents problems. Here we try to address this.

Lemma 19. *Let α be the partition (l^m), and suppose that β is a partition of lm with each $\beta_i \leqslant l$ and with $\beta_1 + \cdots + \beta_m \geqslant l(m-1)$. Let $T_1, \ldots, T_a$ be the row standard α-tableaux of type β in which all entries equal to i occur in row i, for $1 \leqslant i \leqslant m$. Then:*

1. *there is a unique semi-standard α-tableau T_0 of type β in which all the entries greater than m lie in row m;*

2. $$\sum_{i=1}^{a} \hat{\Theta}_{T_i} = \pm\hat{\Theta}_{T_0},$$
   where the sign depends only upon l, m and $\beta_1, \ldots, \beta_m$;

3. *if T is an α-tableau of type β in which all the entries greater than m occur in row m and in which the first $m-1$ rows form a semi-standard (l^{m-1})-tableau, then $\hat{\Theta}_T$ is a scalar multiple of $\hat{\Theta}_{T_0}$.*

Proof.

1. In T_0, the entries greater than m must occupy (in increasing order) the rightmost places of row m. Now the remainder of T_0 must be column standard and filled with entries equal to or less than m, and so the entries in row i must all be either i or $i+1$, for $1 \leqslant i < m$. This tells us how to construct T_0: we must put all the 1s in row 1, from the left, and fill in the remainder of row 1 with 2s. The remaining 2s go at the left of row 2, which must then be filled with 3s, and so on. This constructs T_0 uniquely; that T_0 is semi-standard follows from the condition $\beta_i \leqslant l$.

2. Apply Lemma 7 repeatedly to T_0, moving the entries equal to m into row m, then moving the entries equal to $m-1$ into row $m-1$, and so on. The binomial coefficients which arise are all equal to 1, and each T_i arises uniquely. The signs occurring are all equal to

$$(-1)^{\sum_{j=2}^{m}(T_0)_{j-1}^{j}}.$$

3. We examine the smallest entry g in row m of S. If $g = m$, then S is row equivalent to T_0, and we are done. Otherwise, suppose there are h entries in row m equal to g. Since the entries in row g of S all equal g or $g + 1$, Lemma 7 implies that $\hat{\Theta}_S$ is a scalar multiple of $\hat{\Theta}_{S'}$, where S' is obtained from S by moving the entries equal to g from row m to row g, and moving h entries equal to $g + 1$ from row g to row m. S' then satisfies the hypotheses of the lemma, and the smallest entry in row m of S' is $g + 1$, so we are done by induction.

□

Lemma 19 allows us to 'group together' our pseudo-standard tableaux. Given a λ-tableau T of type μ and an integer $1 \leqslant i \leqslant r$, let $T(i)$ be the tableau consisting of the $m_i - m_{i-1}$ rows of length l_i. Say that T is *quasi-standard* if:

- each $T(i)$ is semi-standard;
- the entries of $T(i)$ are all strictly greater than m_{i-1};
- any entries of $T(i)$ which are greater than m_i occur in the bottom row of $T(i)$.

Let $Q(\lambda, \mu)$ denote the set of quasi-standard λ-tableaux of type μ. For example, the quasi-standard $(5, 3^2, 2)$-tableaux of type $(3^3, 2^2)$ are

$$\begin{array}{l}1\,1\,1\,5\,5\\2\,2\,2\\3\,3\,3\\4\,4\end{array}\quad\begin{array}{l}1\,1\,1\,4\,5\\2\,2\,2\\3\,3\,3\\4\,5\end{array}\quad\begin{array}{l}1\,1\,1\,3\,5\\2\,2\,2\\3\,3\,5\\4\,4\end{array}\quad\begin{array}{l}1\,1\,1\,3\,5\\2\,2\,2\\3\,3\,4\\4\,5\end{array}\quad\begin{array}{l}1\,1\,1\,2\,5\\2\,2\,3\\3\,3\,5\\4\,4\end{array}$$

$$\begin{array}{l}1\,1\,1\,2\,5\\2\,2\,3\\3\,3\,4\\4\,5\end{array}\quad\begin{array}{l}1\,1\,1\,4\,4\\2\,2\,2\\3\,3\,3\\5\,5\end{array}\quad\begin{array}{l}1\,1\,1\,3\,4\\2\,2\,2\\3\,3\,5\\4\,5\end{array}\quad\begin{array}{l}1\,1\,1\,3\,4\\2\,2\,2\\3\,3\,4\\5\,5\end{array}\quad\begin{array}{l}1\,1\,1\,2\,4\\2\,2\,3\\3\,3\,5\\4\,5\end{array}$$

$$\begin{array}{l}1\,1\,1\,2\,4\\2\,2\,3\\3\,3\,4\\5\,5\end{array}\quad\begin{array}{l}1\,1\,1\,3\,3\\2\,2\,2\\3\,5\,5\\4\,4\end{array}\quad\begin{array}{l}1\,1\,1\,3\,3\\2\,2\,2\\3\,4\,5\\4\,5\end{array}\quad\begin{array}{l}1\,1\,1\,3\,3\\2\,2\,2\\3\,4\,4\\5\,5\end{array}\quad\begin{array}{l}1\,1\,1\,2\,3\\2\,2\,3\\3\,5\,5\\4\,4\end{array}$$

$$\begin{array}{l}1\,1\,1\,2\,3\\2\,2\,3\\3\,4\,5\\4\,5\end{array}\quad\begin{array}{l}1\,1\,1\,2\,3\\2\,2\,3\\3\,4\,4\\5\,5\end{array}\quad\begin{array}{l}1\,1\,1\,2\,2\\2\,3\,3\\3\,5\,5\\4\,4\end{array}\quad\begin{array}{l}1\,1\,1\,2\,2\\2\,3\,3\\3\,4\,5\\4\,5\end{array}\quad\begin{array}{l}1\,1\,1\,2\,2\\2\,3\,3\\3\,4\,4\\5\,5\end{array}.$$

We aim to express f in terms of quasi-standard homomorphisms. Define an equivalence relation on the set of λ-tableaux of type μ by saying that two tableaux are equivalent if one can be obtained from the other by re-arranging entries within rows and moving entries between rows of the same length. Given an equivalence class C, suppose C contains the pseudo-standard tableaux $T_1, \dots, T_c$, with $c \geqslant 1$. Then C contains a unique quasi-standard tableau T_0: the rows of length l_i contain at most l_i entries equal to any particular value, and at most $s \leqslant l_i$ entries greater than m_i; these can be re-arranged as in Lemma 19(1). Now we apply Lemma 19 for each i, bearing in mind Lemma 4. We get

$$\hat{\Theta}_{T_0} = \pm \sum_{i=1}^{c} \hat{\Theta}_{T_i}.$$

Since the homomorphisms corresponding to $T_1, \dots, T_c$ occur with the same coefficient in f, we have proved the following.

Lemma 20. *The map $f : S^\lambda \to M^\mu$ can be written as a $\mathbb{Z}$-linear combination $\sum_{S \in Q(\lambda,\mu)} c_S \hat{\Theta}_S$ of quasi-standard homomorphisms, with each coefficient c_S being $\pm f(T)$ for some pseudo-standard tableau T.*

For example, each of the quasi-standard $(5, 3^2, 2)$-tableaux of type $(3^3, 2^2)$ above is equivalent to a unique pseudo-standard tableau, except for $\begin{array}{ccccc}1&1&1&2&3\\2&2&3\\3&4&5\\4&5\end{array}$, where we have

$$\hat{\Theta}_{\begin{array}{ccccc}1&1&1&2&3\\2&2&3\\3&4&5\\4&5\end{array}} = -\hat{\Theta}_{\begin{array}{ccccc}1&1&1&2&3\\2&2&4\\3&3&5\\4&5\end{array}} - \hat{\Theta}_{\begin{array}{ccccc}1&1&1&2&3\\2&2&5\\3&3&4\\4&5\end{array}}.$$

However, we are still in difficulty: the quasi-standard homomorphisms are not linearly independent either. Notice that, among the quasi-standard $(5, 3^2, 2)$-tableaux of type $(3^3, 2^2)$ above, the tableaux

$$\begin{array}{ccccc}1&1&1&3&3\\2&2&2\\3&5&5\\4&4\end{array}, \quad \begin{array}{ccccc}1&1&1&2&3\\2&2&3\\3&5&5\\4&4\end{array}, \quad \begin{array}{ccccc}1&1&1&2&2\\2&3&3\\3&5&5\\4&4\end{array}$$

are not semi-standard, and in fact

$$\hat{\Theta}_{\begin{array}{ccccc}1&1&1&3&3\\2&2&2\\3&5&5\\4&4\end{array}} = -\hat{\Theta}_{\begin{array}{ccccc}1&1&1&3&3\\2&2&2\\3&4&5\\4&5\end{array}} - \hat{\Theta}_{\begin{array}{ccccc}1&1&1&3&3\\2&2&2\\3&4&4\\5&5\end{array}}.$$

Note that this problem – having quasi-standard tableaux which are not semi-standard – arises whenever we have $l_i - l_{i+1} < s$. We content ourselves with the following: let T be a pseudo-standard or quasi-standard λ-tableau of type μ, and say that T is *nice* if, for each $i \geqslant 1$, the number of entries greater than m_i in the rows of length l_i is at most $l_i - l_{i+1}$. For instance, of the quasi-standard $(5, 3^2, 2)$-tableaux of type $(3^3, 2^2)$ above, the first eleven are nice while the last nine are not. Note that a nice quasi-standard tableau is necessarily semi-standard.

Proposition 21. *The homomorphisms corresponding to nice quasi-standard tableaux are linearly independent of the other quasi-standard homomorphisms.*

Proof. Take a quasi-standard λ-tableau T of type μ which is not nice. It suffices to show that when $\hat{\Theta}_T$ is expressed as a linear combination of semi-standard homomorphisms $\hat{\Theta}_{T_1}, \dots, \hat{\Theta}_{T_a}$ (which we can do, by Lemma 2), none of the T_j is a nice quasi-standard tableau.

Let i be minimal such that the number of entries greater than m_i in row m_i is greater than $l_i - l_{i+1}$. Let t be the $(l_i, l_{i+1}^{m_{i+1}-m_i}, \dots, l_r^{m_r-m_{r-1}})$-tableau consisting of rows $m_i, \dots, m_r$ of T. Note that the entries of t are all at least m_i. Express $\hat{\Theta}_t$ as a linear combination $\sum_{j=1}^{a} \beta_j \hat{\Theta}_{t_j}$ of semi-standard homomorphisms, and let T_j be the λ-tableau which agrees with T on rows $1, \dots, m_i - 1$ and with t_j on rows $m_i, \dots, m_r$. Then by Lemma 4 we have

$$\hat{\Theta}_T = \sum_{j=1}^{a} \beta_i \hat{\Theta}_{T_j}.$$

Moreover, for each j either T_j is semi-standard or $\hat{\Theta}_{T_j} = 0$. Indeed, the entries of T_j are all at least m_i, while the entries in the first l_i places of row $m_i - 1$ of T are all at most m_i (note that we might have $m_i - 1 = m_{i-1}$, but in this case we know the minimality of i implies that only the last $l_{i-1} - l_i$ entries of row m_{i-1} can be greater than m_i). So the only way T_j can fail to be semi-standard is if rows m_{i-1} and m_i contain an entry m_i in the same column, in which case $\hat{\Theta}_{T_j} = 0$.

It remains to see that none of the T_j is a nice quasi-standard tableau. But each t_j contains fewer than l_{i+1} entries equal to m_i, and so there must be more than $l_i - l_{i+1}$ entries greater than m_i in row m_i. And so, even if T_j is quasi-standard, it is not nice. □

We can now state our main result.

Theorem 22. *Suppose $\hat{\lambda}$ and $\hat{\mu}$ are partitions of $n + m$ such that $[\hat{\mu}]$ is obtained from $[\hat{\lambda}]$ by moving s nodes from row a to row b, where $a < b$. Define*

$$\lambda = (\lambda_a - \lambda_b, \dots, \lambda_{b-1} - \lambda_b), \qquad \mu = (\mu_a - \lambda_b, \dots, \mu_{b-1} - \lambda_b, s)$$

and suppose that λ and μ are partitions of n.

If we can choose $e > 0$ and $\gamma_1, \dots, \gamma_r \in \mathbb{Z}$ so that

- *for some nice pseudo-standard λ-tableau T, the coefficient $f(T)$ is not divisible by p^e, while*
- *all the coefficients $g(S)$ appearing in Theorem 12 are divisible by p^e,*

then

$$\mathrm{Hom}_{\mathbb{F}_p \mathfrak{S}_n}(S^\lambda, S^\mu) \neq 0$$

and hence

$$\mathrm{Hom}_{\mathbb{F}_p \mathfrak{S}_{n+m}}(S^{\hat{\lambda}}, S^{\hat{\mu}}) \neq 0.$$

Proof. This follows by combining Theorem 9, Theorem 12, Lemma 17, Lemma 20 and Proposition 21. □

Remark. In practice, Theorem 22 is most useful when we choose

$$\gamma_i = \begin{cases} l_0 - l_i + m_i + s - 1 & (i \leqslant a) \\ -l_i + m_i - m_r + s - 1 & (i > a) \end{cases}$$

for some $0 \leqslant a \leqslant r$. Given d and t, we find that $\psi^t_d f = 0$ except when $d = m_a$. With luck, the pseudo-standard homomorphisms corresponding to this value of d can be analysed in a similar way to that used in our proof of Theorem 8.

Example. We illustrate our results with a small example which is not covered by Theorem 8. Let $p = 3$, $\lambda = (7, 3)$ and $\mu = (4, 3^2)$, so that we have $r = 1$ and $s = 3$. We choose $\gamma_1 = -1$. Then the nice quasi-standard λ-tableau

$$T = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&3\\\hline 2&3&3\\\cline{1-3}\end{array}$$

has $f(T) = -2$, which is not divisible by 3. Now we look at the coefficients appearing in Theorem 12. For $d = 2$, each of these coefficients is divisible by $l_r - s + 1 + \gamma_r = 0$, while for $d = 1$, each of the $g(S)$ is divisible by $l_0 + s - l_1 + m_1 - 1 - \gamma_1 = 6$; so all the $g(S)$ are divisible by 3. Hence

$$\mathrm{Hom}_{\mathbb{F}_3\mathfrak{S}_{10}}(S^{(7,3)}, S^{(4,3^2)}) \neq 0.$$

It is possible to obtain more information by examining those quasi-standard tableaux which are not nice: say that a quasi-standard λ-tableau T of type μ is *good* if the tableau formed by rows $1, m_1, \ldots, m_r$ of T is semi-standard. Then we have the following.

Proposition 23. *The homomorphisms corresponding to good quasi-standard λ-tableaux of type μ are linearly independent.*

Proof. We may copy the proof of [4, Lemma 13.11], in which it is proved that the semi-standard homomorphisms are linearly independent. This only depends on the fact that the semi-standard tableaux are row standard and that each $\sim_{\mathrm{col}}$ class of tableaux contains at most one semi-standard tableau. The same is true of the good quasi-standard tableaux. □

If we could find a 'straightening' result describing how to express homomorphisms corresponding to non-good quasi-standard tableaux in terms of those corresponding to good quasi-standard tableaux, then it would be possible to strengthen our results. We leave this for a future paper.

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