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General vertices in ordinary quivers for symmetric group algebras

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Abstract

In [11], Martin and Russell construct part of the ordinary quiver of the principal block of $\mathfrak{S}_{np}$ in characteristic p; they define the notion of a *general vertex*, and show that around general vertices, the quiver assumes an n-dimensional lattice-like structure. Here we use different methods to find more vertices in the quiver of this general type.

1 Introduction

Throughout this paper, we let p be a prime, and k be a field of characteristic p. Let $\mathfrak{S}_m$ denote the symmetric group on m letters. Recall that the *ordinary quiver* of a k-algebra A is a quiver with vertices indexed by the isomorphism classes of simple modules for A, with the number of arrows from M to N being $\dim_k(\mathrm{Ext}^1_A(M,N))$. In the case of the symmetric group, all simple modules are self-dual, and so we draw the ordinary quiver as a multigraph, with an edge indicating an arrow in each direction.

We shall be concerned with the principal block of the symmetric group $\mathfrak{S}_{np}$, where $n \leqslant p$. The ordinary quiver for $k\mathfrak{S}_p$ is well known; the quiver for $k\mathfrak{S}_{2p}$ was constructed by Martin in [9], and that for $k\mathfrak{S}_{3p}$ by Martin and Russell in [10]. In these prototypical cases a lattice-like structure is observed: the quiver for $k\mathfrak{S}_p$ is linear, while part of the quiver for $k\mathfrak{S}_{2p}$ resembles a lattice of squares, and part of that for $k\mathfrak{S}_{3p}$ a lattice of cubes. In [11], Martin and Russell define a 'general' vertex of the quiver using the $\langle n^p\rangle$ abacus notation and prove that such a lattice structure exists in general. Their main result may be stated as follows.

Theorem 1.1. [11, Theorem 4.2] *Let $n \geqslant 1$, and let $\lambda = \langle a_1,\dots,a_n\rangle$ be a general vertex in the principal p-block B of $k\mathfrak{S}_{np}$. Then $\dim_k \mathrm{Ext}^1_B(D^\lambda, D^\mu)$ equals 1 if μ is one of the $2n$ partitions labelled $\langle a_1,\dots,a_r \pm 1,\dots,a_n\rangle$ for $1 \leqslant r \leqslant n$, and 0 otherwise.*

We shall re-prove this result and extend it to include other vertices of the quiver. Using $^{(2)}$ to indicate a bead of weight two on the abacus, we define a 'p-general' vertex and a 'semi-general' vertex, and prove the following.

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Theorem 1.2.

1. *Let $n \geqslant 2$, and let $\lambda = \langle a_1, \ldots, a_{n-1}, p\rangle$ be a p-general vertex in the principal p-block B of $k\mathfrak{S}_{np}$. Then* $\dim_k \mathrm{Ext}^1_B(D^\lambda, D^\mu)$ *equals* 1 *if μ is one of the $2n$ partitions labelled $\langle a_1, \ldots, a_{n-1}, p-1\rangle$, $\langle a_1^{(2)}, a_2, \ldots, a_{n-1}\rangle$ or $\langle a_1, \ldots, a_r \pm 1, \ldots, a_{n-1}, p\rangle$for $1 \leqslant r \leqslant n-1$, and zero otherwise.*

2. *Let $n \geqslant 2$, and let $\lambda = \langle a_1, \ldots, a_{j-1}, a_j^{(2)}, a_{j+1}, \ldots, a_{n-1}\rangle$ be a semi-general vertex in the principal p-block B of $k\mathfrak{S}_{np}$. Then* $\dim_k \mathrm{Ext}^1_B(D^\lambda, D^\mu)$ *equals* 1 *if μ is one the $2n$ (or $2n-1$, if $j = n-1$) labelled as follows:*

   - $\langle a_1, \ldots, a_r \pm 1, \ldots, a_{j-1}, a_j^{(2)}, a_{j+1}, \ldots, a_{n-1}\rangle$*, where $1 \leqslant r \leqslant j-1$;*
   - $\langle a_1, \ldots, a_{j-1}, (a_j \pm 1)^{(2)}, a_{j+1}, \ldots, a_{n-1}\rangle$*;*
   - $\langle a_1, \ldots, a_{j-1}, a_j^{(2)}, a_{j+1}, \ldots, a_r \pm 1, \ldots, a_{n-1}\rangle$*, where $j+1 \leqslant r \leqslant n-1$;*
   - $\langle a_1, \ldots, a_{j-2}, a_{j-1}^{(2)}, a_j, \ldots, a_{n-1}\rangle$ *(provided $j > 1$);*
   - $\langle a_1, \ldots, a_{n-1}, p\rangle$ *(if $j = 1$);*
   - $\langle a_1, \ldots, a_j, a_{j+1}^{(2)}, a_{j+1}, \ldots, a_{n-1}\rangle$ *(provided $j < n-1$);*

   otherwise, $\mathrm{Ext}^1_B(D^\lambda, D^\mu) = 0$.

Hence, by allowing a_n to go to p and then introducing weight 2 beads, we find further vertices of general type before reaching a 'wall' of the quiver. We shall make our terms precise later. The technique of induction and restriction used in [11] breaks down for semi-general vertices, so we adopt a new one, by examining the effect of the Mullineux algorithm on various types of partitions.

In the last two parts of the paper, we examine the structures of Specht modules and projective modules corresponding to general vertices.

1.1 Symmetric group representations

The salient points of the representation theory of the symmetric groups over fields of arbitrary characteristic may be found in James's book [4]; here we recall some results not found there.

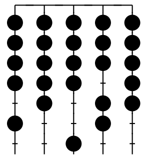
1.1.1 The abacus

In what follows, we make extensive use of James's abacus: we take an abacus with p vertical runners labelled $1, \ldots, p$. We then denote the top position on runner i by $i-1$, the next position down by $i-1+p$, and so on. Given a partition λ and an integer r equal to or greater than the number of non-zero parts of λ, we define the *beta-numbers* for λ to be the values $\beta_i = \lambda_i + r - i$, for $i = 1, \ldots, r$. We then display λ on the abacus by placing a bead at position β_i for each i. Clearly, choosing a different value of r will give a different abacus display. Nakayama's conjecture then tells us that S^λ and S^μ lie in the same block of $k\mathfrak{S}_m$ (we shall frequently abuse notation by saying that λ and μ lie in the same block) if and only if λ and μ can be displayed on abacuses with the same numbers of beads on corresponding runners.

Given an abacus display for a partition λ, the partition whose abacus display we obtain by moving all the beads on the abacus up as far as they will go we call the *p-core* of λ; the p-core is a partition of $m - \omega p$ for some $\omega \geqslant 0$ which we call the *weight* of the block.

We shall concentrate largely on the principal block B of $k\mathfrak{S}_{np}$. Take $r = np$. The p-core of partitions in B is the partition of zero, and so B has an abacus with n beads on each runner. We shall use the

$\langle n^p \rangle$-notation to describe partitions in B: we write $a^{(j)}$ to denote a bead moved down j places on runner a, and a^j to denote j beads each moved down one place on runner a. For example, when $p = n = 5$, the partition $(8, 5, 3^3, 2, 1)$ has an abacus display



which we denote $\langle 1, 3^{(2)}, 4^2 \rangle$.

Since the abacus is traditionally drawn with runners going downwards, we shall refer to *lower* beads to mean those with higher beta-numbers, i.e. those lower down the abacus diagram. We number the rows of the abacus consecutively starting at $1 - n$, so that if there are no weight beads on a runner, then the lowest bead is in row 0. We shall also talk of position i of row j to mean the intersection of runner i and row j, i.e. the position corresponding to the beta-number $(n + j - 1)p + i - 1$.

We frequently abuse notation by identifying a partition, its abacus display and the corresponding vertex of the quiver.

1.1.2 Branching rules

We frequently employ the classical Branching Rule ([4], Theorem 9.3), which describes the induced and restricted Specht modules. We also use the modular branching rules found by Kleshchev [2] to describe induced and restricted simple modules. Since these are less well-known, we describe them here in terms of the abacus.

Let D^λ be a simple module lying in a block A of $k\mathfrak{S}_m$, and take an abacus display for λ. Say that a bead b on runner i and in row r of the display is:

- *normal* if there is no bead immediately to the left of b and if for every $j \geqslant 1$ the number of beads on runner i in rows $r + 1, \ldots, r + j$ is at least the number of beads on runner $i - 1$ in rows $r + 1, \ldots, r + j$;
- *good* if b is the highest normal bead on runner i;
- *conormal* if there is no bead immediately to the right of b and if for every $j \geqslant 1$ the number of beads on runner i in rows $r - 1, \ldots, r - j$ is at least the number of beads on runner $i + 1$ in rows $r - 1, \ldots, r - j$;
- *cogood* if b is the lowest conormal bead on runner i.

Let A^+ be the block of $k\mathfrak{S}_{m+1}$ whose abacus is obtained by moving a bead from runner i to runner $i + 1$, and let A^- be the block of $k\mathfrak{S}_{m-1}$ whose abacus is obtained by moving a bead from runner i to runner $i - 1$. If b is normal, let λ_b be the partition obtained by moving b one place to its left, and if b is conormal, let λ^b be the partition obtained by moving b one place to its right. With these definitions, the following holds.

Theorem 1.3. [2, Theorems E,E′]

1. *$D^\lambda \downarrow^A_{A^-} = 0$ if there are no normal beads on runner i. Otherwise $D^\lambda \downarrow^A_{A^-}$ is an indecomposable module with simple cosocle and socle both isomorphic to D^{λ_b}, where b is the unique good bead on runner i; $D^\lambda \downarrow^A_{A^-}$ is simple if and only if b is the only normal bead on runner i.*

2. *$D^\lambda\uparrow_A^{A^+} = 0$ if there are no conormal beads on runner i. Otherwise $D^\lambda\uparrow_A^{A^+}$ is an indecomposable module with simple cosocle and socle both isomorphic to D^{λ^b}, where b is the unique cogood bead on runner i; $D^\lambda\uparrow_A^{A^+}$ is simple if and only if b is the only conormal bead on runner i.*

1.1.3 Schaper's formula

We shall make use of Schaper's formula [13], which provides a method for estimating the decomposition numbers $[S^\lambda : D^\mu]$. Given partitions λ and μ of m with $\lambda \rhd \mu$, define $H(\lambda,\mu)$ to be the set of ordered pairs (g,h), where

- g is a rim hook of the Young diagram $[\lambda]$ of λ;
- h is a rim hook of the Young diagram $[\mu]$ of μ;
- $[\lambda] \setminus g = [\mu] \setminus h$.

Now define

$$c_{\lambda,\mu} = \sum_{(g,h)\in H(\lambda,\mu)} (-1)^{l(g)+l(h)+1} \nu_p(|g|),$$

where $|g|$ is the number of nodes of g and $l(g)$ its leg length.

A weak version of Schaper's formula may now be stated as follows.

Proposition 1.4. *Let μ and ν be partitions of m, with ν p-regular and $\nu \rhd \mu$. Then*

$$[S^\mu : D^\nu] \leqslant \sum_{\lambda \rhd \mu} c_{\lambda,\mu}[S^\lambda : D^\nu],$$

and the left-hand side is zero only if the right-hand side is.

In fact, the Specht module S^μ has a certain filtration

$$S^\mu = S_0 \geqslant S_1 \geqslant \dots$$

in which S_0/S_1 equals D^μ if μ is p-regular, and zero otherwise and in which the other quotients S_i/S_{i+1} are self-dual (and hence, if their composition factors are distinct, semi-simple). The expression on the right-hand side of the inequality in Proposition 1.4 is then equal to

$$\sum_{i\geqslant 0} i\left[\frac{S_i}{S_{i+1}} : D^\nu\right].$$

An important consequence of Schaper's formula is a refinement of the above statement that $[S^\mu : D^\lambda] = 0$ unless $\lambda \unrhd \mu$. Given a prime p and partitions λ,μ of m, write $\lambda \succcurlyeq \mu$ if an abacus display for μ can be obtained from an abacus display for λ by moving a bead up from position $i + wp$ to position i and then moving a bead down from position j to position $j + wp$, where i, j, w are non-negative integers with $i \geqslant j$. Extend $\succcurlyeq$ transitively to obtain a partial order on the set of partitions of n, of which $\unrhd$ is a refinement. Schaper's formula has the following corollary.

Proposition 1.5. *Let λ, μ be partitions of n with λ p-regular. Then $[S^\mu : D^\lambda] = 0$ unless $\lambda \succcurlyeq \mu$.*

We shall use this 'Schaper dominance' order exclusively from now on.

1.2 The alternating representation

We denote by sgn the one-dimensional representation of $\mathfrak{S}_m$ which sends each permutation π to its signature $(-1)^\pi$. Given any partition λ, we define the conjugate partition λ' by

$$\lambda'_i = \max\{j \mid \lambda_j \geqslant i\}.$$

We then have the following result.

Theorem 1.6. [4, Theorem 8.15] *Over any field, $S^\lambda \otimes \mathrm{sgn}$ is isomorphic to the dual of $S^{\lambda'}$.*

This result will prove very useful later; note that there is an easy way to find the conjugate of a partition displayed on an abacus: we simply replace all the beads with spaces and all the spaces with beads, and then rotate the resulting diagram through 180 degrees. In particular, using the $\langle p^n \rangle$-notation for the principal block of $k\mathfrak{S}_{np}$, we simply change each term $a^{(s)}$ to $\tilde{a}^s$, where $\tilde{a} = p + 1 - a$, and vice versa.

A corollary of Theorem 1.6 is that if D^λ lies in a block with weight ω and p-core ν, then $D^\lambda \otimes \mathrm{sgn}$ will lie in the weight ω block with p-core ν'. We say that such blocks are conjugate.

Since sgn is one-dimensional, $M \otimes \mathrm{sgn}$ will be irreducible for any irreducible $k\mathfrak{S}_m$-module M. This gives a bijection from the set of simple modules to itself; in characteristic zero, the Specht modules are simple and self-dual, and so we have $S^\lambda \otimes \mathrm{sgn} \cong S^{\lambda'}$. But in odd positive characteristic the situation is more complicated. We need a bijection * from the set of p-regular partitions to itself such that

$$D^\lambda \otimes \mathrm{sgn} = D^{\lambda^*}$$

for all p-regular λ. Mullineux [12] described a bijection f and conjectured that $f(\lambda) = \lambda^*$; the conjecture was proved by Ford and Kleshchev [3], by using the equivalent algorithm given by Kleshchev in [7]. We now describe the algorithm.

For each p-regular partition λ we construct a sequence of partitions $\lambda = \lambda^0, \ldots, \lambda^u = (0)$, where λ^i is a partition of some $m_i < m$, and λ^{i+1} is obtained from λ^i by 'removing the p-rim'; on the abacus, this is achieved by the following.

1. Let x be the greatest occupied position in the abacus display of λ^i.
2. If there is no unoccupied position less than x in the display, then stop. Otherwise, let y be
   - the greatest unoccupied position less than x on the same runner as x, if there are any, or
   - the least unoccupied position in the display, if not.

   Move the bead at position x to position y.
3. Let x be the greatest occupied position less than y in the abacus, and return to step 2.

It is clear that this procedure will eventually produce the partition (0). Given the partitions $\lambda^0, \ldots, \lambda^u$, define the *Mullineux symbol* for λ to be the matrix

$$\begin{pmatrix} r_1 & \ldots & r_u \\ s_1 & \ldots & s_u \end{pmatrix}$$

where r_i is the number of non-zero parts of λ^{i-1} and $s_i = m_i - m_{i-1}$, i.e. the length of the p-rim removed to get from λ_{i-1} to λ^i. (We sometimes write $\binom{r}{s}^i$ in the Mullineux symbol to indicate i consecutive columns equal to $\binom{r}{s}$.)

Mullineux shows that a given Mullineux symbol corresponds to at most one partition; in fact λ can be recovered from its Mullineux symbol by reconstructing λ^{i-1} from λ^i according to the following algorithm.

1. Let x be the r_ith greatest occupied position in the abacus display of λ^i.
2. Let y be the least unoccupied position in the abacus display such that $y > x$ and $y - x \equiv s_i \pmod{p}$. Move the bead at position x to position y.
3. If there is no occupied position greater than y in the abacus display, then stop. Otherwise, let x be the least occupied position greater than y.
4. Let y be the least unoccupied position greater than x on the same runner of the abacus as x; move the bead at position x to position y. Return to step 3.

Now define the conjugate Mullineux symbol of $((r_1, \dots, r_u), (s_1, \dots, s_u))$ to be $((r'_1, \dots, r'_u), (s_1, \dots, s_u))$, where

$$r'_i = \begin{cases} s_i - r_i & (p \mid s_i) \\ s_i - r_i + 1 & (p \nmid s_i); \end{cases}$$

this function is evidently self-inverse. It turns out that if $((r_1, \dots, r_u), (s_1, \dots, s_u))$ corresponds to a p-regular partition λ of n, then $((r'_1, \dots, r'_u), (s_1, \dots, s_u))$ also corresponds to a p-regular partition of n, which we call $f(\lambda)$. We then have

$$f(\lambda) = \lambda^*$$

for all λ. We also observe that $f(\lambda^i) = (f(\lambda))^i$ for all i.

2 Applying the Mullineux algorithm

We now apply Mullineux's algorithm to find $f(\lambda)$ for certain partitions λ lying in the principal block of $k\mathfrak{S}_{np}$; we use the $\langle n^p \rangle$-notation throughout.

2.1 n-rim partitions

Take an integer $1 \leqslant r \leqslant n$, and suppose that $1 \leqslant a_1 < \dots < a_r \leqslant p$ are integers. Writing $n = \alpha r + \beta$ with $0 \leqslant \beta < r$, we define the corresponding *n-rim partition*

$$(a_1, \dots, a_r)_n$$

to be

$$\langle a_1^{(\alpha+1)}, \dots, a_\beta^{(\alpha+1)}, a_{\beta+1}^{(\alpha)}, \dots, a_r^{(\alpha)} \rangle.$$

(We call this an n-rim partition because, as we shall see, it always has exactly n p-rims.)

Now define the *down-set* of $\{a_1, \dots, a_r\}$ to be $\{b_1, \dots, b_n\}$, where $b_1 < \dots < b_n \leqslant p$ are the greatest integers such that

$$b_j < a_j$$

for $1 \leqslant j \leqslant r$ and

$$b_j \neq a_l$$

for all j, l. We then have the following.

Lemma 2.1. *Given an n-rim partition $(a_1, \ldots, a_r)_n$, let $\{b_1, \ldots, b_n\}$ be the corresponding down-set, and let i be maximal such that $a_i < b_{n+1-i}$. Then the partition $(a_1, \ldots, a_r)_n$ has Mullineux symbol*

$$\left(\begin{smallmatrix} \tilde{a}_1 & \cdots & \tilde{a}_i \\ p+b_n-a_1 & \cdots & p+b_{n+1-i}-a_i \end{smallmatrix} \begin{pmatrix} \max(\tilde{a}_{i+1}, \tilde{b}_{n+1-i}) \\ p \end{pmatrix}^{n-2i} \begin{smallmatrix} \tilde{b}_{n+1-i} & \cdots & \tilde{b}_n \\ p+a_i-b_{n+1-i} & \cdots & p+a_1-b_n \end{smallmatrix} \right).$$

Proof. Write $\lambda = (a_1, \ldots, a_r)_n$. The first space in the abacus for λ occurs at position a_1 in row 0, so λ has $\tilde{a}_1$ parts. Removing the first p-rim involves moving the lowest bead on runner a_β (or a_r if $\beta = 0$) up one place, and then moving a bead from position b_n to position a_1 in row 0. This gives a p-rim of length $p + b_n - a_1$, and the remaining partition λ^1 has its first space at position a_2 in row 0, and hence has $\tilde{a}_2$ parts. We then move the lowest bead on runner $a_{\beta-1}$ up one place, and a bead from b_{n-1} to a_2 in row 0. We continue in this way until the partition λ_i, where (since i is maximal such that $a_i < b_{n+1-i}$) every bead is to the left of every space in row 0 of the abacus, and the spaces occur at $a_{i+1}, \ldots, a_r$ and $b_{n+1-i}, \ldots, b_n$. Hence the number of parts of λ^i is $\max(\tilde{a}_{i+1}, \tilde{b}_{n+1-i})$. The next few steps in the Mullineux algorithm each consist in moving the lowest bead up one place; this gives a p-rim of length p and does not change the number of parts of the partition, except possibly the last time. We can remove such p-rims until the lowest bead is on row 1 and runner a_i, since the position on row 0 and runner a_i is occupied. This must be the partition λ^{n-i}, so there are $n - 2i$ p-rims of length exactly p. The partition λ^{n-i} then has spaces in row 0 at $b_{n+1-i}, \ldots, b_n$ and beads in row 1 at $a_1, \ldots, a_i$. Hence the last i p-rim removals consist of moving the bead at position a_j ($j = i, i-1, \ldots, 1$) in row 1 to position b_{n+1-j} in row 0, and the remainder of the Mullineux symbol is as indicated. □

In certain circumstances, this enables us to find $f(\lambda)$ for an n-rim partition λ.

Proposition 2.2. *Suppose that $1 \leqslant a_1 < \cdots < a_r \leqslant p$, and that the down-set $\{b_1, \ldots, b_n\}$ for $\{a_1, \ldots, a_r\}$ is positive, i.e. $b_1 \geqslant 1$. Then*

$$f((a_1, \ldots, a_r)_n) = \langle \tilde{b}_n, \ldots, \tilde{b}_1 \rangle.$$

Proof. Write $c_j = \tilde{b}_{n+1-j}$ for $1 \leqslant j \leqslant n$, and let $\{d_1, \ldots, d_n\}$ be the down-set for $\{c_1, \ldots, c_n\}$. Then we have $d_{n+1-j} = \tilde{a}_j$ for $1 \leqslant j \leqslant r$, and with i as in Lemma 2.1, we have

$$c_i < d_{n+1-i}, \quad c_{i+1} > d_{n-i}.$$

Hence, by Lemma 2.1, the Mullineux symbol of $(c_1, \ldots, c_n)_n = \langle \tilde{b}_n, \ldots, \tilde{b}_1 \rangle$ is

$$\left(\begin{smallmatrix} \tilde{c}_1 & \cdots & \tilde{c}_i \\ p+d_n-c_1 & \cdots & p+d_{n+1-i}-c_i \end{smallmatrix} \begin{pmatrix} \max(\tilde{c}_{i+1}, \tilde{d}_{n+1-i}) \\ p \end{pmatrix}^{n-2i} \begin{smallmatrix} \tilde{d}_{n+1-i} & \cdots & \tilde{d}_n \\ p+c_i-d_{n+1-i} & \cdots & p+c_1-d_n \end{smallmatrix} \right).$$

We claim that this is the conjugate Mullineux symbol to that for $(a_1, \ldots, a_r)_n$ described in Lemma 2.1. This follows from the definition of c_j and d_j, provided either

$$\max(a_i, b_{n-i}) = p - \max(\tilde{a}_{i+1}, \tilde{b}_{n+1-i})$$

or $n - 2i = 0$. Assuming $n - 2i > 0$, we have $b_{n+1-i} \geqslant b_{i+1}$; from the definition of the b_j, there must then be a string of consecutive integers each of which is contained in $\{a_1, \ldots, a_r\} \cup \{b_1, \ldots, b_n\}$ and which includes a_{i+1} and b_{i+1}. Hence the integer immediately below $\min(a_{i+1}, b_{n+1-i})$ is contained in $\{a_1, \dots, a_r\} \cup \{b_1, \dots, b_n\}$; since

$$a_i < b_{n+1-i}, b_{n-i} < a_{i+1},$$

this integer must be either a_i or b_{n-i}. Hence we have

$$\min(a_{i+1}, b_{n+1-i}) = \max(a_i, b_{n-i}) + 1,$$

as required. □

2.2 Modified n-rim partitions

Next we examine the effect of the Mullineux map on certain partitions in blocks other than B; this will facilitate the Mullineux operation for more complicated partitions of B.

Given

$$1 \leqslant a_1 < \dots < a_e < s < a_{e+1} < \dots < a_f < t < a_{f+1} < \dots < a_r \leqslant p,$$

define the *modified n-rim partition* $(a_1, \dots, a_r)_n^{(s,t)}$ as follows:

- construct the partition $(a_1, \dots, a_e, s, a_{e+1}, \dots, a_r)_n$ of B;
- move a bead from position t to position s in row 0.

Thus $(a_1, \dots, a_r)_n^{(s,t)}$ lies in the weight $n-1$ block whose abacus has $n+1$ beads on the sth runner, $n-1$ on the tth runner, and n on every other runner; the core of this block is $(s, 1^{p-t})$.

We define the *modified down-set* for $(a_1, \dots, a_r)_n^{(s,t)}$ to be the set $\{b_1, \dots, b_n\}$, where $b_1 < \dots < b_n \leqslant p$ are maximal integers such that:

- $b_j < a_j$ for $j = 1, \dots, r$;
- $b_j \neq a_l$ for all j,l;
- $b_j \neq s, t$ for all j.

Lemma 2.3. *Suppose that $(a_1, \dots, a_r)_n^{(s,t)}$ is a modified n-rim partition whose down-set $\{b_1, \dots, b_n\}$ is positive, i.e. $b_1 \geqslant 1$. Then*

$$f((a_1, \dots, a_r)_n^{(s,t)}) = (\tilde{b}_n, \dots, \tilde{b}_1)_n^{(\tilde{t},\tilde{s})}.$$

This lemma can be proved combinatorially like Proposition 2.2, but we prefer to use restriction between blocks of the symmetric groups and exploit the truth of Mullineux's conjecture. Given $s < t$, define blocks $C_0, \dots, C_{t-s}$ of $\mathfrak{S}_{np}, \dots, \mathfrak{S}_{np-t+s}$ respectively by letting $C_0 = B$, and moving a bead from runner $t-i+1$ to runner $t-i$ to obtain C_i from C_{i-1}. Thus the blocks $C_{t-1}, \dots, C_{t-s}$ all have weight $n-1$, and have cores $(t-1, 1^{p-t}), \dots, (s, 1^{p-t})$ respectively. We also define blocks $D_0, \dots, D_{t-s}$ by letting $D_0 = B$ and moving a bead from runner $\tilde{t}+i$ to runner $\tilde{t}+i-1$ to obtain D_i from D_{i-1}. Thus $D_1, \dots, D_{t-s}$ all have weight $n-1$ and have cores $(\tilde{t}, 1^{p-1-\tilde{t}}), \dots, (\tilde{t}, 1^{p-\tilde{s}})$. In particular, D_i is the conjugate block to C_i for all i.

Since C_i and D_i are conjugate, we have, for any module M lying in B,

$$M{\downarrow_{C_1}} \cdots {\downarrow_{C_{t-s}}} \otimes \mathrm{sgn} \cong (M \otimes \mathrm{sgn}){\downarrow_{D_1}} \cdots {\downarrow_{D_{t-s}}}.$$

Proof of Lemma 2.3. Define the n-rim partition

$$\lambda = (a_1, \ldots, a_e, a_{e+1} - 1, \ldots, a_f - 1, t, a_{f+1}, a_r)_n$$

in B. Since $a_f < t$, the down-set for λ will include $t - 1$. Thus, to construct the down-set, we may construct the down-set ignoring runners t and $t - 1$, and then add $t - 1$ to the resulting set. But this is just the same as constructing the down-set of $\{a_1, \ldots, a_r\}$ ignoring runners s and t, subtracting 1 from any elements lying between s and t, and then inserting $t - 1$. Thus, the down-set for λ is

$$\{b_1, \ldots, b_g, b_{g+1} - 1, \ldots, b_h - 1, t - 1, b_{h+1}, \ldots, b_n\},$$

where $b_g < s < b_{g+1}$ and $b_h < t < b_{h+1}$. This is a positive down-set, so by Proposition 2.2,

$$f(\lambda) = \langle \tilde{b}_n, \ldots, \tilde{b}_{h+1}, \tilde{t} + 1, \tilde{b}_h + 1, \ldots, \tilde{b}_{g+1} + 1, \tilde{b}_g, \ldots, \tilde{b}_1 \rangle;$$

call this latter partition μ.

Defining

$$\overline{\lambda} = (a_1, \ldots, a_r)_n^{(s,t)}, \overline{\mu} = (\tilde{b}_n, \ldots, \tilde{b}_1)_n^{(\tilde{t},\tilde{s})},$$

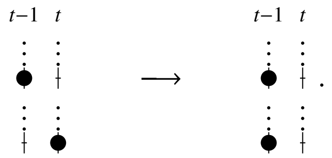
we claim that

$$D^{\lambda}{\downarrow_{C_1}} \cdots {\downarrow_{C_{t-s}}} \cong D^{\overline{\lambda}}$$

and

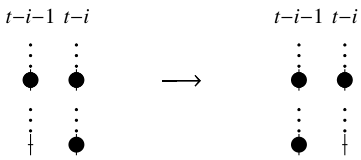
$$D^{\mu}{\downarrow_{D_1}} \cdots {\downarrow_{D_{t-s}}} \cong D^{\overline{\mu}};$$

the lemma will then follow. We use Theorem 1.3. Restricting λ from C_0 to C_1, we have a bead of positive weight on runner t, but no weight on runner $t - 1$. Thus the weight bead simply moves one place to the left, and $D^{\lambda}{\downarrow_{C_1}}$ is the simple module corresponding to this new abacus display:

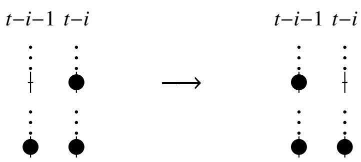


When restricting between C_i and C_{i+1} for $i \geqslant 1$, one of the following three situations occurs; in each case, $D^{\lambda}{\downarrow_{C_1}} \cdots {\downarrow_{C_{i+1}}}$ is simple, by Theorem 1.3.

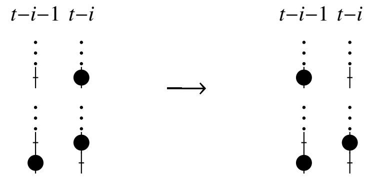
1.



2.



3.



Thus we see that the weight bead on runner t moves down as far as runner a_f, the weight bead on runner $a_j - 1$ moves down to runner a_{j-1} ($e+2 \leqslant j \leqslant f$), and the weight bead on runner $a_{e+1} - 1$ moves to runner s. This gives the partition $(a_1, \dots, a_r)_n^{(s,t)} = \overline{\lambda}$. Restricting D^μ is similar (easier, in fact). □

2.3 Partitions of types *X* and *Y*

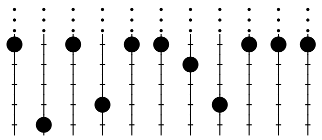
Now we define two more types of partition in B whose Mullineux conjugates we shall find using Lemma 2.3.

Define a type X partition as follows. Take $r < n-1$ and choose $1 \leqslant c_1 < \cdots < c_r \leqslant p$. Also choose $1 \leqslant u \leqslant p$ such that

- $u \neq c_j$ for all j, and
- if $r \geqslant \frac{n-1}{2}$, then $u < c_{n-1-r}$.

Form the $(n-1)$-rim partition $(c_1, \dots, c_r)_{n-1}$, and then move a bead on runner u from row 0 to row 1. Call the resulting partition $X_n(c_1, \dots, c_r; u)$.

For example, with $p = 11$, $n = 11$, $r = 3$, $(c_1, c_2, c_3) = (2, 4, 8)$, $u = 7$ we get the partition

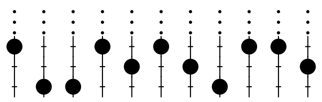
.

Define a partition of type Y as follows. Choose $\frac{n-1}{2} \leqslant r \leqslant n-2$, and choose $1 \leqslant c_1 < \cdots < c_r \leqslant p$. Also choose $1 \leqslant u \leqslant p$ such that

- $u \neq c_j$ for all j, and
- if $r = n-2$, then $u < c_r$.

Form the $(n-2)$-rim partition $(c_1, \dots, c_r)_{n-2}$, and then move a bead on runner u from row 0 to row 1. Now define y to be minimal such that $c_y > u$ (or put $y = 1$ if $u > c_r$), and move the lowest bead on runner c_y down one row. Call the resulting partition $Y_n(c_1, \dots, c_r; u)$.

With $p = 11$, $n = 9$, $r = 5$, $(c_1, c_2, c_3, c_4, c_5) = (2, 3, 5, 8, 11)$, $u = 7$ we get the partition

.

Remarks.

1. The partitions are carefully defined in order to have exactly $n-1$ p-rims. The second condition on u in each case ensures that they do not just reduce to n-rim partitions.

2. Note that $X_n(c_1,\dots,c_r;u)$ and $Y_n(c_1,\dots,c_r;u)$ may coincide. In particular, if $r = n-2$ and $X_n(c_1,\dots,c_r;u)$ is defined, then $Y_n(c_1,\dots,c_r;u)$ is defined and equals $X_n(c_1,\dots,c_r;u)$.

In order to find the Mullineux conjugates of partitions of these types, we define a special type of down-set. Given $c_1,\dots,c_r,u$ satisfying the conditions for either a partition of type X or of type Y, we first define t to be the greatest integer such that $t < u$, $t \neq c_j$ for all j. We then define $b_1 < \cdots < b_{n-2} \leqslant p$ to be the greatest integers such that $b_j < c_j$, $b_j \neq c_l$, $b_j \neq t$ for all j, l. Note that we allow $b_j = u$.

Proposition 2.4. *Let $\lambda = X_n(c_1,\dots,c_r;u)$ or $Y_n(c_1,\dots,c_r;u)$, and suppose that the down-set $\{b_1,\dots,b_{n-2}\}$ of λ is positive, i.e. $b_1 \geqslant 1$. Then*

$$f(\lambda) = \langle \tilde{b}_{n-2},\dots,\tilde{b}_1,\tilde{t}^2\rangle.$$

Note that t need not be smaller than b_1, so the runners on the right-hand side may not be in ascending order.

The same proof covers both types of partitions.

Proof. First assume $t < c_1$. Then removing the first p-rim involves moving the lowest bead (on runner c_{n-r-1} in type X, or runner c_y in type Y) up one row, and then moving the bead on runner u from row 1 to row 0. Hence the first column of the Mullineux symbol is

$$\begin{matrix}\tilde{c}_1\\ 2p\end{matrix},$$

and the remaining partition λ^1 is the $(n-2)$-rim partition

$$(c_1,\dots,c_r)_{n-2}.$$

Since $t < c_1$, the down-set of this is $\{b_2,\dots,b_{n-2},t\}$, which is positive, so by Proposition 2.2 we have

$$f(\lambda)^1 = f(\lambda^1) = (\tilde{b}_{n-2},\dots,\tilde{b}_2,\tilde{t})_{n-2},$$

where the runners on the right-hand side need not be in ascending order. To find $f(\lambda)$ we must add a p-rim in accordance with the first column

$$\begin{matrix}p-1+c_1\\ 2p\end{matrix}$$

of the conjugate Mullineux symbol. To obtain $p-1+c_1$ parts, we move a bead from position $\tilde{c}_1+1$ on row -1; but $\tilde{c}_1+1=\tilde{t}$, and there is no bead in position $\tilde{t}$ in row 0; so we move the bead down one row, and we then move the next bead to the right of this in row 0. This must lie in position $\tilde{b}_1$. We move this bead down one row to obtain $f(\lambda)$ as indicated.

Now we assume $t > c_1$. Removing the first rim involves moving two beads up one row as above, and then moving a bead from position t to position c_1 in row 0. Thus the first column of the Mullineux symbol is

$$\begin{matrix}\tilde{c}_1\\ 2p+t-c_1\end{matrix},$$

and the remaining partition λ^1 is the modified $(n-2)$-rim partition

$$(c_2,\dots,c_r)_{n-2}^{(c_1,t)}.$$

The modified down-set for this is $\{b_2, \dots, b_{n-2}\}$, which is positive, so we have

$$f(\lambda)^1 = f(\lambda^1) = (\tilde{b}_{n_2}, \dots, \tilde{b}_2)^{(\tilde{t},\tilde{c}_1)}_{n-2}.$$

To find $f(\lambda)$ we add a p-rim in accordance with the first column

$$\begin{matrix} p + \tilde{t} \\ 2p + t - c_1 \end{matrix}$$

of the conjugate Mullineux symbol. To obtain $p + \tilde{t}$ parts, we must move a bead from position $\tilde{t}$ in row -1 to runner $\tilde{c}_1$; the highest space on this runner is in row 0, so we move the bead here; we must then move the first bead to the right of this in row 0 down one row; this lies on runner $\tilde{b}_1$. Hence $f(\lambda)$ is as indicated. □

3 General vertices

We are now in a position to employ the Mullineux map in order to provide information about decomposition numbers and Ext-spaces.

Definition. Let B be the principal block of $k\mathfrak{S}_{np}$, with the $\langle n^p \rangle$ abacus notation.

A vertex of the quiver of B is said to be *general* if it has the form $\langle a_1, \dots, a_n \rangle$, where $1 < a_1 < \dots < a_n \leqslant p - 1$ and $a_{i+1} - a_i \geqslant 3$ for all i.

A vertex of the quiver of B is said to be *p-general* if it has the form $\langle a_1, \dots, a_{n-1}, p \rangle$, where $1 < a_1 < \dots < a_{n-1} \leqslant p - 3$ and $a_{i+1} - a_i \geqslant 3$ for all i.

A vertex of the quiver of B is said to be *semi-general* if it has the form $\langle a_1, \dots, a_{j-1}, a_j^{(2)}, a_{j+1}, \dots, a_{n-1} \rangle$ for some $1 \leqslant j \leqslant n - 1$, with $1 < a_1 < \dots < a_{n-1} \leqslant p - 2$ and $a_{i+1} - a_i \geqslant 3$ for all i.

We make some basic observations using standard representation theory of the symmetric groups. Recall from Proposition 1.4 that if $[S^\mu : D^\lambda] \neq 0$, then $\lambda \trianglerighteq \mu$.

Lemma 3.1. *If λ and μ are partitions of m with λ p-regular and $[S^\mu : D^\lambda] \neq 0$, then $\mu \trianglerighteq \lambda^{*\prime}$.*

Proof. By Theorem 1.6 (and since the simple modules are self-dual), we have $[S^{\mu'} : D^{\lambda^*}] \neq 0$, so that $\lambda^* \trianglerighteq \mu'$. But conjugation of partitions exactly reverses the order $\trianglerighteq$, so $\mu \trianglerighteq \lambda^{*\prime}$ as required. □

The following theorem is a special case of a general result [1, Proposition 1.9.6] from modular representation theory.

Theorem 3.2. *Let λ be a p-regular partition, and let $P(D^\lambda)$ denote the projective cover of D^λ. Then*

$$P(D^\lambda) \sim \sum_\mu [S^\mu : D^\lambda] S^\mu.$$

Now if $\operatorname{Ext}^1_{k\mathfrak{S}_m}(D^\lambda, D^\mu) \neq 0$, then D^μ appears as a composition factor of the second Loewy layer of $P(D^\lambda)$. Hence either D^μ is a composition factor of $\operatorname{rad}(S^\lambda)$, or D^μ is an irreducible quotient of some S^ν with $\nu \neq \lambda$, $[S^\nu : D^\lambda] \neq 0$. In the latter case, if ν is p-regular, then we must have $\mu = \nu$. Thus we have the following.

Proposition 3.3. *Suppose a partition λ of m has the property that all partitions ν with $\lambda \geqslant \nu \geqslant \lambda^{*\prime}$ are p-regular. Then D^λ does not self-extend, and for any simple module D^μ we have*

$$\dim_k \operatorname{Ext}^1_{k\mathfrak{S}_m}(D^\lambda, D^\mu) \leqslant [S^\lambda : D^\mu] + [S^\mu : D^\lambda].$$

In particular, for any simple module D^μ with $\operatorname{Ext}^1_{k\mathfrak{S}_m}(D^\lambda, D^\mu) \neq 0$, either

$$\lambda > \mu \geqslant \lambda^{*\prime}$$

or

$$\mu > \lambda \geqslant \mu^{*\prime}.$$

We now show that our various types of general vertices have the property specified in Proposition 3.3.

Proposition 3.4. *Let B be the principal p-block of $\mathfrak{S}_{np}$, with $\langle n^p \rangle$ abacus notation.*

1. *Suppose $\langle a_1, \ldots, a_n \rangle$ is a general or a p-general vertex in B. Then*

$$\langle a_1, \ldots, a_n \rangle^{*\prime} = \langle a_1 - 1, \ldots, a_n - 1 \rangle.$$

2. *Suppose $\langle a_1^{(2)}, \ldots, a_{n-1} \rangle$ is a semi-general vertex in B. Then*

$$\langle a_1^{(2)}, \ldots, a_{n-1} \rangle^{*\prime} = \langle a_1 - 1, \ldots, a_{n-1} - 1, p \rangle.$$

3. *Suppose $\langle a_1, \ldots, a_{j-1}, a_j^{(2)}, a_{j+1}, \ldots, a_{n-1} \rangle$ is a semi-general vertex in B, with $j > 1$. Then*

$$\langle a_1, \ldots, a_{j-1}, a_j^{(2)}, a_{j+1}, \ldots, a_{n-1} \rangle^{*\prime} =$$

$$\langle a_1 - 1, \ldots, a_{j-2} - 1, (a_{j-1} - 1)^{(2)}, a_j - 1, \ldots, a_{n-1} - 1 \rangle.$$

Hence for λ a general, p-general or semi-general vertex, all partitions $\nu \geqslant \lambda^{*\prime}$ have at most p parts, and are p-regular.

Proof. $\langle a_1, \ldots, a_n \rangle$ is the n-rim partition $(a_1, \ldots, a_n)_n$; since the a_i differ by at least three, the corresponding down-set $\{b_1, \ldots, b_n\}$ has $b_i = a_i - 1$. Hence by Proposition 2.2,

$$\langle a_1, \ldots, a_n \rangle^* = \langle \widetilde{a_n - 1}, \ldots, \widetilde{a_1 - 1} \rangle,$$

implying the result.

The semi-general vertex with $j = 1$ is also an n-rim partition, and is dealt with similarly. A semi-general vertex with $j > 1$ is of type Y; in fact

$$\langle a_1, \ldots, a_{j-1}, a_j^{(2)}, a_{j+1}, \ldots, a_{n-1} \rangle = Y_n(a_1, \ldots, \widehat{a_{j-1}}, \ldots, a_{n-1}; a_{j-1}).$$

Since the a_i differ by at least three, this has $t = a_{j-1} - 1$ and down-set $\{a_1 - 1, \ldots, \widehat{a_{j-1} - 1}, \ldots, a_{n-1} - 1\}$. The result follows from Proposition 2.4. □

This result enables us immediately to confine the possible μ with $\lambda \geqslant \mu$ and $\operatorname{Ext}^1_B(D^\lambda, D^\mu) \neq 0$ to a very small set.

Proposition 3.5.

1. *Suppose $\lambda = \langle a_1, \dots, a_n \rangle$ is a general or a p-general vertex, and that $\lambda > \mu \geqslant \lambda^{*\prime}$. Then μ has the form*
$$\langle e_1, \dots, e_n \rangle,$$
where e_i equals a_i or $a_i - 1$.

2. *Suppose $\lambda = \langle a_1^{(2)}, a_2, \dots, a_{n-1} \rangle$ is a semi-general vertex, and that $\lambda > \mu \geqslant \lambda^{*\prime}$. Then μ has the form*
$$\langle (e_1)^{(2)}, e_2, \dots, e_{n-1} \rangle$$
or
$$\langle e_1, \dots, e_{n-1}, p \rangle,$$
where e_i equals a_i or $a_i - 1$.

3. *Suppose $\langle a_1, \dots, a_{j-1}, a_j^{(2)}, a_{j+1}, \dots, a_{n-1} \rangle$ is a semi-general vertex, with $j > 1$, and that $\lambda > \mu \geqslant \lambda^{*\prime}$. Then μ has the form*
$$\langle e_1, \dots, e_{j-1}, (e_j)^{(2)}, e_{j+1}, \dots, e_{n-1} \rangle$$
or
$$\langle e_1, \dots, e_{j-2}, (e_{j-1})^{(2)}, e_j, \dots, e_{n-1} \rangle,$$
where e_i equals a_i or $a_i - 1$.

A somewhat harder task is to find the possible neighbours D^μ of D^λ for which $\mu > \lambda$; we now undertake this.

Lemma 3.6. *Suppose λ is a semi-general or a p-general vertex, and that $\mu > \lambda \geqslant \mu^{*\prime}$. Then μ is an n-rim partition.*

Proof. Since $\mu > \lambda$, μ has at most one weight bead on each runner; in particular, μ has fewer than p parts. If the first p-rim of μ has length at least $2p$, then μ^* will have more than p parts, i.e. the first part of $\mu^{*\prime}$ will be greater than p. This contradicts $\lambda \geqslant \mu^{*\prime}$, so the first p-rim of μ has size less than $2p$. Thus if μ has a bead of weight w on runner r, it cannot have a bead of weight $0 < w' < w$ on a runner $r' < r$, or a bead of weight $0 < w' < w - 1$ on runner $r' > r$. Hence μ is an n-rim partition. □

Given this restriction, we can be more precise.

Proposition 3.7. *Suppose $\lambda = \langle a_1, \dots, a_n \rangle$ is a general vertex, and that $\mu > \lambda \geqslant \mu^{*\prime}$. Then μ has the form $\langle c_1, \dots, c_n \rangle$, where:*

- *$a_i \leqslant c_i \leqslant a_i + 2$ for all i, and*
- *if $c_i = a_i + 2$, then $c_{i+1} = c_i + 1$.*

Proof. Let $\mu = (c_1, \dots, c_r)_n$. Since $\mu > \lambda$, we must have $c_i \geqslant a_i$ for $i = 1, \dots, r$. Hence if $\{b_1, \dots, b_n\}$ is the down-set for μ, then $b_1 \geqslant a_1$, so $\{b_1, \dots, b_n\}$ is positive. Thus $\mu^{*\prime} = \langle b_1, \dots, b_n \rangle$, and we must have $a_i \geqslant b_i$.

If $r < n$, then either $b_n = p$ or $c_r = p$. The former case contradicts $\lambda \geqslant \mu^{*\prime}$, so assume the latter. Let s be maximal such that $b_{s+1} > c_s$ (this condition is to be treated as vacuous in the case $s = 0$). Then the (disjoint) sets $\{c_{s+1},\dots,c_r\}$ and $\{b_{s+1},\dots,b_n\}$ must constitute a set of consecutive integers, whose largest value is $c_r = p$; thus

$$b_{s+1} = p + 1 - n - r + 2s.$$

Now

$$b_{s+1} \leqslant a_{s+1} \leqslant p + 2 - 3(n - s),$$

so we have

$$s + 1 \geqslant 2n - r,$$

which gives a contradiction. Hence $r = n$.

Now suppose $c_i > b_{i+\delta}$ for some $\delta > 0$. Let $s < i$ be maximal such that $b_{s+1} > c_s$, and let $t > i$ be minimal such that $b_{t+1} > c_t$. Then the sets $\{b_{s+1},\dots,b_t\}$ and $\{c_{s+1},\dots,c_t\}$ comprise a set of consecutive integers, i.e. $c_t - b_{s+1} = 2(t - s) - 1$. Comparison with the inequality $a_t - a_{s+1} \geqslant 3(t - s - 1)$ yields $t - s \leqslant 2$. Hence $\delta \leqslant 1$, and if $\delta = 1$ we must have $b_i, b_{i+1}, c_i, c_{i+1}$ as consecutive integers; this can only happen if $b_i = a_i$ and $c_{i+1} = a_{i+1}$.

If $c_i < b_{i+1}$ and $c_{i-1} < b_i$, then $b_i = c_i - 1$, which can happen only if c_i equals a_i or $a_i + 1$. □

Example. Let $p \geqslant 11$, and let λ be the general vertex $\langle 2, 6, 9\rangle$. From what we have seen so far, if D^λ extends D^μ, then μ must be one of

$$\langle 2,6,8\rangle, \langle 2,5,9\rangle, \langle 2,5,8\rangle,$$

$$\langle 1,6,9\rangle, \langle 1,6,8\rangle, \langle 1,5,9\rangle, \langle 1,5,8\rangle$$

or one of

$$\langle 2,6,10\rangle, \langle 2,7,9\rangle, \langle 2,7,10\rangle,$$

$$\langle 3,6,9\rangle, \langle 3,6,10\rangle, \langle 3,7,9\rangle, \langle 3,7,10\rangle,$$

$$\langle 2,8,9\rangle, \langle 3,8,9\rangle.$$

Proposition 3.8. *Suppose $\langle a_1,\dots,a_{n-1},p\rangle$ is a p-general vertex, and that $\mu \succ \lambda \succcurlyeq \mu^{*\prime}$. Then μ has one of the following two forms:*

1. *$\langle c_1,\dots,c_{n-1},p\rangle$, where*
   - *$a_i \leqslant c_i \leqslant a_i + 2$, for all i, and*
   - *if $c_i = a_i + 2$, then $c_{i+1} = c_i + 1$ (where we take $c_n = p$);*
2. *$\langle c_1^{(2)}, c_2,\dots,c_{n-1}\rangle$, where*
   - *$a_i \leqslant c_i \leqslant a_i + 2$, for all i, and*
   - *if $c_i = a_i + 2$, then $c_{i+1} = c_i + 1$.*

Proof. Let $\mu = (c_1,\dots,c_r)_n$. Again, we must have $c_i \geqslant a_i$ for $i = 1,\dots,r$, so the down-set $\{b_1,\dots,b_n\}$ of $(c_1,\dots,c_r)_n$ is positive and $\mu^{*\prime} = \langle b_1,\dots,b_n\rangle$.

Suppose $r < n-1$. Then either $b_{n-1} \geqslant p-2$ or $c_{r-1} \geqslant p-2$. The former contradicts $b_{n-1} \leqslant a_{n-1}$, so assume the latter; again, we take a maximal s such that $b_{s+1} > c_s$, whereupon the sets $\{b_{s+1},\dots,b_n\}$ and

$\{c_{s+1}, \dots, c_r\}$ constitute a set of consecutive integers with greatest value p (which equals either b_n or c_r). Hence $b_{s+1} = p + 1 - n - r + 2s$; comparison with $a_{s+1} \leqslant p + 3 - 3(n - s)$ yields $s + 2 \geqslant 2n - r \geqslant n + 2$, which gives a contradiction. Hence $r = n$ or $n - 1$.

If $r = n$, then $\mu = \langle c_1, \dots, c_n\rangle$, and we must have $c_i \geqslant a_i$ for all i, and $c_n = p$. If $r = n - 1$, then $\mu = \langle c_1^{(2)}, c_2, \dots, c_{n-1}\rangle$, and c_i must exceed a_i for all i. The other conditions on the c_i follow exactly as in the proof of Proposition 3.7. □

For semi-general vertices, the situation is more complicated.

Lemma 3.9. *Suppose that $\lambda = \langle a_1, \dots, a_{j-1}, a_j^{(2)}, \dots, a_{n-1}\rangle$ is a semi-general vertex, and that $\mu > \lambda \geqslant \mu^{*\prime}$. Then μ is either an n-rim partition or a partition of type X or type Y.*

Proof. If the first p-rim of μ has size at least $3p$, then (since μ has fewer than p parts) μ^* has at least $2p$ parts, i.e. the first part of $\mu^{*\prime}$ is at least $2p$. But this contradicts $\lambda \geqslant \mu^{*\prime}$; so the first p-rim of μ has size less than $3p$. If it has size less than $2p$, then μ is an n-rim partition, as in the proof of Lemma 3.6. So assume the first p-rim of μ has size at least $2p$ but less than $3p$. If the second p-rim of μ has size at least $2p$, then $(\mu^1)^*$ has at least p parts; attaching a p-rim involves adding at least one to each part, so μ^* has at least p parts of size at least two. Hence $\mu^{*\prime}$ has second part at least p, which again contradicts $\lambda \geqslant \mu^{*\prime}$. So the second p-rim of μ has size less than $2p$.

Since μ has at most one weight bead on each vertex, removing the first p-rim of μ must consist of moving two beads, b and c say, up one row each, and then possibly moving a bead across in row 0. Hence the beta-numbers corresponding to b and c differ by more than p; suppose b is the lower of the two beads. c cannot lie in row 3 or lower, since then μ^1 would have two weight beads whose beta-numbers differed by more than p, so the second p-rim of μ would have size at least $2p$. If c lies in row 2, then either there is a space immediately above c in the abacus for μ^1, in which case the second p-rim will have size at least $2p$, or there is no weight bead to the left of c, so that in removing the p-rim of μ a bead is moved across row 0 to the space above c. But in the latter case suppose that c lies on runner a_1, and that the next weight bead to the right of c lies on runner a_2. Then μ^1 has $\tilde{a}_2$ parts, and the p-rim of μ^1 has size at least $2p + a_1 - a_2$, so that μ^1* has at least $p - 1 + a_1$ parts; this gives a contradiction, as above. So c lies in row 1.

Now there cannot be any beads in between positions $b - p$ and c; if b exceeds c by more than $2p$, then there cannot be any weight beads less than c either, in which case we have a partition of type X, with u being the runner on which c lies. If b exceeds c by less than $2p$ and some other weight bead exceeds c by more than p, then again μ is of type X. If b exceeds c by less than $2p$ and no other weight bead exceeds c by p, then the partition is of type Y, with u being the runner on which c lies, and c_y the runner on which b lies. □

Proposition 3.10. *Suppose that $\lambda = \langle a_1, \dots, a_{j-1}, a_j^{(2)}, a_{j+1}, \dots, a_{n-1}\rangle$ is a semi-general vertex, and that $\mu > \lambda \geqslant \mu^{*\prime}$. Then μ has one of the following two forms:*

1. *$\langle c_1, \dots, c_{j-1}, c_j^{(2)}, c_{j+1}, \dots, c_{n-1}\rangle$, where*
   - *$a_i \leqslant c_i \leqslant a_i + 2$, for all i, and*
   - *if $c_i = a_i + 2$, then $i \neq j - 1$ and either $c_{i+1} = c_i + 1$ or $j = 1$, $i = n - 1$, $a_{n-1} = p - 2$;*
2. *$\langle c_1, \dots, c_j, c_{j+1}^{(2)}, c_{j+2}, \dots, c_{n-1}\rangle$, where*

- $a_i \leqslant c_i \leqslant a_i + 2$, *for all i, and*
- *if* $c_i = a_i + 2$, *then* $i \neq j - 2$ *and* $c_{i+1} = c_i + 1$.

Proof.

1. Suppose first of all that $\mu = (c_1, \ldots, c_r)_n$ is an n-rim partition. Since $c_i \geqslant a_i$ for $i = 1, \ldots, r$, the down-set $\{b_1, \ldots, b_n\}$ for μ is positive, so $\mu^{*\prime} = \langle b_1, \ldots, b_n \rangle$.

   If $r < n - 1$, then either $b_{n-1} = p - 1$ or $c_r \geqslant p - 1$. The former contradicts $\lambda \geqslant \mu^{*\prime}$, so assume the latter, and let s be maximal such that $c_s < b_{s+1}$; we get a contradiction, as in the proofs of Propositions 3.7 and 3.8. So $r \geqslant n - 1$; $\mu > \lambda$ implies $r \leqslant n - 1$, whence $r = n - 1$, and $\mu = \langle c_1^{(2)}, c_2, \ldots, c_{n-1} \rangle$.

   [$j > 1$] We must have $c_1 \geqslant a_j$, so b_1 cannot equal $c_1 - 1$; in fact $b_1 \leqslant c_1 - 3$. Thus there is some $i \geqslant 3$ with $b_i < c_1$. Let s be minimal such that $c_s < b_{s+1}$.

   If $s < n - 1$ or $c_{n-1} < b_n$, then the sets $\{b_1, \ldots, b_s\}$ and $\{c_1, \ldots, c_s\}$ comprise a set of consecutive integers, i.e. $c_s - b_1 = 2s - 1$. But $c_s \geqslant a_s \geqslant a_1 + 3(s-1) \geqslant b_1 + 3(s-1)$, which yields $s \leqslant 2$; contradiction.

   If $s = n - 1$ and $c_{n-1} > b_n$, then the sets $\{b_1, \ldots, b_n\}$ and $\{c_1, \ldots, c_{n-1}\}$ comprise a set of consecutive integers whose highest value is $c_{n-1} = p$. Thus $b_1 = p + 2 - 2n$; but $b_1 \leqslant a_1 \leqslant p - 2 - 3(n-2)$, which gives $n \leqslant 2$; contradiction.

   [$j = 1$] We have $c_i \geqslant a_i \geqslant b_i$ for all i; suppose $c_i > b_{i+\delta}$ for some $\delta > 0$. Let $s < i$ be maximal such that $b_{s+1} > c_s$, and let $t > i$ be minimal such that $b_{t+1} > c_t$ (this condition is to be treated as vacuous in the case $t = n$. If $t \leqslant n - 1$, then, as in the proof of Proposition 3.7, we get $t - s \leqslant 2$. If $t = n$, then the sets $\{b_{s+1}, \ldots, b_n\}$ and $\{c_{s+1}, \ldots, c_{n-1}\}$ comprise a set of consecutive integers with $c_{n-1} = p$, so we get $b_{s+1} = p + 1 - 2n + 2s + 1$; comparison with $a_{s+1} \leqslant p + 4 - 3n + 3s$ yields $s \geqslant n - 2$. In any case, we have $\delta = 1$, and the possible μ are as described.

2. Next we must consider partitions μ of type X or type Y. Suppose that μ is of the form $X_n(c_1, \ldots, c_r; u)$ or $Y_n(c_1, \ldots, c_r; u)$ and that $r < n - 2$. Then, since $\mu \geqslant \lambda$, the down-set $\{b_1, \ldots, b_{n-2}, t\}$ for μ is positive, and so $\mu^{*\prime} = \langle b_1, \ldots, b_{n-2}, t^{(2)} \rangle$ (where the runners need not be in ascending order). Since $r < n-2$, one of b_{n-1}, c_r equals p. Let g be minimal such that the set $\{g, g+1, \ldots, p\}$ is contained in $\{c_1, \ldots, c_r, b_1, \ldots, b_{n-1}, t\rangle$ and such that $g \neq t$. Then g equals some b_i with $i \leqslant r$, and the sets $\{b_i, \ldots, b_{n-2}\}$ and $\{c_i, \ldots, c_r\}$, possibly together with t, comprise $\{g, \ldots, p\}$. So

$$b_i = \begin{cases} p - n - r + 2i & (b_i < t) \\ p - n - r + 2i + 1 & (b_i > t). \end{cases}$$

   Since $\lambda \geqslant \mu^{*\prime}$, we must have

$$b_i \leqslant \begin{cases} a_i & (b_i < t) \\ a_{i+1} & (b_i > t); \end{cases}$$

   comparison with $a_i \leqslant p + 1 - 3(n-i)$ yields $i + 1 \geqslant 2n - r$ in either case, which is a contradiction. Hence $r = n - 2$, and (from Remark (2) following the definition of type Y partitions) μ is of type Y.

We relabel the runners $c_1, \ldots, c_{n-2}, u$ as $c_1, \ldots, c_{n-1}$ in ascending order, so that we may write

$$\mu = \langle c_1, \ldots, c_{k-1}, c_k^{(2)}, c_{k+1}, \ldots, c_{n-1}\rangle,$$

where c_{k-1} is the runner previously called u. We must then have $c_i \geqslant a_i$ for all i, and $c_k \geqslant a_j$.

Next we claim that k must equal j or $j+1$. If $k < j$, then, since $b_{k-1} \leqslant a_k$, we have $c_k - b_{k-1} \geqslant 3$, so (by the definition of the down-set, and recalling that c_k is the runner previously called c_{k-1}) b_k must be less than c_k. By taking a minimal $t > k$ such that $c_t < b_t$, we derive a contradiction in the same manner as that used in earlier proofs. If $k > j+1$, then we have $c_{k-1} \geqslant a_{k-1}$ but $t \leqslant a_j$, which implies that $c_{k-2} > b_{k-1}$. By taking a maximal $s < k-1$ such that $c_s < b_{s+1}$, we can derive a contradiction. Hence k equals j or $j+1$. The restrictions on the c_i can now be found exactly as in Part 1 of this proof.

□

4 Restriction to $\mathfrak{S}_{np-1}$

In this section, we show which of the possible extensions of general vertices actually exist, and complete the proof of Theorems 1.1 and 1.2. We use restriction to blocks of $k\mathfrak{S}_{np-1}$, as in [11]; we cannot use the techniques of [11] for the whole proof, since for semi-general vertices the restricted modules are not always simple.

Take $2 \leqslant s \leqslant p$. As in [11, Section 1], we let B_s be the block of $k\mathfrak{S}_{np-1}$ obtained from B by moving a bead from runner s to runner $s-1$ of the np-bead abacus. We also recall the following.

Definition. [11, Definition 1.3] Let $\tilde{\Lambda}_s = \{\tilde{\lambda} \in B_s \mid S^{\tilde{\lambda}}\uparrow^B$ has exactly two Specht factors$\}$. Then if $\tilde{\Lambda}_s = \{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m\}$ and $\{\lambda_1, \mu_1, \ldots, \lambda_m, \mu_m\}$ is the set of partitions of B such that

$$S^{\lambda_i}\downarrow_{B_s} \cong S^{\mu_i}\downarrow_{B_s} \cong S^{\tilde{\lambda}_i}$$

and $\lambda_i > \mu_i$, define $\Lambda_s = \{\lambda_1, \ldots, \lambda_m\}$ and $M_s = \{\mu_1, \ldots, \mu_m\}$.

Definition. [11, Definition 1.4] Let $\Theta_s : \Lambda_s \to \{$partitions of $B_s\}$ be the map $\lambda \mapsto \tilde{\lambda}$, where

$$S^{\lambda}\downarrow_{B_s} \cong S^{\tilde{\lambda}}.$$

Theorem 4.1. *Let $\lambda \in \Lambda_s$ and let $\tilde{\lambda} = \Theta_s(\lambda)$. Let $\mu \in M_s$ be such that $S^{\mu}\downarrow_{B_s} \cong S^{\tilde{\lambda}}$. Then $D^{\tilde{\lambda}}\uparrow^B$ has two composition factors isomorphic to D^{λ} and one composition factor isomorphic to D^{μ}. Any other irreducible factor D^{ρ} of $D^{\tilde{\lambda}}\uparrow^B$ has $\rho \notin \Lambda_s$.*

Proof. This is simply the second part of [11, Theorem 1.6]. □

Lemma 4.2. *Let λ, μ be as in Theorem 4.1. If $\lambda > \rho$, $D^{\rho}\downarrow_{B_s} = 0$ and $\mathrm{Ext}^1_B(D^{\lambda}, D^{\rho}) \neq 0$, then $\rho \geqslant \mu$.*

Proof. If $D^{\tilde{\lambda}}\uparrow^B$ has no composition factor isomorphic to D^{ρ}, then we have

$$\begin{aligned}\mathrm{Ext}^1_B(D^{\lambda}, D^{\rho}) &= \mathrm{Ext}^1_B(\mathrm{cosoc}(D^{\tilde{\lambda}}\uparrow^B), D^{\rho})\\ &\leqslant \mathrm{Ext}^1_B(D^{\tilde{\lambda}}\uparrow^B, D^{\rho})\\ &= \dim_k \mathrm{Ext}^1_{B_s}(D^{\tilde{\lambda}}, D^{\rho}\downarrow_{B_s})\\ &= 0,\end{aligned}$$

a contradiction. Hence D^ρ appears as a composition factor of $D^{\tilde{\lambda}}\uparrow^B$. But the latter is a quotient of $S^{\tilde{\lambda}}\uparrow^B \sim S^\lambda + S^\mu$. So D^ρ is a composition factor of S^μ. □

This immediately rules out a lot of the possible extensions of general vertices.

Corollary 4.3.

1. *Let $\lambda = \langle a_1, \ldots, a_n \rangle$ be a general vertex in B, and suppose that ρ takes one of the following forms:*

   (a) $\langle a_1 - \epsilon_1, \ldots, a_n - \epsilon_1 \rangle$, *where at least two of the ϵ_i are positive;*

   (b) $\langle a_1 + \epsilon_1, \ldots, a_n + \epsilon_n \rangle$, *where at least two of the ϵ_i are positive or some ϵ_i equals two.*

   Then D^λ does not extend D^ρ.

2. *Let $\lambda = \langle a_1, \ldots, a_{n-1}, p \rangle$ be a p-general vertex in B, and suppose that ρ has one of the following forms:*

   (a) $\langle a_1 - \epsilon_1, \ldots, a_{n-1} - \epsilon_{n-1}, p - \epsilon_n \rangle$, *where at least two of the ϵ_i are positive;*

   (b) $\langle a_1 + \epsilon_1, \ldots, a_{n-1} + \epsilon_{n-1}, p \rangle$, *where at least two of the ϵ_i are positive, or some ϵ_i equals two;*

   (c) $\langle (a_1 + \epsilon_1)^{(2)}, a_2 + \epsilon_2, \ldots, a_{n-1} + \epsilon_{n-1} \rangle$, *where at least one of the ϵ_i is positive.*

   Then D^λ does not extend D^ρ.

3. *Let $\lambda = \langle a_1, \ldots, a_{j-1}, a_j^{(2)}, a_{j+1}, \ldots, a_{n-1} \rangle$ be a semi-general vertex in B, and suppose that ρ has one of the following forms:*

   (a) $\langle a_1 - \epsilon_1, \ldots, a_{j-1} - \epsilon_{j-1}, (a_j - \epsilon_j)^{(2)}, a_{j+1} - \epsilon_{j+1}, \ldots, a_{n-1} - \epsilon_{n-1} \rangle$, *where at least two of the ϵ_i are positive;*

   (b) $\langle a_1 + \epsilon_1, \ldots, a_{j-1} + \epsilon_{j-1}, (a_j + \epsilon_j)^{(2)}, a_{j+1} + \epsilon_{j+1}, \ldots, a_{n-1} + \epsilon_{n-1} \rangle$, *where at least two of the ϵ_i are positive, or some ϵ_i equals two;*

   (c) $(j = 1)$ $\langle a_1 - \epsilon_1, \ldots, a_{n-1} - \epsilon_{n-1}, p \rangle$, *where at least one of the ϵ_i is positive;*

   (d) $(j > 1)$ $\langle a_1 - \epsilon_1, \ldots, a_{j-2} - \epsilon_{j-2}, (a_{j-1} - \epsilon_{j-1})^{(2)}, a_j - \epsilon_j, \ldots, a_{n-1} - \epsilon_{n-1} \rangle$, *where at least one of the ϵ_i is positive;*

   (e) $(j < n-1)$ $\langle a_1 + \epsilon_1, \ldots, a_j + \epsilon_j, (a_{j+1} + \epsilon_{j+1})^{(2)}, a_{j+2} + \epsilon_{j+2}, \ldots, a_{n-1} + \epsilon_{n-1} \rangle$, *where at least one of the ϵ_i is positive.*

   Then D^λ does not extend D^ρ.

Proof. Suppose $\lambda = \langle a_1, \ldots, a_n \rangle$, $\rho = \langle a_1 - \epsilon_1, \ldots, a_n - \epsilon_n \rangle$, and that $\epsilon_i = \epsilon_l = 1$. Then $\lambda \succ \rho$, and $D^\rho\downarrow_{B_{a_i}} = 0$. Now $D^\lambda\downarrow_{B_{a_i}} \cong D^{\tilde{\lambda}}$, where

$$\tilde{\lambda} = \langle a_1, \ldots, \widehat{a_i}, \ldots, a_n \rangle$$

in $\langle n^{a_i - 2}, n+1, n-1, n^{p-a_i} \rangle$ notation. The corresponding partition in M_{a_i} is $\mu = \langle a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n \rangle$, and $\rho \not\succeq \mu$; now apply Lemma 4.2. The other cases follow similarly; in those cases where $\rho \succ \lambda$, the rôles of λ and ρ must be interchanged. □

We now show that for general and p-general vertices, all the possible remaining extensions do in fact exist.

Theorem 4.4. *Let $\lambda = \langle a_1, \ldots, a_n\rangle$ be a general or a p-general vertex in B, and denote by λ_i^+, λ_i^- the partitions $\langle a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_n\rangle$, $\langle a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n\rangle$ respectively (in the case $a_n = p$, $i = n$, we put $\lambda_n^+ = \langle a_1^{(2)}, a_2, \ldots, a_{n-1}\rangle$). Then*

$$\mathrm{Ext}^1_B(D^\lambda, D^{\lambda_i^+}) \cong \mathrm{Ext}^1_B(D^\lambda, D^{\lambda_i^-}) \cong k,$$

and

$$D^\lambda{\downarrow_{B_{a_i}}}{\uparrow^B} \cong \begin{array}{c} D^\lambda \\ D^{\lambda_i^+} D^{\lambda_i^-} \\ D^\lambda \end{array}.$$

Proof. From Propositions 3.5, 3.7 and 3.8 and Corollary 4.3 we have seen that the only possible neighbours of λ in the quiver of B are the partitions λ_i^+, λ_i^-, for $i = 1, \ldots, n$. Fixing a particular i, write $\tilde{\lambda} = \Theta_{a_i}(\lambda)$ and $D_i = D^{\tilde{\lambda}}{\uparrow^B}$. Then D_i has cosocle isomorphic to D^λ, so that all composition factors of the second Loewy layer of D_i must extend D^λ. By Theorem 4.1, such factors can only be isomorphic to $D^{\lambda_i^+}$ or $D^{\lambda_i^-}$, and $[D_i : D^{\lambda_i^-}] = 1$.

Claim. $[D_i : D^{\lambda_i^+}] = 1$.

Proof. We use Schaper's formula: using the notation of 1.1.3, we have

$$c_{\lambda_i^+,\lambda} = c_{\lambda,\lambda_i^-} = 1,$$

$$c_{\lambda_i^+,\lambda_i^-} = -1;$$

there is no partition ν other than λ with $\lambda_i^+ > \nu > \lambda_i^-$, so we get

$$[S^\lambda : D^{\lambda_i^+}] = 1, \quad [S^{\lambda_i^-} : D^{\lambda_i^+}] = 0,$$

whence

$$[S^{\tilde{\lambda}}{\uparrow^B} : D^{\lambda_i^+}] = 1.$$

Thus $[D_i : D^{\lambda_i^+}] = 1$ unless $D^{\lambda_i^+}$ is a composition factor of $D^{\tilde{\nu}}{\uparrow^B}$ for some factor $D^{\tilde{\nu}}$ of $\mathrm{rad}(S^{\tilde{\lambda}})$. If this is the case, let ν be the element of Λ_{a_i} with $\Theta_{a_i}(\nu) = \tilde{\nu}$, and let ν^- be the corresponding element of M_{a_i}. $D^{\lambda_i^+}$ is then a factor of $S^{\tilde{\nu}}{\uparrow^B} \sim S^\nu + S^{\nu^-}$; $\nu > \lambda$, so we have

$$\lambda_i^+ > \nu^- > \lambda_i^-;$$

but M_{a_i} contains no such partition ν^-. □

The structure of D_i now follows, since it is self-dual. Hence the spaces $\mathrm{Ext}^1_B(D^\lambda, D^{\lambda_i^+}), \mathrm{Ext}^1_B(D^\lambda, D^{\lambda_i^-})$ are non-zero; that they are one-dimensional follows from Proposition 3.3, since we have $[S^\lambda : D^{\lambda_i^+}] = 1$ (from above) and $[S^{\lambda_i^-} : D^\lambda] = 1$ (using the Schaper's formula coefficients above). □

Remark. The reader who is wary of using Schaper's formula in the above proof may instead care to prove that $[D_i : D^{\lambda_i^+}] = 1$ using the Mullineux involution.

We can do almost as well for the semi-general vertices.

Theorem 4.5. *Let $\lambda = \langle a_1, \dots, a_{j-1}, a_j^{(2)}, a_{j+1}, \dots, a_{n-1}\rangle$ be a semi-general vertex in B, and for $i = 1, \dots, n-1$, denote by λ_i^+, λ_i^- the partitions obtained from λ by changing a_i to $a_i + 1$, $a_i - 1$ respectively. Then*

$$\mathrm{Ext}^1_B(D^\lambda, D^{\lambda_i^+}) \cong \mathrm{Ext}^1_B(D^\lambda, D^{\lambda_i^-}) \cong k,$$

and

$$D^\lambda\!\downarrow_{B_{a_i}}\!\uparrow^B \cong \begin{matrix} & D^\lambda & \\ D^{\lambda_i^+} & & D^{\lambda_i^-} \\ & D^\lambda & \end{matrix}\ .$$

Proof. The proof is exactly as for Theorem 4.4. □

Thus, in order to determine completely the extensions of simple modules labelled by a semi-general partition $\lambda = \langle a_1, \dots, a_{j-1}, a_j^{(2)}, a_{j+1}, \dots, a_{n-1}\rangle$, we need only to know whether D^λ extends D^{λ^+}, D^{λ^-}, where λ^+ and λ^- are obtained from λ by changing j to $j+1$, $j-1$ respectively (except in the case $j = 1$, where we define $\lambda^- = \langle a_1, \dots, a_{n-1}, p\rangle$, and the case $j = n-1$, where we do not define λ^+). To do this, we use the restriction method of [11], and induction on n; with what we already know, however, there will be much less notation to set up.

Theorem 4.6. *Define λ, λ^+ (if $j < n-1$) and λ^- as above. Then*

$$\mathrm{Ext}^1_B(D^\lambda, D^{\lambda^-}) \cong k,$$

and if $j < n-1$ then

$$\mathrm{Ext}^1_B(D^\lambda, D^{\lambda^+}) \cong k.$$

Proof. We need only consider $\mathrm{Ext}^1_B(D^\lambda, D^{\lambda^-})$; $\mathrm{Ext}^1_B(D^\lambda, D^{\lambda^+})$ will then follow, since λ^+ is a semi-general partition with $(\lambda^+)^- = \lambda$. First of all, consider restriction and induction between the blocks B and B_{a_1}. Put $\tilde\lambda = \Theta_{a_1}(\lambda)$, $\tilde\lambda^- = \Theta_{a_1}(\lambda^-)$. Recall from Theorem 4.5 that

$$D^{\tilde\lambda}\!\uparrow^B \cong \begin{matrix} & D^\lambda & \\ D^{\lambda_1^+} & & D^{\lambda_1^-} \\ & D^\lambda & \end{matrix}\ .$$

Now we know that D^{λ^-} does not extend either $D^{\lambda_1^+}$ or $D^{\lambda_1^-}$, so if $\mathrm{Ext}^1_B(D^{\tilde\lambda}\!\uparrow^B, D^{\lambda^-}) \neq 0$, then $\mathrm{Ext}^1_B(D^\lambda, D^{\lambda^-})$ will be non-zero (and hence one-dimensional) as well. The Eckmann-Shapiro relations tell us that

$$\mathrm{Ext}^1_B(D^{\tilde\lambda}\!\uparrow^B, D^{\lambda^-}) \cong \mathrm{Ext}^1_{B_{a_1}}(D^{\tilde\lambda}, D^{\tilde\lambda^-}),$$

and we claim that the latter space is one-dimensional.

We restrict $D^{\tilde\lambda}$ and $D^{\tilde\lambda^-}$ through a sequence of blocks $A_1, \dots, A_p$ defined as follows:

- $A_1 = B_{a_1}$;
- for $2 \leqslant i \leqslant a_1 - 1$, the abacus for A_i is obtained from that for A_{i-1} by moving a bead from runner $a_1 + 1 - i$ to runner $a_1 - i$;
- for $a_1 \leqslant i \leqslant p$, A_i is obtained from A_{i-1} by moving a bead from runner $i+1$ to runner i (where runner $p+1$ is taken to mean runner 1).

In particular, note that A_p is the principal block of $k\mathfrak{S}_{(n-1)p}$. By applying Theorem 1.3 we find that $D^{\tilde\lambda}$ and $D^{\tilde\lambda^-}$ restrict to simple modules. Define the partitions $\check\lambda$, $\check\lambda^-$ of A_p by

$$\check\lambda = \begin{cases} \langle a_2-1,\dots,a_{j-1}-1,(a_j-1)^{(2)},a_{j+1}-1,\dots,a_{n-1}-1\rangle & (j>1)\\ \langle a_2-1,\dots,a_{n-1}-1,p\rangle & (j=1)\end{cases}$$

and

$$\check\lambda^- = \begin{cases} \langle a_2-1,\dots,a_{j-2}-1,(a_{j-1}-1)^{(2)},a_j-1,\dots,a_{n-1}-1\rangle & (j>2)\\ \langle a_2-1,\dots,a_{n-1}-1,p\rangle & (j=2)\\ \langle a_2-1,\dots,a_{n-1}-1,p-1\rangle & (j=1).\end{cases}$$

Then the following hold:

$$\begin{aligned} D^{\tilde\lambda}{\downarrow_{A_2}}\cdots{\downarrow_{A_p}} &\cong D^{\check\lambda},\\ D^{\tilde\lambda^-}{\downarrow_{A_2}}\cdots{\downarrow_{A_p}} &\cong D^{\check\lambda^-},\\ D^{\check\lambda}{\uparrow^{A_{p-1}}}\cdots{\uparrow^{A_1}} &\cong D^{\tilde\lambda},\\ D^{\check\lambda^-}{\uparrow^{A_{p-1}}}\cdots{\uparrow^{A_1}} &\cong D^{\tilde\lambda^-}.\end{aligned}$$

Now $\check\lambda$ is a semi-general or a p-general vertex of the quiver of $k\mathfrak{S}_{(n-1)p}$ according as $j>1$ or $j=1$, while $\check\lambda^-$ is a semi-general, p-general or general vertex according as $j>2$, $j=2$ or $j=1$. Hence by Theorem 4.4 and by induction on n we have

$$\begin{aligned} k &\cong \mathrm{Ext}^1_{A_p}(D^{\check\lambda},D^{\check\lambda^-})\\ &\cong \mathrm{Ext}^1_{B_{a_1}}(D^{\tilde\lambda},D^{\tilde\lambda^-}).\end{aligned}$$ □

5 Further general vertices

For $k\mathfrak{S}_m$-modules M, N, we have

$$\mathrm{Ext}^1_{k\mathfrak{S}_m}(M\otimes\mathrm{sgn},N\otimes\mathrm{sgn})\cong\mathrm{Ext}^1_{k\mathfrak{S}_m}(M,N);$$

hence we can apply the Mullineux map to our general vertices and their neighbours and immediately find some more vertices of general type. We derive the following. We assume throughout that $n\geqslant 3$; a modified statement exists in the case $n=2$.

Theorem 5.1.

1. *Let λ be a partition of B labelled $\langle 1,a_2,\dots,a_n\rangle$, where $4\leqslant a_2<\dots<a_n\leqslant p$ and $a_{i+1}-a_i\geqslant 3$ for all i. Then λ is attached in the quiver of B to exactly $2n$ vertices, labelled as follows:*
   - *$\langle 1,a_2,\dots,a_r\pm 1,\dots,a_n\rangle$, where $2\leqslant r\leqslant n-1$;*

- $\langle 2, a_2, \ldots, a_n \rangle$;
- $\langle 1, a_2, \ldots, a_n - 1 \rangle$;
- $\langle 1, a_2, \ldots, a_n + 1 \rangle$, *provided* $a_n < p$;
- $\langle 1^{(2)}, a_2, \ldots, a_n \rangle$, *if* $a_n = p$;
- $\langle a_2, \ldots, a_{n-1}, a_n^2 \rangle$.

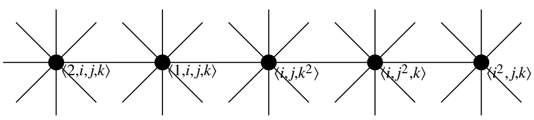
2. *Let* λ *be a partition of* B *labelled* $\langle a_2, \ldots, a_{n-1}, a_n^2 \rangle$, *where* $3 \leqslant a_2 < \cdots < a_n \leqslant p$ *and* $a_{i+1} - a_i \geqslant 3$ *for all* i. *Then* λ *is attached in the quiver of* B *to exactly* $2n$ *vertices, labelled as follows:*

   - $\langle a_2, \ldots, a_r \pm 1, \ldots, a_{n-1}, a_n^2 \rangle$, *where* $2 \leqslant r \leqslant n - 1$;
   - $\langle a_2, \ldots, a_{n-1}, (a_n - 1)^2 \rangle$;
   - $\langle a_2, \ldots, a_{n-1}, (a_n + 1)^2 \rangle$, *provided* $a_n < p$;
   - $\langle 1, a_2^{(2)}, a_3, \ldots, a_{n-1} \rangle$ *if* $a_n = p$;
   - $\langle a_2, \ldots, a_{n-2}, a_{n-1}^2, a_n \rangle$;
   - $\langle 1, a_2, \ldots, a_n \rangle$.

3. *Let* λ *be a partition of* B *labelled* $\langle a_2, \ldots, a_{j-1}, a_j^{(2)}, a_{j+1}, \ldots, a_n \rangle$, *where* $3 \leqslant j \leqslant n - 1$, $3 \leqslant a_2 < \cdots < a_n \leqslant p$ *and* $a_{i+1} - a_i \geqslant 3$ *for all* i. *Then* λ *is attached in the quiver of* B *to precisely* $2n$ *(if* $j > 3$*) or* $2n - 1$ *(if* $j = 3$*) vertices, labelled as follows:*

   - $\langle a_2, \ldots, a_r \pm 1, \ldots, a_{j-1}, a_j^2, a_{j+1}, \ldots, a_n \rangle$, *where* $2 \leqslant r \leqslant j - 1$;
   - $\langle a_2, \ldots, a_{j-1}, (a_j \pm 1)^2, a_{j+1}, \ldots, a_n \rangle$;
   - $\langle a_2, \ldots, a_{j-1}, a_j^2, a_{j+1}, \ldots, a_r \pm 1, \ldots, a_n \rangle$, *where* $j + 1 \leqslant r \leqslant n$;
   - $\langle a_2, \ldots, a_{j-1}, a_j^2, a_{j+1}, \ldots, a_n - 1 \rangle$;
   - $\langle a_2, \ldots, a_{j-1}, a_j^2, a_{j+1}, \ldots, a_n + 1 \rangle$, *provided* $a_n < p$;
   - $\langle a_2^{(2)}, a_3, \ldots, a_{j-1}, a_j^2, a_{j+1}, \ldots, a_{n-1} \rangle$ *if* $a_n = p$;
   - $\langle a_2, \ldots, a_{j-2}, a_{j-1}^2, a_j, \ldots, a_n \rangle$, *provided* $j > 3$;
   - $\langle a_2, \ldots, a_j, a_{j+1}^2, a_{j+2}, \ldots, a_n \rangle$.

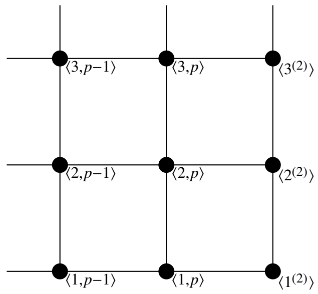
Hence, by applying the Mullineux map, we obtain another 'wall' of the quiver, orthogonal to the first. For $n = 4$, a cross-section of part of the quiver looks as follows:



;

here each vertex is also attached to the corresponding vertices labelled with $i \pm 1, j, k$ or $i, j \pm 1, k$ or $i, j, k \pm 1$.

A natural question to ask is what the quivers look like where these walls meet; in the case $n = 2$, we have



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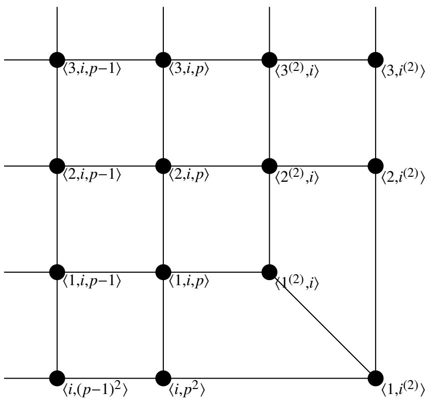
where the Mullineux involution corresponds to reflection in the diagonal. But for $n = 3$, the lattice structure begins to break down; a cross-section is



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where each vertex is also attached to the corresponding vertices labelled with $i \pm 1$, and the Mullineux involution corresponds to reflecting the diagram in the diagonal and changing i to $\tilde{\imath} + 1$.

6 The Specht module corresponding to a general vertex

In this section, we determine the module structure of the Specht module S^λ, where $\lambda = \langle a_1, \dots, a_n\rangle$ is a general vertex.

Proposition 6.1. *Suppose $\lambda = \langle a_1, \dots, a_n\rangle$ is a general vertex in B. Then*

$$[S^\lambda : D^\mu] = \begin{cases} 1 & \text{(if } \mu \text{ is of the form } \langle c_1, \dots, c_n\rangle \text{ with each } c_i \text{ equal to } a_i \text{ or } a_i + 1) \\ 0 & \text{(otherwise).} \end{cases}$$

Proof. If $[S^\lambda : D^\mu] > 0$, then we must have $\mu \trianglerighteq \lambda \succ \mu^{*\prime}$; recall from Proposition 3.7 that this only happens when μ has the form $\langle c_1, \dots, c_n\rangle$, where $a_i \leqslant c_i \leqslant a_i + 2$, and if $c_i = a_i + 2$ then $c_{i+1} = c_i + 1$. Such partitions μ have the property that if $\mu \trianglerighteq \nu \trianglerighteq \mu^{*\prime}$, then ν is p-regular. Now recall that the projective cover $P(D^\mu)$ has a Specht filtration whose factors are precisely those Specht modules containing D^μ as a composition factor, with multiplicities. The part of the decomposition matrix corresponding only to p-regular partitions is invertible, and so if we can find any Specht filtration of $P(D^\mu)$ in which the factors S^ν all have ν p-regular, then those factors will be precisely the Specht modules containing D^μ as a composition factor, with multiplicities. We proceed to find such a filtration.

Let μ be as above, and suppose first of all that each c_i equals a_i or $a_i + 1$. We form a sequence of blocks $B = A_0, A_1, \dots, A_n$ by moving a bead from runner c_i to runner $c_i - 1$ to obtain A_i from A_{i-1}. By Kleshchev's branching rules, we find that

$$D^\mu\!\downarrow_{A_1} \cdots\!\downarrow_{A_n}$$

is a simple module D^ξ. ξ is a p-core, and so $D^\xi = S^\xi$ is projective. Hence

$$P = D^\xi\!\uparrow^{A_{n-1}} \cdots\!\uparrow^B$$

is also projective; by the Branching Rule we find a Specht filtration for P; each factor is of the form $S^{\langle e_1,\dots,e_n\rangle}$ where e_i equals c_i or c_{i-1}, and each such S^ν occurs once; in particular, S^λ occurs once. We claim that $P = P(D^\mu)$; since all the Specht factors correspond to p-regular partitions, any simple module occurring in the cosocle of P must have the form D^ν with ν as above. But by Kleshchev's branching rules,

$$D^\nu\!\downarrow_{A_1} \cdots\!\downarrow_{A_n} \cong \begin{cases} D^\xi & (\nu = \mu) \\ 0 & (\text{otherwise}), \end{cases}$$

so the claim follows by Frobenius reciprocity. Hence D^μ occurs once as a composition factor of S^λ.

Now suppose that for some i we have $c_i = a_i + 2$. Form a sequence of blocks $B = A_0, \dots, A_n$ slightly differently; to obtain A_i from A_{i-1}, we move a bead from runner c_i to runner $c_i - 2$ if either $c_{i-1} = c_i - 1$ or $c_{i+1} = c_i + 1$, and from runner c_i to runner $c_i - 1$ otherwise. Again we find that

$$D^\mu\!\downarrow_{A_1} \cdots\!\downarrow_{A_n} = D^\xi$$

where ξ is a p-core. Defining

$$P = D^\xi\!\uparrow^{A_{n-1}} \cdots\!\uparrow^B$$

as before, we apply the Branching Rule to find a Specht filtration for the projective module P (which includes $P(D^\mu)$ as a summand); the Specht factors again all correspond to p-regular partitions, but none of them is S^λ. Hence $[S^\lambda : D^\mu] = 0$. □

We proceed to find the submodule structure of S^λ explicitly. Since the composition factors are distinct, the submodule lattice is distributive, and we may represent the structure of S^λ by means of a quiver with vertices corresponding to the composition factors, and an arrow from D^μ to D^ν if and only if S^λ has a subquotient isomorphic to a non-split extension of D^μ by D^ν.

We assume now that $a_i - a_{i-1} \geqslant 4$ for all i; this means that all the composition factors of S^λ correspond to general vertices, which makes the following proof easier. We believe that our results hold when some of the $a_i - a_{i-1}$ equal 3, but this requires further analysis of the Ext-quiver of B. The full subquiver of the Ext-quiver of B corresponding to those simple modules which are composition factors of S^λ has the structure of an n-cube: there is an edge from $\langle c_1, \dots, c_n\rangle$ to $\langle d_1, \dots, d_n\rangle$ if and only if $c_i \neq d_i$ for exactly one value of i. We find that in fact this n-cube structure holds in S^λ, i.e. every possible extension (or its dual) occurs as a subquotient of S^λ.

Proposition 6.2. *Suppose that $\langle a_1, \dots, a_n\rangle$ is a general vertex in B with $a_i - a_{i-1} \geqslant 4$ for each i. Suppose also that $c_1, \dots, \widehat{c_l}, \dots, c_n$ are such that each c_i equals a_i or a_{i+1}. Then S^λ has a subquotient isomorphic to the non-split extension of $D^{\langle c_1,\dots,c_{l-1},a_l,c_{l+1},\dots,c_n\rangle}$ by $D^{\langle c_1,\dots,c_{l-1},a_l+1,c_{l+1},\dots,c_n\rangle}$.*

Proof. We wish to use Schaper's formula to provide some information about the structure of S^λ. Let $D^{\langle d_1,\dots,d_n\rangle}$ be a composition factor of S^λ; to find the bound for $[S^\lambda : D^{\langle d_1,\dots,d_n\rangle}]$ provided by Schaper's formula, we need to know which partitions μ have $c_{\mu,\lambda} \neq 0$ and $[S^\mu : D^{\langle d_1,\dots,d_n\rangle}] > 0$; these are precisely those partitions $\langle e_1,\dots,e_n\rangle$ for which $a_i \leqslant e_i \leqslant d_i$ for each i, with $e_i = a_i + 1$ for exactly one value of i. Such a partition has $[S^{\langle e_1,\dots,e_n\rangle} : D^{\langle d_1,\dots,d_n\rangle}] = 1$ by Proposition 6.1 and $c_{\langle e_1,\dots,e_n\rangle,\lambda} = 1$. Hence the bound for the multiplicity of $D^{\langle d_1,\dots,d_n\rangle}$ in S^λ equals the number of such partitions $\langle e_1,\dots,e_n\rangle$, i.e. the number of i such that $d_i = a_i + 1$. So S^λ has a filtration

$$S^\lambda = S_0 \geqslant \dots \geqslant S_n = 0$$

in which S_j/S_{j+1} is the direct sum of those $D^{\langle d_1,\dots,d_n\rangle}$ for which $d_i = a_i + 1$ for exactly j values of i.

A consequence of this is that S^λ does have a subquotient isomorphic to a (possibly split) extension M of $D^{\langle c_1,\dots,c_{i-1},a_i,c_{i+1},\dots,c_n\rangle}$ by $D^{\langle c_1,\dots,c_{i-1},a_i+1,c_{i+1},\dots,c_n\rangle}$. To show that M is in fact non-split, we consider restriction to $\mathfrak{S}_{np-n+1}$.

Form the sequence of blocks $B = A_0,\dots,A_{n-1}$ by moving a bead from runner c_i to runner $c_i - 1$ to obtain A_i from A_{i-1} if $i < l$, or from c_{i+1} to $c_{i+1} - 1$ if $i \geqslant l$. By the classical Branching rule, we find that $S^\lambda{\downarrow_{A_1}}\cdots{\downarrow_{A_{n-1}}}$ has just one Specht factor, and is therefore indecomposable. By Kleshchev's modular Branching rules, we find that $D^\mu{\downarrow_{A_1}}\cdots{\downarrow_{A_{n-1}}}$ is zero for all factors D^μ of S^λ other than $D^{\langle c_1,\dots,c_{l-1},a_l,c_{l+1},\dots,c_n\rangle}$ and $D^{\langle c_1,\dots,c_{l-1},a_l+1,c_{l+1},\dots,c_n\rangle}$, which restrict to simple modules. Thus

$$M{\downarrow_{A_1}}\cdots{\downarrow_{A_{n-1}}} \cong S^\lambda{\downarrow_{A_1}}\cdots{\downarrow_{A_{n-1}}},$$

which is a non-split extension of two simple modules. Hence M is non-split. □

We have now determined the structure of the Specht module corresponding to a general vertex λ: from what we know about the quiver of B, the only extensions of simple modules which can occur as subquotients of S^λ are those occurring in Proposition 6.2 and their duals. But since the composition factors of S^λ occur with multiplicity 1, the extensions in Proposition 6.2 are the only ones occurring in S^λ.

7 The projective cover of a simple module corresponding to a general vertex

In this section we determine the structure of the projective cover of D^λ, where $\lambda = \langle a_1,\dots,a_n\rangle$ is a general vertex. Although the module retains the structure of a $2n$-cube in some sense, the submodule lattice is not distributive, and we content ourselves with determining the Loewy series of the projective cover.

From the proof of Proposition 6.1, we know that $P(D^\lambda)$ is filtered by the Specht modules $S^{\langle c_1,\dots,c_n\rangle}$, where each c_i equals a_i or a_{i-1}. Moreover, we know that $S^{\langle c_1,\dots,c_n\rangle}$ lies above $S^{\langle d_1,\dots,d_n\rangle}$ in $P(D^\lambda)$ only if $\langle c_1,\dots,c_n\rangle \geqslant \langle d_1,\dots,d_n\rangle$, i.e. if $c_i \geqslant d_i$ for all i.

By restricting attention to the case where $a_1 \geqslant 3$ and $a_i - a_{i-1} \geqslant 5$ for all i, we ensure that all composition factors of these Specht modules correspond to general vertices. Again, the authors believe that the same result holds if some $a_i - a_{i-1}$ equals 4. From the previous section we know the module structure of each Specht factor. In particular, we know that the jth Loewy layer of $S^{\langle c_1,\dots,c_n\rangle}$ consists of the modules $D^{\langle d_1,\dots,d_n\rangle}$ where each d_i equals c_i or c_{i+1} and $d_i = c_{i+1}$ for exactly j values of i. Thus each composition factor of $P(D^\lambda)$ has the form $D^{\langle d_1,\dots,d_n\rangle}$, where each d_i equals a_i, a_i-1 or a_i+1; call such a partition *λ-close*. If μ is λ-close, denote by $s(\mu)$ the number of i for which $d_i = a_i - 1$, and by $t(\mu)$ the number of i for which $d_i = a_i + 1$. D^μ then appears $2^{n-s(\mu)-t(\mu)}$ times as a composition factor of $P(D^\lambda)$.

Notice that for λ-close partitions σ and τ, we can only have $\mathrm{Ext}^1_B(D^\sigma, D^\tau) \neq 0$ if $|s(\sigma)-s(\tau)| \leqslant 1$, $|t(\sigma)-t(\tau)| \leqslant 1$ and $|(s(\sigma)-t(\sigma))-(s(\tau)-t(\tau))| \leqslant 1$.

We determine the Loewy structure of $P(D^\lambda)$ as follows.

Proposition 7.1. *Suppose that $\lambda = \langle a_1,\dots,a_n\rangle$ is a general vertex with $a_1 \geqslant 3$ and $a_i - a_{i-1} \geqslant 5$ for $i \geqslant 2$. For S^μ a Specht factor of $P(D^\lambda)$, the rth Loewy layer of S^μ is contained in the $(r+s(\mu))$th Loewy layer of $P(D^\lambda)$.*

Proof. We proceed by induction on r and on $s(\mu)$. If $s(\mu) = 0$, then $\mu = \lambda$; but S^λ is a quotient of $P(D^\lambda)$, so the result holds. In particular, for every λ-close partition ν with $s(\nu) = 0$, the $(t(\nu)+1)$th Loewy layer of $P(D^\lambda)$ contains a copy of D^ν.

Applying the Mullineux map to λ, we find that $\lambda^* = \langle \tilde{a}_n,\dots,\tilde{a}_1\rangle$, where $\tilde{a}_i = p+2-a_i$. We have $3 \leqslant \tilde{a}_n \leqslant \dots \leqslant \tilde{a}_1 < p$ and $\tilde{a}_i - \tilde{a}_{i+1} \geqslant 5$, so the results of this section apply to λ^* also. Hence for every λ^*-close partition ξ with $s(\xi) = 0$, $P(D^{\lambda^*})$ has a copy of D^ξ in its $(t(\xi)+1)$th Loewy layer. But λ-close and λ^*-close partitions correspond under the Mullineux map, with $s(\xi^*) = t(\xi)$, $t(\xi^*) = s(\xi)$. Thus for every λ-close partition ν with $t(\nu) = 0$, there is a copy of D^ν in the $(s(\nu)+1)$th Loewy layer of $P(D^\lambda)$. We claim that this is the cosocle of S^ν. If not, then suppose it lies in the Specht factor S^π of $P(D^\lambda)$. We must then have $s(\pi) \geqslant s(\nu)$, and the cosocle D^π of S^π must lie in some higher Loewy layer of $P(D^\lambda)$, i.e. in at most the $s(\nu)$th Loewy layer. But for two λ-close partitions σ, τ, D^σ can only extend D^τ if $|s(\sigma)-s(\tau)| \leqslant 1$; in particular, any factor D^σ of the rth Loewy layer of $P(D^\lambda)$ must have $s(\sigma) \leqslant r-1$. This gives a contradiction, and our claim is proven; this deals with the case $r = 1$ of the proposition.

Now suppose that $r > 1$ and $s(\mu) > 0$, and consider a factor D^ξ of the rth Loewy layer of S^ν. The $(r-1)$th layer of S^ν lies in the $(r+s(\mu)-1)$th layer of $P(D^\lambda)$, and so D^ξ lies in at least the $(r+s(\mu))$th layer; it can only lie in a lower layer if it extends some module which we already know lies in the $(r+s(\mu))$th layer or lower, i.e. if there is some λ-close partition π with $t(\pi) = 0$, $s(\pi) < s(\nu)$, and a factor D^σ in at least the $(s(\nu)-s(\pi)+r)$th layer of S^π such that $\mathrm{Ext}^1_B(D^\xi, D^\sigma) \neq 0$. But then $s(\sigma)-t(\sigma) \leqslant s(\pi)-(s(\nu)-s(\pi)+r-1)$, while $s(\xi)-t(\xi) = s(\nu)-r+1$; thus $s(\sigma)-t(\sigma)$ and $s(\xi)-t(\xi)$ differ by at least 2, and so D^σ does not extend D^ξ; contradiction. The result follows. □

Corollary 7.2. *Let λ be as in Proposition 7.1. Then $P(D^\lambda)$ has Loewy length $2n+1$ and is stable, i.e. its Loewy series is the same as its socle series.*

Proof. Denote by $\mathbb{I}(x \in X)$ the indicator function of a finite set X. Now, by examining the Loewy layers of the Specht factors of $P(D^\lambda)$ and using Proposition 7.1, we can easily verify the following. The composition factors of the rth Loewy layer of $P(D^\lambda)$ correspond to pairs (S,T) of subsets of $\{1,\dots,n\}$ with $|S|+|T| = r-1$. The correspondence is via

$$(S,T) \leftrightarrow D^{\langle b_1,\dots,b_n\rangle},$$

where

$$b_i = a_i - \mathbb{I}(i \in S) + \mathbb{I}(i \in T).$$

The composition factor $D^{\langle b_1,\dots,b_n\rangle}$ is contained in the Specht factor $S^{\langle c_1,\dots,c_n\rangle}$, where $c_i = a_i - \mathbb{I}(i \in S)$.

Hence, for a λ-close partition μ, the number of copies of D^μ in the rth Loewy layer of $P(D^\lambda)$ is

$$\binom{n - s(\mu) - t(\mu)}{\frac{r-1-s(\mu)-t(\mu)}{2}},$$

or zero if $r-1$ and $s(\mu)+t(\mu)$ have different parities. In particular, for $r > 2n+1$, the rth Loewy layer is zero, and the multiplicities of D^μ in the rth and $2n+2-r$th Loewy layers are equal; a correspondence between the rth and $2n+2-r$th Loewy layers may be given by sending the pair (S, T) to $(\overline{T}, \overline{S})$. □

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