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A generalisation of core partitions

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Abstract

Suppose s and t are coprime natural numbers. A theorem of Olsson says that the t -core of an s -core partition is again an s -core. We generalise this theorem, showing that the s -weight of the t -core of a partition λ is at most the s -weight of λ . Then we consider the set $C_{s,t}$ of partitions for which equality holds, which we call $[s:t]$ -cores; this set has interesting structure, and we expect that it will be the subject of future study. We show that the set of $[s:t]$ -cores is a union of finitely many orbits for an action of a Coxeter group of type $\tilde{A}_{s-1} \times \tilde{A}_{t-1}$ on the set of partitions. We also consider the problem of constructing an $[s:t]$ -core with specified s -core and t -core.

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1 Introduction

In this paper we study the combinatorics of integer partitions. If s is a natural number, an s -core (often referred to in the literature as an s -core partition) is a partition with no rim hooks of length s . The s -core of an arbitrary partition λ is obtained by repeatedly removing rim s -hooks from λ , and the s -weight of λ is the number of rim hooks removed. The set of all s -cores, which we denote C_s , has geometric structure related to type A alcove geometry, and has applications in representation theory and number theory.

Now suppose t is another natural number which is prime to s . A recent trend in partition combinatorics has been to compare s -cores and t -cores. Anderson [A] enumerated (s, t) -cores, i.e. partitions which are both s - and t -cores, and Fishel and Vazirani [FV] explored the connection between (s, t) -cores and the associated alcove geometry. Several authors [K, OS, V, F] have studied the properties of the largest (s, t) -core, which is denoted $\kappa_{s,t}$. The present author explored another avenue in [F], considering the t -core of an arbitrary s -core; by a theorem of Olsson, the t -core of an s -core is again an s -core, so we have a natural map from s -cores to (s, t) -cores. Exploiting natural actions of the Coxeter group W_s of type $\tilde{A}_{s-1}$ on C_s reveals interesting symmetry in this map.

In the present paper we generalise Olsson's theorem, showing that replacing any partition with its t -core does not increase its s -weight. We define an $[s:t]$ -core to be a partition for which equality holds in this statement, and explore the family $C_{s:t}$ of $[s:t]$ -cores, which plays a kind of dual role to the family of (s, t) -cores. We show that $C_{s:t}$ is a union of orbits for a certain action of $W_s \times W_t$, with each orbit containing a unique (s, t) -core. We then consider the problem of constructing a partition with a given s -core σ and a given t -core τ ; we show that if the t -core of σ coincides with the s -core of τ , then there is a unique $[s:t]$ -core with s -core σ and t -core τ , and we give a simple method for constructing this partition. This leads to an alternative characterisation of an $[s:t]$ -core as a partition which is uniquely determined by its size, its s -core and its t -core. Finally we consider the orbit of $W_s \times W_t$ containing $\kappa_{s,t}$, showing that this is naturally in bijection with $C_s \times C_t$.

In the next section we recall basic definitions and simple results, largely to fix conventions and notation. In Section 3 we define the group W_s and study its actions on integers and partitions; some of this material has not appeared in this form before. In Sections 4 to 6 we prove our main results. We finish with some brief comments in Section 7.

2 Basic definitions

2.1 Conventions and notation

In this section we set out some basic conventions that we use throughout the paper. As usual, $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{N}_0$ the set of non-negative integers. We shall often consider the set $\mathbb{Z}/s\mathbb{Z}$, where $s \in \mathbb{N}$, and we use the formal convention that

$$\mathbb{Z}/s\mathbb{Z} = \{a + s\mathbb{Z} \mid a \in \mathbb{Z}\},$$

where as usual

$$a + s\mathbb{Z} = \{a + sb \mid b \in \mathbb{Z}\}$$

for an integer a . We adopt the standard convention that $(a + s\mathbb{Z}) + b = (a + b) + s\mathbb{Z}$ for any $a, b \in \mathbb{Z}$, but we adopt an unusual convention for multiplication, namely that

$$(a + s\mathbb{Z})b = (ab) + s\mathbb{Z}.$$

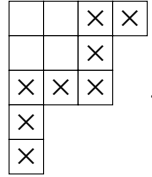
If X is a set, a $\mathbb{Z}/s\mathbb{Z}$ -tuple of elements of X is simply a function $i \mapsto x_i$ from $\mathbb{Z}/s\mathbb{Z}$ to X , and we may write such a tuple in the form $(x_i \mid i \in \mathbb{Z}/s\mathbb{Z})$. A *multiset* of elements of X is a subset of X with

possibly repeated elements (i.e. a function from X to $\mathbb{N}_0$). We write a multiset by writing the elements (with multiplicity) surrounded by square brackets. Given a $\mathbb{Z}/s\mathbb{Z}$ -tuple $(x_i \mid i \in \mathbb{Z}/s\mathbb{Z})$, we denote the associated multiset $[x_i \mid i \in \mathbb{Z}/s\mathbb{Z}]$.

2.2 Partitions

In this paper, a *partition* is an infinite weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers with finite sum; we write $|\lambda|$ for this sum. When writing partitions, we omit zeroes and group together equal parts with a superscript, so the partition $(4, 3, 3, 1, 1, 0, 0, \dots)$ is written as $(4, 3^2, 1^2)$. The partition $(0, 0, \dots)$ is denoted $\emptyset$, and the set of all partitions is denoted $\mathcal{P}$.

The *Young diagram* of a partition λ is the set of all pairs $(i, j) \in \mathbb{N}^2$ for which $j \leq \lambda_i$; we often identify λ with its Young diagram, so for example we may write $\lambda \subseteq \mu$ to mean that $\lambda_i \leq \mu_i$ for all i . The *rim* of λ is the set of all $(i, j) \in \lambda$ such that $(i+1, j+1) \notin \lambda$. We draw the Young diagram as an array of boxes in the plane; for example, the Young diagram of $(4, 3^2, 1^2)$ (with the elements of the rim marked $\times$) is as follows:



2.3 Cores

Suppose λ is a partition and $s \in \mathbb{N}$. A *rim s -hook* of λ is a subset of the rim of λ of size exactly s which is connected (i.e. comprises consecutive elements of the rim) and which is removable in the sense that it can be removed to leave the Young diagram of a partition. For example, in the example $\lambda = (4, 3^2, 1^2)$ above, the set $\{(2, 3), (3, 3), (3, 2)\}$ is a rim 3-hook.

The *s -core* of λ is the partition obtained from λ by repeatedly removing rim s -hooks until none remain. For example, the 3-core of the partition $\lambda = (4, 3^2, 1^2)$ above is $(4, 2)$. It is well known (and follows from Lemma 2.2 below) that the s -core of λ is independent of the choice of rim hook removed at each stage. The *s -weight* of λ is the number of rim s -hooks removed to reach the s -core of λ , i.e. $\frac{1}{s}(|\lambda| - |\text{cor}_s \lambda|)$. We write $\text{cor}_s \lambda$ for the s -core of λ , and $\text{wt}_s \lambda$ for the s -weight. Note that these definitions remain valid in the case $s = 1$, although this case is seldom considered in the literature. In this case, we have $\text{cor}_1 \lambda = \emptyset$ for any λ , and hence $\text{wt}_1 \lambda = |\lambda|$.

We say that λ is an *s -core* if $\text{cor}_s \lambda = \lambda$ (or equivalently if $\text{wt}_s \lambda = 0$), and we write C_s for the set of all s -cores. This set has been studied at length; it enjoys a rich geometric structure, and has applications in representation theory and number theory.

A trend in recent years has been to compare s -cores and t -cores, where s and t are distinct positive integers. We define an *(s, t) -core* to be a partition which is both an s -core and a t -core. It is known that there are finitely many (s, t) -cores if and only if s and t are coprime; in this case, the number of (s, t) -cores is precisely $\frac{1}{s+t} \binom{s+t}{s}$ [A, Theorems 1 & 3].

2.4 Beta-sets

A useful way to understand partitions, and in particular s -cores, is via beta-sets. Given a partition λ and an integer r , we define the *beta-set* $\mathcal{B}_r^\lambda$ to be the infinite set of integers

$$\mathcal{B}_r^\lambda = \{\lambda_i - i + r \mid i \in \mathbb{N}\}.$$

We shall mostly consider the beta-set $\mathcal{B}_0^\lambda$, which we denote simply $\mathcal{B}^\lambda$.

Note that $\mathcal{B}_r^\lambda$ is bounded above and its complement in $\mathbb{Z}$ is bounded below. Conversely, if we are given a subset $\mathcal{B}$ of $\mathbb{Z}$ which is bounded above and whose complement is bounded below, then we have $\mathcal{B} = \mathcal{B}_r^\lambda$ for some (uniquely defined) partition λ and integer r : we let r be the number of positive integers in $\mathcal{B}$ minus the number of non-positive integers not in $\mathcal{B}$; then, writing the elements of $\mathcal{B}$ as $b_1 > b_2 > \dots$ and setting $\lambda_i = b_i + i - r$ for each i , we have a partition λ , and $\mathcal{B} = \mathcal{B}_r^\lambda$.

Later we will need the following simple lemma.

Lemma 2.1. *Suppose λ and μ are partitions and $r \in \mathbb{Z}$. If $\mathcal{B}_r^\lambda \subseteq \mathcal{B}_r^\mu$, then $\lambda = \mu$.*

Proof. Choose N sufficiently large that $\lambda_N = \mu_N = 0$. Then $\mathcal{B}_r^\lambda$ and $\mathcal{B}_r^\mu$ both contain all integers less than or equal to $r - N$. Moreover, $\mathcal{B}_r^\lambda$ contains exactly $N - 1$ elements greater than $r - N$, namely $\lambda_1 - 1 + r, \dots, \lambda_{N-1} - (N - 1) + r$, and similarly $\mathcal{B}_r^\mu$ contains exactly $N - 1$ elements greater than $r - N$. Since $\mathcal{B}_r^\lambda \subseteq \mathcal{B}_r^\mu$, we get $\mathcal{B}_r^\lambda = \mathcal{B}_r^\mu$, and hence $\lambda = \mu$. $\square$

The main advantage of beta-sets is the easy identification and classification of s -cores. The following lemma is due to James, and is the main motivation for introducing beta-sets.

Lemma 2.2. *Suppose λ is a partition, $r \in \mathbb{Z}$ and $s \in \mathbb{N}$. Then λ is an s -core if and only if there is no $b \in \mathcal{B}_r^\lambda$ such that $b - s \notin \mathcal{B}_r^\lambda$; moreover, the beta-set $\mathcal{B}_r^{\text{cor}_s \lambda}$ can be obtained from $\mathcal{B}_r^\lambda$ by repeatedly replacing integers b with $b - s$ as far as possible.*

Example. Take $\lambda = (4, 3^2, 1^2)$. Then $\mathcal{B}^\lambda = \{3, 1, 0, -3, -4, -6, -7, \dots\}$. Taking $s = 5$, we see that there are two integers $a \in \mathcal{B}^\lambda$ such that $a - s \notin \mathcal{B}^\lambda$, namely 3 and 0. Replacing 3 with -2 or 0 with -5 corresponds to removing a rim 5-hook from λ , and making both of these replacements yields the set $\{1, -2, -3, -4, \dots\}$, which is the beta-set of the partition $(2) = \text{cor}_5 \lambda$.

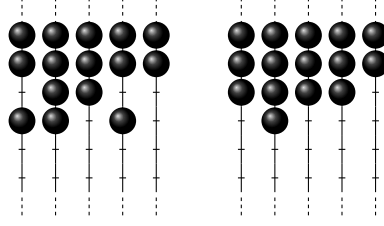
A very helpful way to visualise a beta-set of a partition is via James's *abacus*. Take an abacus with s infinite vertical runners, numbered $0, \dots, s - 1$ from left to right, and mark positions on the runners labelled by the integers, such that position x is immediately below position $x - s$ for all x , and position x is immediately to the right of position $x - 1$ whenever $s \nmid x$. For example, when $s = 3$ the positions are marked as follows.

0	1	2
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
-6	-5	-4
$\vdots$	$\vdots$	$\vdots$
-3	-2	-1
$\vdots$	$\vdots$	$\vdots$
0	1	2
$\vdots$	$\vdots$	$\vdots$
3	4	5
$\vdots$	$\vdots$	$\vdots$
6	7	8
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$

Now given a partition λ and an integer r , place a bead on the abacus at position x for each $x \in \mathcal{B}_r^\lambda$. The resulting configuration is called an *abacus display* for λ . Lemma 2.2 can now be stated as follows: λ is an s -core if and only if every bead in an abacus display for λ has a bead immediately above it; if λ is not an s -core, an abacus display for the s -core of λ can be obtained by sliding the beads up their runners as far as possible.

Example. Taking $\lambda = (4, 3^2, 1^2)$ and $s = 5$ as in the previous example, we obtain the following abacus displays for λ and $\text{cor}_5 \lambda$. (Note that we typically omit the labels of runners and positions when

drawing abacus displays.)



Lemma 2.2 implies the following two results.

Corollary 2.3. *Suppose λ is a partition, $r \in \mathbb{Z}$ and $s \in \mathbb{N}$. Then $\text{wt}_s \lambda$ equals the number of pairs $(x, l) \in \mathbb{Z} \times \mathbb{N}$ such that $x \in \mathcal{B}_r^\lambda \nexists x - ls$.*

Corollary 2.4. *Suppose $r, s \in \mathbb{N}$ and $\lambda \in \mathcal{P}$. Then $\text{cor}_s \lambda = \text{cor}_s(\text{cor}_{rs} \lambda)$. In particular, if λ is an s -core then λ is an rs -core.*

2.5 s -sets

Now we define the s -set of a partition λ ; this is a set of s integers which provides a useful encoding of the s -core of λ .

Suppose λ is an s -core. Given $i \in \mathbb{Z}/s\mathbb{Z}$, we define $\Gamma_i \lambda$ to be the smallest integer in i which is not in $\mathcal{B}^\lambda$. Then by Lemma 2.2 we have

$$\mathcal{B}^\lambda \cap i = \{\Gamma_i \lambda - ks \mid k \in \mathbb{N}\}.$$

Following [F], we refer to the set $\{\Gamma_i \lambda \mid i \in \mathbb{Z}/s\mathbb{Z}\}$ as the s -set of λ ; this set consists of s integers which are pairwise incongruent modulo s , and which sum to $\frac{1}{2}s(s-1)$. Conversely, any set of s integers with these properties is the s -set of some s -core. In terms of the abacus, the s -set consists of those unoccupied positions x such that $x - s$ is occupied, in the abacus display for λ with $r = 0$.

Now suppose λ is an arbitrary partition. We define $\Gamma_i \lambda = \Gamma_i(\text{cor}_s \lambda)$ for each i , and refer to the s -set of λ meaning the s -set of $\text{cor}_s \lambda$.

2.6 Quotients and the abacus

Next we define the s -quotient of a partition λ . Given $j \in \mathbb{Z}/s\mathbb{Z}$, write $j = i + s\mathbb{Z}$ for some integer i , and consider the set $\mathcal{B}^\lambda \cap j$. By subtracting i from each of the integers in this set and dividing by s , we obtain a set of integers which is bounded above and whose complement in $\mathbb{Z}$ is bounded below. This set is therefore a beta-set of some partition, which we denote $\lambda^{(j)}$. It is very easy to see that this partition is independent of the choice of i . We define the s -quotient $\text{quo}_s \lambda$ to be the $\mathbb{Z}/s\mathbb{Z}$ -tuple $(\lambda^{(j)} \mid j \in \mathbb{Z}/s\mathbb{Z})$.

Example. Suppose $s = 3$ and $\lambda = (5, 4^2, 3, 2, 1^2)$. Then

$$\mathcal{B}^\lambda = \{4, 2, 1, -1, -3, -5, -6, -8, -9, \dots\},$$

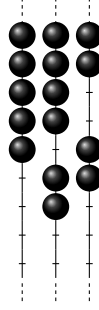
so

$$\mathcal{B}^\lambda \cap (1 + 3\mathbb{Z}) = \{4, 1, -5, -8, -11, \dots\}.$$

Subtracting 1 from each element and dividing by 3, we obtain the set $\{1, 0, -2, -3, -4, \dots\}$. This is the beta-set $\mathcal{B}_1^{(1^2)}$, so we have $\lambda^{(1+3\mathbb{Z})} = (1^2)$. In a similar way we find that $\lambda^{(3\mathbb{Z})} = \emptyset$ and $\lambda^{(2+3\mathbb{Z})} = (2^2)$.

The s -quotient of a partition can easily be visualised on the abacus. Taking the abacus display for λ with s runners and with $r = 0$, examine runner i in isolation; this can be regarded as a 1-runner abacus display for a partition, and this partition is $\lambda^{(i+s\mathbb{Z})}$. In other words, $\lambda_l^{(i+s\mathbb{Z})}$ equals the number of unoccupied positions above the l th lowest bead on runner i .

Example. Taking $\lambda = (5, 4^2, 3, 2, 1^2)$ and $s = 3$ as in the previous example, we obtain the following abacus display, from which we can see that the 3-quotient of λ is as given above.



Note that a partition λ is determined by its s -core and s -quotient. To see this, let $\sigma = \text{cor}_s \lambda$, and for each $j \in \mathbb{Z}/s\mathbb{Z}$ consider the set $\mathcal{B}^\sigma \cap j$. By Lemma 2.2 we have

$$\mathcal{B}^\sigma \cap j = \{\Gamma_j \sigma - is \mid i \in \mathbb{N}\}.$$

So if we let $v = \mu^{(j)}$, then

$$\mathcal{B}^\lambda \cap j = \{\Gamma_j \sigma + v_i s - is \mid i \in \mathbb{N}\}.$$

Applying this for all j , we see that $\mathcal{B}^\lambda$ (and hence λ) is determined by σ and $\text{quo}_s \lambda$. Moreover, we can see that for any s -core σ and any $\mathbb{Z}/s\mathbb{Z}$ -tuple $v = (v^{(j)} \mid j \in \mathbb{Z}/s\mathbb{Z})$ of partitions, there is a partition with s -core σ and s -quotient v .

Quotients of partitions will prove useful below. Two important properties are given in the following lemma.

Lemma 2.5. Suppose $\lambda \in \mathcal{P}$ and $r, s \in \mathbb{N}$.

1. $\text{wt}_{rs} \lambda = \sum_{j \in \mathbb{Z}/s\mathbb{Z}} \text{wt}_r \lambda^{(j)}$; in particular, λ is an rs -core if and only if each component of $\text{quo}_s \lambda$ is an r -core.
2. For each $j \in \mathbb{Z}/s\mathbb{Z}$, we have $(\text{cor}_{rs} \lambda)^{(j)} = \text{cor}_r(\lambda^{(j)})$.

Proof. Both statements follow from Lemma 2.2: removing a rim rs -hook from λ corresponds to reducing some element of $\mathcal{B}^\lambda$ by rs ; this in turn corresponds to reducing an element of the beta-set of one component of $\text{quo}_s \lambda$ by r . $\square$

3 The affine symmetric group

In this section we assume $s \geq 2$. Let W_s denote the Coxeter group of type $\tilde{A}_{s-1}$; this has generators w_i for $i \in \mathbb{Z}/s\mathbb{Z}$, and relations

$$\begin{aligned} w_i^2 &= 1 && \text{for each } i, \\ w_i w_j &= w_j w_i && \text{if } j \neq i \pm 1, \\ w_i w_j w_i &= w_j w_i w_j && \text{if } j = i + 1 \neq i - 1. \end{aligned}$$

Now suppose t is another positive integer which is prime to s , and set $sot = \frac{1}{2}(s-1)(t-1)$. We define the *level t action* of W_s on $\mathbb{Z}$ by

$$w_i n = \begin{cases} n+t & (n \in (i-1)t - sot) \\ n-t & (n \in it - sot) \\ n & (\text{otherwise}) \end{cases} \quad \text{for each } i \in \mathbb{Z}/s\mathbb{Z}.$$

(Recall from Section 2 our convention: if $i = a + s\mathbb{Z}$, then it means $at + s\mathbb{Z}$.)

This action is faithful for every t ; in fact, the image of the level 1 action is often taken as a concrete definition of W_s .

It is easy to see that if $\mathcal{B}$ is a subset of $\mathbb{Z}$ which is bounded above and whose complement in $\mathbb{Z}$ is bounded below, then the same is true of $w\mathcal{B}$ for any $w \in W_s$; moreover, the number of non-negative elements minus the number of negative non-elements is the same in $\mathcal{B}$ and $w\mathcal{B}$. Hence we have an action of W_s on $\mathcal{P}$, given by

$$\mathcal{B}^{w_i \lambda} = w_i \mathcal{B}^\lambda.$$

We refer to this action as the *level t action* of W_s on $\mathcal{P}$, and we refer to an orbit under this action as a *level t orbit*.

Remark. The level 1 action of W_s on $\mathcal{P}$ is well known, and was first addressed by Lascoux [L]. In [F], the author introduced the level t action of W_s on $\mathbb{Z}$ and on the set of s -cores (which is a union of orbits for the action on $\mathcal{P}$), but with a slight difference from the definition above, in that the terms $-sot$ do not appear in the definition in [F]. This makes little practical difference, since the two versions of the action are equivalent to each other via a diagram automorphism of W_s . However, we prefer the version above in this paper; although slightly more complicated to define, it turns out to be more helpful, as we shall see in Proposition 6.5. It also respects conjugation of partitions, in the sense that $(w_i \lambda)' = w_{-i}(\lambda')$ for any i and any λ , where λ' denotes the conjugate (i.e. transpose) partition.

Example. Take $s = 2$, $t = 3$, so that $sot = 1$. Let $\lambda = (4, 1)$, which has beta-set $\mathcal{B}^\lambda = \{3, -1, -3, -4, \dots\}$. To calculate $w_{1+2\mathbb{Z}}\lambda$, we add 3 to every odd element of this beta-set and subtract 3 from every even element; we obtain $\{6, 2, 0, -2, -4, -6, -7, -8, \dots\} = \mathcal{B}^{(7, 4, 3, 2, 1)}$, so $w_{1+2\mathbb{Z}}\lambda = (7, 4, 3, 2, 1)$. In a similar way, we calculate $w_{2\mathbb{Z}}\lambda = (1^2)$. Note that $w_{1+2\mathbb{Z}}\lambda$ is obtained by adding four rim 3-hooks to λ , while $w_{2\mathbb{Z}}\lambda$ is obtained by removing a rim 3-hook. In general, the effect of w_i acting on a partition λ at level t can be described in terms of simultaneously adding and removing rim t -hooks; we leave the reader to work out the details.

We now give some invariants of the level t action of W_s . In Proposition 3.3 we shall use these to give an explicit criterion for when two partitions lie in the same level t orbit.

Lemma 3.1. Suppose $\lambda \in \mathcal{P}$ and $w \in W_s$, and define $w\lambda$ using the level t action. Then:

1. $\text{cor}_t(w\lambda) = \text{cor}_t\lambda$;
2. $\text{quo}_s(w\lambda)$ is the same as $\text{quo}_s\lambda$ with the components re-ordered;
3. $\text{wt}_s(w\lambda) = \text{wt}_s\lambda$;
4. $\text{cor}_s(w\lambda) = w(\text{cor}_s\lambda)$.

Proof. Since the relations occurring in all four parts are transitive, we may assume $w = w_i$ for $i \in \mathbb{Z}/s\mathbb{Z}$. Write $j = it - sot$.

1. We obtain $\mathcal{B}^{w_i\lambda}$ from $\mathcal{B}^\lambda$ by adding t to every element of $\mathcal{B}^\lambda \cap (j-t)$ and subtracting t from every element of $\mathcal{B}^\lambda \cap j$. But we may as well ignore those pairs of integers $b, b-t$ which both lie in $\mathcal{B}^\lambda$ and for which $b \in j$. Since all but finitely many elements $b \in \mathcal{B}^\lambda$ satisfy $b-t, b+t \in \mathcal{B}^\lambda$, this means that we can get from $\mathcal{B}^\lambda$ to $\mathcal{B}^{w_i\lambda}$ in a finite sequence of moves, where each move is either increasing an element by t or decreasing an element by t . In other words, we can get from λ to $w_i\lambda$ by adding and removing finitely many t -hooks. So λ and $w_i\lambda$ have the same t -core.
2. For any $l \in \mathbb{Z}/s\mathbb{Z} \setminus \{j, j-t\}$, we have $\mathcal{B}^{w_i\lambda} \cap l = \mathcal{B}^\lambda \cap l$, so that $(w_i\lambda)^{(l)} = \lambda^{(l)}$. On the other hand, we have

$$\begin{aligned}\mathcal{B}^{w_i\lambda} \cap (j-t) &= \{b-t \mid b \in \mathcal{B}^\lambda \cap j\} \\ \mathcal{B}^{w_i\lambda} \cap j &= \{b+t \mid b \in \mathcal{B}^\lambda \cap (j-t)\},\end{aligned}$$

which gives $(w_i\lambda)^{(j-t)} = \lambda^{(j)}$ and $(w_i\lambda)^{(j)} = \lambda^{(j-t)}$.

3. This follows from (2) and Lemma 2.5(1) (taking $r = 1$ in that lemma).
4. From the definition of the s -quotient, the largest element of $\mathcal{B}^\lambda \cap l$ is $s\lambda_1^{(l)} + \Gamma_l\lambda - s$. Taking $l = j$ and applying w_i , the largest element of $\mathcal{B}^{(w_i\lambda)} \cap (j-t)$ is $s\lambda_1^{(j)} + \Gamma_j\lambda - s - t$, and this must equal $s(w_i\lambda)_1^{(j-t)} + \Gamma_{j-t}(w_i\lambda) - s$. From the proof of part (2) we have $(w_i\lambda)^{(j-t)} = \lambda^{(j)}$, and we deduce that $\Gamma_{j-t}(w_i\lambda) = \Gamma_j\lambda - t$. Similarly $\Gamma_j(w_i\lambda) = \Gamma_{j-t}\lambda + t$, and obviously $\Gamma_l(w_i\lambda) = \Gamma_l\lambda$ for $l \neq j, j-t$.

In particular, the s -set of $w_i\lambda$ is determined by the s -set of λ . Since λ and $\text{cor}_s\lambda$ have the same s -set, so do $w_i\lambda$ and $w_i(\text{cor}_s\lambda)$. By (3) $w_i(\text{cor}_s\lambda)$ is an s -core, and since there is a unique s -core with a given s -set, we therefore have $w_i(\text{cor}_s\lambda) = \text{cor}_s(w_i\lambda)$. $\square$

Next we give a criterion for determining when two partitions lie in the same level t orbit; to do this, we shall need to cite a result from [F] which gives a condition for two s -cores to lie in the same level t orbit. (Note that although a slightly different level t action is used in that paper, it differs from our action only by an automorphism of W_s , and so the orbits for the two actions are the same.)

Proposition 3.2 [F, Proposition 4.1 & Corollary 4.5]. *Suppose λ and μ are s -cores, and that the multisets*

$$[\Gamma_i\lambda + t\mathbb{Z} \mid i \in \mathbb{Z}/s\mathbb{Z}], \quad [\Gamma_i\mu + t\mathbb{Z} \mid i \in \mathbb{Z}/s\mathbb{Z}]$$

are equal. Then $\text{cor}_t\lambda = \text{cor}_t\mu$, and λ and μ lie in the same level t orbit.

For our more general result, we make a definition which combines the s -quotient of a partition with its s -set modulo t .

Suppose $\lambda \in \mathcal{P}$, with s -set $\{\Gamma_i\lambda \mid i \in \mathbb{Z}/s\mathbb{Z}\}$ and s -quotient $(\lambda^{(i)} \mid i \in \mathbb{Z}/s\mathbb{Z})$. We define the t -weighted s -quotient of λ to be the multiset

$$Q_s^t\lambda = [(\Gamma_i\lambda + t\mathbb{Z}, \lambda^{(i)}) \mid i \in \mathbb{Z}/s\mathbb{Z}]$$

of elements of $\mathbb{Z}/t\mathbb{Z} \times \mathcal{P}$.

Example. Take $s = 4$ and $\lambda = (10, 8, 7, 5, 2, 1^4)$. Then we have

$$\begin{aligned}\Gamma_{4\mathbb{Z}}\lambda &= 0, & \Gamma_{1+4\mathbb{Z}}\lambda &= 9, & \Gamma_{2+4\mathbb{Z}}\lambda &= 2, & \Gamma_{3+4\mathbb{Z}}\lambda &= -5, \\ \lambda^{(4\mathbb{Z})} &= (2), & \lambda^{(1+4\mathbb{Z})} &= (1), & \lambda^{(2+4\mathbb{Z})} &= (2), & \lambda^{(3+4\mathbb{Z})} &= (1).\end{aligned}$$

So the 7-weighted 4-quotient of λ is

$$[(7\mathbb{Z}, (2)), (2+7\mathbb{Z}, (1)), (2+7\mathbb{Z}, (2)), (2+7\mathbb{Z}, (1))].$$

Proposition 3.3. *Suppose $\lambda, \mu \in \mathcal{P}$, and s and t are coprime positive integers. Then λ and μ lie in the same level t orbit of W_s if and only if they have the same t -weighted s -quotient.*

Proof. For the ‘only if’ part, we may assume that $\mu = w_i \lambda$ for $i \in \mathbb{Z}/s\mathbb{Z}$, and we write $j = it - sot$. From the proof of Lemma 3.1 we have

$$\mu^{(l)} = \begin{cases} \lambda^{(j)} & (l = j - t) \\ \lambda^{(j-t)} & (l = j) \\ \lambda^{(l)} & (\text{otherwise}), \end{cases} \quad \Gamma_l \mu = \begin{cases} \Gamma_j \lambda - t & (l = j - t) \\ \Gamma_{j-t} \lambda + t & (l = j) \\ \Gamma_j \lambda & (\text{otherwise}). \end{cases}$$

So λ and μ have the same t -weighted s -quotient.

For the ‘if’ part, assume that λ and μ have the same t -weighted s -quotient. Since $\text{cor}_s \lambda$ and λ have the same s -set (by definition) and the s -quotient of $\text{cor}_s \lambda$ has all components equal to $\emptyset$, we see that $\text{cor}_s \lambda$ and $\text{cor}_s \mu$ also have the same t -weighted s -quotient. So by Proposition 3.2 $\text{cor}_s \lambda$ and $\text{cor}_s \mu$ have the same t -core ξ , and lie in the same level t orbit as ξ ; that is, there are $w, x \in W_s$ such that $w \text{cor}_s \lambda = \xi = x \text{cor}_s \mu$. By Lemma 3.1(4) we have $\text{cor}_s(w\lambda) = \xi = \text{cor}_s(x\mu)$, so by replacing λ and μ with $w\lambda$ and $x\mu$ (and using the ‘only if’ part above), we may assume that λ and μ both have s -core ξ , with ξ being an (s, t) -core. So we have $\Gamma_i \lambda = \Gamma_i \mu = \Gamma_i \xi$ for every i . Now given any $r \in \mathbb{Z}/t\mathbb{Z}$, let

$$\mathcal{X}_r = \{i \in \mathbb{Z}/s\mathbb{Z} \mid \Gamma_i \xi \in r\}.$$

Then the fact that λ and μ have the same t -weighted s -quotient simply means that the multisets

$$[\lambda^{(i)} \mid i \in \mathcal{X}_r] \quad \text{and} \quad [\mu^{(i)} \mid i \in \mathcal{X}_r]$$

are equal for each r .

Now since ξ is an (s, t) -core, it follows from the proof of [F, Proposition 4.1] that the $\Gamma_i \xi$ lying in a given congruence class modulo t form an arithmetic progression with common difference t . So given r , there are $a \in \mathbb{Z}/s\mathbb{Z}$ and $m \in \mathbb{N}_0$ such that

$$\mathcal{X}_r = a + t, a + 2t, \dots, a + mt$$

and there is an integer c such that

$$\Gamma_{a+bt} \xi = c + bt$$

for $b = 1, \dots, m$. Now given any $1 < b \leq m$, let $i \in \mathbb{Z}/s\mathbb{Z}$ be such that $it - sot = a + bt$. Then (from the first paragraph of this proof) the effect of applying w_i to λ is to fix all the $\Gamma_i \lambda$, and to interchange $\lambda^{(a+(b-1)t)}$ and $\lambda^{(a+bt)}$, fixing all other parts of the s -quotient of λ . So by applying elements of W_s , we can re-order $\lambda^{(a+t)}, \dots, \lambda^{(a+mt)}$ arbitrarily without affecting the s -set of λ or the rest of $\text{quo}_s \lambda$. By doing this for every r , we can apply elements of W_s to transform λ into μ . $\square$

4 Generalised cores

Equipped with our definitions and basic results concerning the action of W_s , we now come to our main object of study.

4.1 Olsson’s theorem

We begin by stating a theorem of Olsson which is the starting point for the work in this paper; this says that the t -core of an s -core is again an s -core, but we phrase this slightly differently.

Theorem 4.1 [O, Theorem 1]. Suppose s, t are coprime positive integers and $\lambda \in \mathcal{P}$. Then

$$\text{wt}_s \lambda = 0 \implies \text{wt}_s(\text{cor}_t \lambda) = 0.$$

Now we can give our first main result, which is a generalisation of Theorem 4.1.

Theorem 4.2. Suppose s, t are coprime positive integers and $\lambda \in \mathcal{P}$. Then

$$\text{wt}_s(\text{cor}_t \lambda) \leq \text{wt}_s \lambda.$$

Proof. We proceed by induction on $\text{wt}_t \lambda$, with the case where λ is a t -core being trivial. Assuming λ is not a t -core, we can find $b \in \mathcal{B}^\lambda$ such that $b - t \notin \mathcal{B}^\lambda$. We define a new partition ν by replacing a with $a - t$ for every $a \in \mathcal{B}^\lambda$ such that $a - t \notin \mathcal{B}^\lambda$ and $a \equiv b \pmod{s}$. Then $\text{cor}_t \nu = \text{cor}_t \lambda$ and $\text{wt}_t \nu < \text{wt}_t \lambda$, so by induction it suffices to show that $\text{wt}_s \nu \leq \text{wt}_s \lambda$.

We use Corollary 2.3 to compare $\text{wt}_s \lambda$ and $\text{wt}_s \nu$. Call a pair $(x, l) \in \mathbb{Z} \times \mathbb{N}$ a *weight pair* for λ if $x \in \mathcal{B}^\lambda \not\equiv x - ls$. If $x \not\equiv b, b - t \pmod{s}$, then clearly (x, l) is a weight pair for λ if and only if it is a weight pair for ν . Now suppose $x \equiv b \pmod{s}$, and consider the two pairs (x, l) and $(x - t, l)$. By considering the sixteen possibilities for the set $\mathcal{B}^\lambda \cap \{x, x - ls, x - t, x - ls - t\}$, we can check that among the two pairs (x, l) and $(x - t, l)$, there are at least as many weight pairs for λ as there are for ν ; hence $\text{wt}_s \nu \leq \text{wt}_s \lambda$. $\square$

With any inequality, it is natural to consider the situation where equality occurs. Hence we make the following definition.

Definition. Suppose s, t are positive integers. A partition λ is an $[s:t]$ -core if

$$\text{wt}_s(\text{cor}_t \lambda) = \text{wt}_s \lambda.$$

We write $C_{s:t}$ for the set of $[s:t]$ -cores.

Trivially $C_{s:t}$ includes all t -cores, and it follows from Theorem 4.1 that $C_{s:t}$ contains all s -cores. However, in general $C_{s:t}$ will include partitions which are neither s - nor t -cores; for example, $(4, 1)$ is a $[2:3]$ -core.

Note that in the above definition we do not assume that s and t are coprime. However, for the rest of this paper we do make this assumption. Given this, it is easy to determine whether a partition is an $[s:t]$ -core from its beta-set.

Proposition 4.3. Suppose $\lambda \in \mathcal{P}$, $r \in \mathbb{Z}$ and s, t are coprime positive integers. Then λ is an $[s:t]$ -core if and only if there do not exist integers d, e, f such that:

- $d \equiv e \pmod{s}$;
- $d \equiv f \pmod{t}$;
- $d, e + f - d \in \mathcal{B}_r^\lambda$;
- $e, f \notin \mathcal{B}_r^\lambda$.

Proof. Since $\mathcal{B}_r^\lambda$ is just a translation of $\mathcal{B}^\lambda$, we may assume $r = 0$. Say that (d, e, f) is a *bad triple* for λ if d, e, f satisfy the conditions in the proposition. First we suppose (d, e, f) is bad, and show that λ is not a t -core. Trivially, we must have either $d > f$ or $e + f - d > e$, either way, we find that there are $x, y \in \mathbb{Z}$ such that $x > y$, $x \equiv y \pmod{t}$ and $x \in \mathcal{B}^\lambda \not\equiv y$. So by Lemma 2.2 λ is not an $(x - y)$ -core, and hence by Corollary 2.4 λ is not a t -core.

So (since every t -core is an $[s:t]$ -core) the proposition is true for λ a t -core. Now we assume λ is not a t -core, and choose $b \in \mathcal{B}^\lambda$ such that $b - t \notin \mathcal{B}^\lambda$. As in the proof of Theorem 4.2, we define a new partition ν by replacing a with $a - t$ for every $a \in \mathcal{B}^\lambda$ such that $a - t \notin \mathcal{B}^\lambda$ and $a \equiv b \pmod{s}$. Now by induction it suffices to show that either:

- $\text{wt}_s \nu = \text{wt}_s \lambda$, and there is a bad triple for ν if and only if there is a bad triple for λ ; or
- $\text{wt}_s \nu < \text{wt}_s \lambda$, and there is a bad triple for λ .

Suppose first that there is no $a \equiv b \pmod{s}$ for which $a \notin \mathcal{B}^\lambda \ni a - t$. Then we have $\nu = w_i \lambda$, where $i \in \mathbb{Z}/s\mathbb{Z}$ is such that $it - s \circ t = b + s\mathbb{Z}$, and w_i is the corresponding generator of W_s acting at level t . So by Lemma 3.1(3) we have $\text{wt}_s \nu = \text{wt}_s \lambda$; and (d, e, f) is a bad triple for λ if and only if $(w_i d, w_i e, w_i f)$ is a bad triple for ν .

Next suppose there is an $a \equiv b \pmod{t}$ for which $a \notin \mathcal{B}^\lambda \ni a - t$. Then $(b, a, b - t)$ is a bad triple for λ , and it remains to show that $\text{wt}_s \nu < \text{wt}_s \lambda$. As in the proof of Theorem 4.2, we consider weight pairs (x, l) . Taking $x = \max\{a, b\}$ and $l = \frac{|a-b|}{t}$, we find that exactly one of (x, l) and $(x - t, l)$ is a weight pair for λ , while neither of them is a weight pair for ν . Using the rest of the argument in the proof of Theorem 4.2, we have $\text{wt}_s \nu < \text{wt}_s \lambda$. $\square$

Corollary 4.4. *If s, t are coprime positive integers, then $C_{s:t} = C_{t:s}$.*

Proof. The condition in Proposition 4.3 is symmetric in s and t . $\square$

This last result (which is very surprising given the definition of $C_{s:t}$) suggests that the set $C_{s:t}$ is worth studying. Our intuition is that $C_{s:t}$, rather than $C_s \cup C_t$, is the ‘correct’ counterpart to $C_s \cap C_t$ (just as one studies the sum of two subspaces of a vector space rather than their union).

We now go on to examine the structure of $C_{s:t}$ with respect to the level t action of W_s .

4.2 Affine symmetric group actions on $C_{s:t}$

We continue to assume that s and t are coprime. Recall that the group W_s acts at level t on $\mathcal{P}$; symmetrically, W_t acts at level s on $\mathcal{P}$. These actions commute (since the actions on $\mathbb{Z}$ commute), and so we have an action of $W_s \times W_t$ on $\mathcal{P}$.

Our first result is that $C_{s:t}$ is a union of orbits for the level t action of W_s .

Proposition 4.5. *Suppose s and t are coprime positive integers. Given $\lambda \in \mathcal{P}$ and $w \in W_s$, define $w\lambda$ using the level t action. If $\lambda \in C_{s:t}$, then $w\lambda \in C_{s:t}$.*

Proof. Using Lemma 3.1(1,3) and the fact that $\lambda \in C_{s:t}$, we have

$$\text{wt}_s(\text{cor}_t(w\lambda)) = \text{wt}_s(\text{cor}_t \lambda) = \text{wt}_s \lambda = \text{wt}_s(w\lambda). \quad \square$$

Interchanging s and t and appealing to Corollary 4.4, we see that $C_{s:t}$ is also a union of orbits for the level s action of W_t . Hence $C_{s:t}$ is a union of orbits for the action of $W_s \times W_t$.

Now we consider these orbits in more detail. We begin by considering just the level t action of W_s .

Proposition 4.6. *Suppose $\lambda \in \mathcal{P}$, and let $\mathcal{O}$ be the orbit containing λ under the level t action of W_s . Then the following are equivalent.*

1. λ is an $[s:t]$ -core.
2. $\mathcal{O}$ contains a t -core.
3. $\mathcal{O}$ contains $\text{cor}_t \lambda$.

Proof. Since every t -core is an $[s:t]$ -core, Proposition 4.5 shows that if O contains a t -core, then λ is an $[s:t]$ -core. So (2) implies (1). Trivially (3) implies (2), so it remains to show that (1) implies (3).

So suppose λ is an $[s:t]$ -core. We can assume that λ is not a t -core, so there is $b \in \mathcal{B}^\lambda$ such that $b - t \notin \mathcal{B}^\lambda$. From the proof of Proposition 4.3, there is no $a \equiv b \pmod{s}$ for which $a - t \in \mathcal{B}^\lambda \not\equiv a$, and if we take $i \in \mathbb{Z}/s\mathbb{Z}$ such that $it - sot = b + s\mathbb{Z}$, then the partition $\nu = w_i\lambda$ satisfies $\text{cor}_t\nu = \text{cor}_t\lambda$ and $\text{wt}_t\nu < \text{wt}_t\lambda$. By Lemma 3.1 ν is also an $[s:t]$ -core, and by induction the orbit containing ν contains $\text{cor}_t\nu$. $\square$

Now we can introduce a connection between $[s:t]$ -cores and (s, t) -cores.

Corollary 4.7. *Let O be an orbit of $W_s \times W_t$ consisting of $[s:t]$ -cores. Then O contains exactly one (s, t) -core.*

Proof. Let λ be a partition in O . Then by Proposition 4.6 $\text{cor}_t\lambda \in O$, and by the same result with s and t interchanged, $\nu := \text{cor}_s(\text{cor}_t\lambda)$ lies in O . By Theorem 4.1 ν is an (s, t) -core.

Now suppose that there is another (s, t) -core in O . We can write this as $xw\nu$, with $w \in W_s$ and $x \in W_t$. By Lemma 3.1(3) we have

$$\text{wt}_t w\nu = \text{wt}_t xw\nu = 0,$$

so

$$w\nu = \text{cor}_t w\nu = \text{cor}_t\nu = \nu,$$

using Lemma 3.1(1). Similarly $x\nu = \nu$, and so $xw\nu = x\nu = \nu$. $\square$

Remarks.

1. From Proposition 4.6 and Corollary 4.7 we see that two $[s:t]$ -cores λ and μ lie in the same orbit of $W_s \times W_t$ if and only if $\text{cor}_s(\text{cor}_t\lambda) = \text{cor}_s(\text{cor}_t\mu)$. But it does not seem to be easy to tell when two arbitrary partitions lie in the same orbit; we would like an analogue of Proposition 3.3, but the author has so far been unable to find one.
2. Corollary 4.7 shows that the number of orbits of $W_s \times W_t$ consisting of $[s:t]$ -cores equals the number of (s, t) -cores; by Anderson's theorem [A, Theorems 1 & 3] this is exactly $\frac{1}{s+t} \binom{s+t}{s}$.

4.3 Further properties of $[s:t]$ -cores

Now we give two more properties of $[s:t]$ -cores which will be useful later.

Lemma 4.8. *Suppose λ is an $[s:t]$ -core. Then:*

1. λ is an st -core;
2. $\text{cor}_s(\text{cor}_t\lambda) = \text{cor}_t(\text{cor}_s\lambda)$.

Proof.

1. If λ is not an st -core, let ν be a partition obtained by removing an st -hook. Then (from Lemma 2.2) we can obtain ν from λ either by successively removing s t -hooks, or by successively removing t s -hooks. Hence we see that

$$\text{cor}_t\nu = \text{cor}_t\lambda, \quad \text{wt}_s\nu = \text{wt}_s\lambda - t.$$

So

$$\text{wt}_s(\text{cor}_t\lambda) = \text{wt}_s(\text{cor}_t\nu) \leq \text{wt}_s\nu < \text{wt}_s\lambda,$$

so λ is not an $[s:t]$ -core.

2. By Theorem 4.1 both $\text{cor}_s(\text{cor}_t\lambda)$ and $\text{cor}_t(\text{cor}_s\lambda)$ are (s, t) -cores, and from Proposition 4.6 they both lie in the same orbit as λ under the action of $W_s \times W_t$. So the result follows from Corollary 4.7. $\square$

Note that the properties in Lemma 4.8 do not characterise $[s:t]$ -cores; for example, the rectangular partition (s^t) satisfies both properties, but if $s, t > 1$ it is not an $[s:t]$ -core.

5 The sum of an s -core and a t -core

Next we consider the possibility of constructing an $[s:t]$ -core with specified s -core and t -core. We continue to assume that s and t are coprime.

5.1 Partitions with a given s -core and t -core

It is a simple exercise using the Chinese Remainder Theorem to show that given an s -core σ and a t -core τ , there are infinitely many partitions λ with $\text{cor}_s\lambda = \sigma$ and $\text{cor}_t\lambda = \tau$. But if we insist that λ be an $[s:t]$ -core, then by Lemma 4.8(2) we need $\text{cor}_t\sigma = \text{cor}_s\tau$. In this case, we have the following result.

Proposition 5.1. *Suppose $\sigma \in C_s$ and $\tau \in C_t$, and that $\text{cor}_t\sigma = \text{cor}_s\tau$. Then there is a unique $[s:t]$ -core λ such that $\text{cor}_s\lambda = \sigma$ and $\text{cor}_t\lambda = \tau$. Moreover, $|\lambda| = |\sigma| + |\tau| - |\text{cor}_s\tau|$, and λ is the unique smallest partition with s -core σ and t -core τ .*

Proof. In this proof we use Lemma 3.1 without comment. Let $\xi = \text{cor}_t\sigma$, and consider the action of $W_s \times W_t$ on $\mathcal{P}$. By Proposition 4.6 we can find $w \in W_s$ and $x \in W_t$ such that $w\xi = \sigma$ and $x\xi = \tau$, and we let $\lambda = w\tau$. Then λ is an $[s:t]$ -core, since it lies in the same orbit as τ . Moreover, $\text{cor}_t\lambda = \text{cor}_t\tau = \tau$, and

$$\text{cor}_s\lambda = \text{cor}_s(wx\xi) = \text{cor}_s(xw\xi) = \text{cor}_s(x\sigma) = \text{cor}_s\sigma = \sigma.$$

Furthermore,

$$\begin{aligned} |\lambda| &= |\text{cor}_s\lambda| + s\text{wt}_s\lambda \\ &= |\sigma| + s\text{wt}_s(w\tau) \\ &= |\sigma| + s\text{wt}_s\tau \\ &= |\sigma| + |\tau| - |\text{cor}_s\tau|. \end{aligned}$$

Now suppose μ is a partition other than λ with s -core σ and t -core τ , and let w, x be as above. Then we have $\text{cor}_t(w^{-1}\mu) = \text{cor}_t\mu = \tau$, but $w^{-1}\mu \neq w^{-1}\lambda = \tau$. So $|w^{-1}\mu| > |\tau|$. Hence

$$\text{wt}_s\mu = \text{wt}_s(w^{-1}\mu) = \frac{|w^{-1}\mu| - |\xi|}{s} > \frac{|\tau| - |\xi|}{s} = \text{wt}_s\tau;$$

so μ is not an $[s:t]$ -core. Furthermore, we see that $\text{wt}_s\mu > \text{wt}_s\lambda$, so $|\mu| > |\lambda|$, and hence $|\lambda|$ is the unique smallest partition with s -core σ and t -core τ . $\square$

We write $\sigma \boxplus \tau$ for the partition λ given by Proposition 5.1.

Remark. We have shown that $\sigma \boxplus \tau$ is the smallest partition with s -core σ and t -core τ in terms of size; it is reasonable to ask whether $\sigma \boxplus \tau$ is smallest in the sense that $\sigma \boxplus \tau \subseteq \mu$ for any partition μ with s -core σ and t -core τ . In fact, this is false: taking $(s, t) = (2, 3)$, we have $(2, 1) \boxplus (2) = (2, 1^3)$; but the partition $(8, 3) \not\supseteq (2, 1^3)$ also has 2-core $(2, 1)$ and 3-core (2) .

Now we derive a corollary which yields another characterisation of $[s:t]$ -cores.

Corollary 5.2. *Suppose s and t are coprime positive integers not both equal to 1 and λ is a partition of n . Then λ is an $[s:t]$ -core if and only if there is no other partition of n with the same s -core and t -core as λ .*

Proof. Suppose λ is an $[s:t]$ -core, and let $\sigma = \text{cor}_s \lambda$ and $\tau = \text{cor}_t \lambda$. Then $\lambda = \sigma \boxplus \tau$, since by Proposition 5.1 this is the unique $[s:t]$ -core with s -core σ and t -core τ . So by the last part of Proposition 5.1, λ is the unique partition of its size with s -core σ and t -core τ .

Conversely, suppose λ is not an $[s:t]$ -core. Then by Proposition 4.3 we can find integers d, e, f such that $d \equiv e \pmod{s}$, $d \equiv f \pmod{t}$ and $d, e + f - d$ are elements of $\mathcal{B}^\lambda$ while e, f are not.

First assume d, e, f and $e + f - d$ are distinct. Define a new partition μ by

$$\mathcal{B}^\mu = \mathcal{B}^\lambda \setminus \{d, e + f - d\} \cup \{e, f\}.$$

Then by Lemma 2.2 μ can be obtained from λ by removing a $|d - e|$ -hook and adding a $|d - e|$ -hook. So $|\mu| = |\lambda|$, and μ has the same s -core as λ , since s divides $d - e$. Alternatively, μ can be obtained from λ by removing a $|d - f|$ -hook and adding a $|d - f|$ -hook, so μ also has the same t -core as λ . So λ is not the unique partition of n with s -core σ and t -core τ .

Now assume d, e, f and $e + f - d$ are not distinct. Then these integers are congruent modulo st , and hence by Lemma 2.2 λ is not an st -core. Since $st > 1$ by assumption, it is easy to find another partition μ of n with the same st -core as λ , and by Corollary 2.4 μ has the same s -core and t -core as λ . $\square$

5.2 Constructing $\sigma \boxplus \tau$

Suppose σ is an s -core and τ is a t -core, with $\text{cor}_t \sigma = \text{cor}_s \tau$. In this section we give a method for constructing the partition $\sigma \boxplus \tau$. Of course, this can be done as in the proof of Proposition 5.1: find $w \in W_s$ such that $\sigma = w \text{cor}_t \sigma$, and then compute $w\tau$. But this is a laborious process; we present here a much quicker method using weighted quotients.

Recall the s -set $\{\Gamma_i \lambda \mid i \in \mathbb{Z}/s\mathbb{Z}\}$ and the s -quotient $(\lambda^{(i)} \mid i \in \mathbb{Z}/s\mathbb{Z})$ of a partition λ .

Lemma 5.3. *Suppose $j, k \in \mathbb{Z}/s\mathbb{Z}$ and τ is a t -core with $\Gamma_j \tau \equiv \Gamma_k \tau \pmod{t}$. Then $\tau^{(j)} = \tau^{(k)}$.*

Proof. Without loss of generality suppose $\Gamma_k \tau > \Gamma_j \tau$. The elements of $\mathcal{B}^\tau \cap j$ are the integers $s(\tau_i^{(j)} - i) + \Gamma_j \tau$, for $i \geq 1$, and similarly for $\mathcal{B}^\tau \cap k$. If $\tau^{(j)} \neq \tau^{(k)}$, then by Lemma 2.1 $\mathcal{B}^{\tau^{(k)}} \not\subseteq \mathcal{B}^{\tau^{(j)}}$, so there is some i such that $\tau_i^{(k)} - i \notin \mathcal{B}^{\tau^{(j)}}$. This means that $s(\tau_i^{(k)} - i) + \Gamma_j \tau \notin \mathcal{B}^\tau$; on the other hand, $(\tau_i^{(k)} - i)s + \Gamma_k \tau \in \mathcal{B}^\tau$, so by Lemma 2.2 τ is not an $(\Gamma_k \tau - \Gamma_j \tau)$ -core, and hence by Corollary 2.4 τ is not a t -core. Contradiction. $\square$

Now recall the t -weighted s -quotient $(\Gamma_i \lambda + t\mathbb{Z}, \lambda^{(i)}) \mid i \in \mathbb{Z}/s\mathbb{Z}$ of λ .

Proposition 5.4. *Suppose σ is an s -core and τ is a t -core and that $\text{cor}_t \sigma = \text{cor}_s \tau$. Then there is a unique partition with s -core σ and with the same t -weighted s -quotient as τ .*

Proof. Let $\xi = \text{cor}_t \sigma$. Then by Lemma 5.3 and Proposition 4.6 ξ has the same t -weighted s -quotient as σ , i.e. there is a bijection $\phi : \mathbb{Z}/s\mathbb{Z} \rightarrow \mathbb{Z}/s\mathbb{Z}$ such that $\Gamma_i \sigma \equiv \Gamma_{\phi(i)} \xi \pmod{t}$ for each i . Since $\xi = \text{cor}_s \tau$, it is therefore possible to construct a partition λ as stated: we just take the partition λ with s -core σ , and with $\lambda^{(i)} = \tau^{(\phi(i))}$ for each i .

By Lemma 5.3 we have $\tau^{(j)} = \tau^{(k)}$ whenever $\Gamma_j \tau \equiv \Gamma_k \tau \pmod{t}$, i.e. whenever $\Gamma_{\phi^{-1}(j)} \sigma \equiv \Gamma_{\phi^{-1}(k)} \sigma \pmod{t}$, so we have no choice in the construction of λ , and λ is unique. $\square$

Proposition 5.5. *Suppose σ is an s -core and τ is a t -core, with $\text{cor}_t \sigma = \text{cor}_s \tau$, and let λ be the partition with s -core σ and with the same t -weighted s -quotient as τ . Then $\lambda = \sigma \boxplus \tau$.*

Proof. By Proposition 3.3, λ and τ lie in the same level t orbit of W_s . Hence by Proposition 4.6 λ is an $[s:t]$ -core and $\text{cor}_t \lambda = \tau$. Since $\text{cor}_s \lambda = \sigma$ by construction, we have $\lambda = \sigma \boxplus \tau$. $\square$

Example. Take $s = 3$, $t = 5$, $\sigma = (7, 5, 4^2, 3^2, 2^2, 1^2)$ and $\tau = (5, 2^4, 1^4)$. Then $\text{cor}_5 \sigma = \text{cor}_3 \tau = (2)$. We have

$$(\Gamma_{3\mathbb{Z}}\tau, \Gamma_{1+3\mathbb{Z}}\tau, \Gamma_{2+3\mathbb{Z}}\tau) = (0, 4, -1), \quad (\tau^{(3\mathbb{Z})}, \tau^{(1+3\mathbb{Z})}, \tau^{(2+3\mathbb{Z})}) = ((1^3), (1), (1)),$$

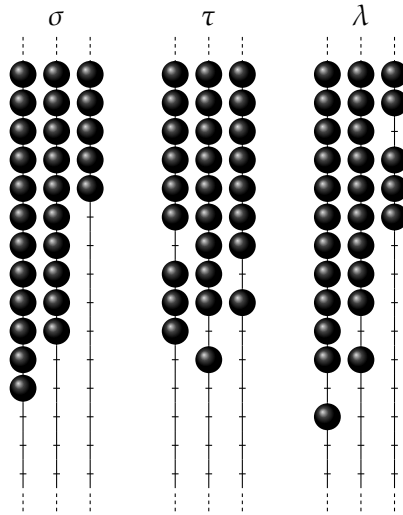
so that the 5-weighted 3-quotient of τ is

$$[(5\mathbb{Z}, (1^3)), (4 + 5\mathbb{Z}, (1)), (4 + 5\mathbb{Z}, (1))].$$

On the other hand,

$$(\Gamma_{3\mathbb{Z}}\sigma, \Gamma_{1+3\mathbb{Z}}\sigma, \Gamma_{2+3\mathbb{Z}}\sigma) = (9, 4, -10);$$

so if λ has 3-core σ (and hence has s -set $\{9, 4, -10\}$), then the only way λ can have the same 5-weighted 3-quotient as τ is if $(\lambda^{(3\mathbb{Z})}, \lambda^{(1+3\mathbb{Z})}, \lambda^{(2+3\mathbb{Z})}) = ((1), (1), (1^3))$. This gives $\lambda = (10, 6^2, 4, 3^2, 2^2, 1^{11})$. The 3-runner abacus displays of σ , τ and λ are as follows.



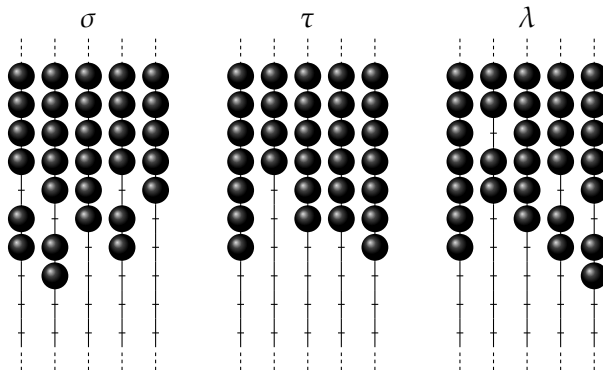
For further illustration, we take the same example with s and t interchanged. The 3-weighted 5-quotient of σ is

$$[(3\mathbb{Z}, (1^2)), (3\mathbb{Z}, (1^2)), (2 + 3\mathbb{Z}, \emptyset), (3\mathbb{Z}, (1^2)), (2 + 3\mathbb{Z}, \emptyset)].$$

On the other hand,

$$(\Gamma_{5\mathbb{Z}}\tau, \Gamma_{1+5\mathbb{Z}}\tau, \Gamma_{2+5\mathbb{Z}}\tau, \Gamma_{3+5\mathbb{Z}}\tau, \Gamma_{4+5\mathbb{Z}}\tau) = (5, -9, 2, 3, 9);$$

so we must take $(\lambda^{(5\mathbb{Z})}, \lambda^{(1+5\mathbb{Z})}, \lambda^{(2+5\mathbb{Z})}, \lambda^{(3+5\mathbb{Z})}, \lambda^{(4+5\mathbb{Z})}) = (\emptyset, (1^2), \emptyset, (1^2), (1^2))$, again yielding $\lambda = (10, 6^2, 4, 3^2, 2^2, 1^{11})$. The 5-runner abacus displays are as follows.



6 The κ -orbit

In this section we examine one particular orbit in $C_{s:t}$, continuing to assume that s and t are coprime. Under this assumption, there are only finitely many (s, t) -cores, and there is a unique largest such. This partition (which is usually denoted $\kappa_{s,t}$) has been studied before; it is known [OS, Theorem 4.1] that $|\kappa_{s,t}| = \frac{1}{24}(s^2 - 1)(t^2 - 1)$, and also that if λ is any (s, t) -core then $\lambda \subseteq \kappa_{s,t}$ [V, Theorem 2.4], [F, Theorem 5.1]. In this section we will consider the $W_s \times W_t$ -orbit containing $\kappa_{s,t}$. We denote this orbit $C_{s:t}^\kappa$, and refer to it as the κ -orbit of $C_{s:t}$. We will see that $C_{s:t}^\kappa$ is naturally in bijection with $C_s \times C_t$.

To begin with, we explain how to construct $\kappa_{s,t}$. Let $\mathcal{B}_{s,t}$ denote the set of integers which cannot be written as a linear combination of s and t with non-negative integer coefficients; this set can be written as

$$\mathcal{B}_{s,t} = \{at - bs \mid a \in \{0, \dots, s-1\}, b \in \mathbb{N}\}.$$

Then $\mathcal{B}_{s,t}$ is bounded above and its complement in $\mathbb{Z}$ is bounded below, so it is a beta-set of a partition, and this partition is $\kappa_{s,t}$. In fact (recalling the integer $sot = \frac{1}{2}(s-1)(t-1)$ from Section 3) $\mathcal{B}_{s,t} = \mathcal{B}_{sot}^{\kappa_{s,t}}$.

The following statement is proved in [F, §5].

Lemma 6.1. *The s -set of $\kappa_{s,t}$ is*

$$\{-sot, t - sot, 2t - sot, \dots, (s-1)t - sot\}.$$

Note in particular that the elements of the s -set of $\kappa_{s,t}$ are congruent modulo t ; in fact, $\kappa_{s,t}$ is the unique (s, t) -core with this property.

Example. Take $s = 3$ and $t = 4$. Then

$$\begin{aligned} \mathcal{B}_{s,t} &= \{-3, -6, -9, \dots\} \cup \{1, -2, -5, \dots\} \cup \{5, 2, -1, \dots\} \\ &= \{5, 2, 1, -1, -2, -3, \dots\} \\ &= \mathcal{B}_3^{(3,1^2)}, \end{aligned}$$

so $\kappa_{s,t} = (3, 1^2)$. The 3-set of this partition is $\{-3, 1, 5\}$, while its 4-set is $\{-3, 0, 3, 6\}$.

Now we consider the κ -orbit $C_{s:t}^\kappa$. We begin by showing that the s - and t -quotients of partitions in this orbit have a particularly nice form.

Proposition 6.2. *Suppose τ is a t -core such that $\text{cor}_s \tau = \kappa_{s,t}$. Then $\text{quo}_s \tau = (\lambda, \dots, \lambda)$ for some t -core λ .*

Proof. By Lemma 6.1, the elements of the s -set of τ (i.e. the s -set of $\kappa_{s,t}$) are congruent modulo t . By Lemma 5.3, this means that the components of $\text{quo}_s \tau$ are all equal. Furthermore, since τ is a t -core, it is an st -core, by Corollary 2.4, and so by Lemma 2.5(1) each component of $\text{quo}_s \tau$ must be a t -core. $\square$

Remark. Let us say that a partition is s -homogeneous if all the components of its s -core are equal; we have just shown that a t -core whose s -core is $\kappa_{s,t}$ is s -homogeneous. However, the condition $\text{cor}_s \tau = \kappa_{s,t}$ is not necessary for a t -core τ to be s -homogeneous; for example, $\tau = \emptyset$ has s -quotient $(\emptyset, \dots, \emptyset)$. However, one can show that if t is prime, then there are only finitely many s -homogeneous t -cores whose s -core is not $\kappa_{s,t}$.

A consequence of Proposition 6.4 is that the construction of $\sigma \boxplus \tau$ is even simpler when σ is an s -core and τ a t -core in the κ -orbit.

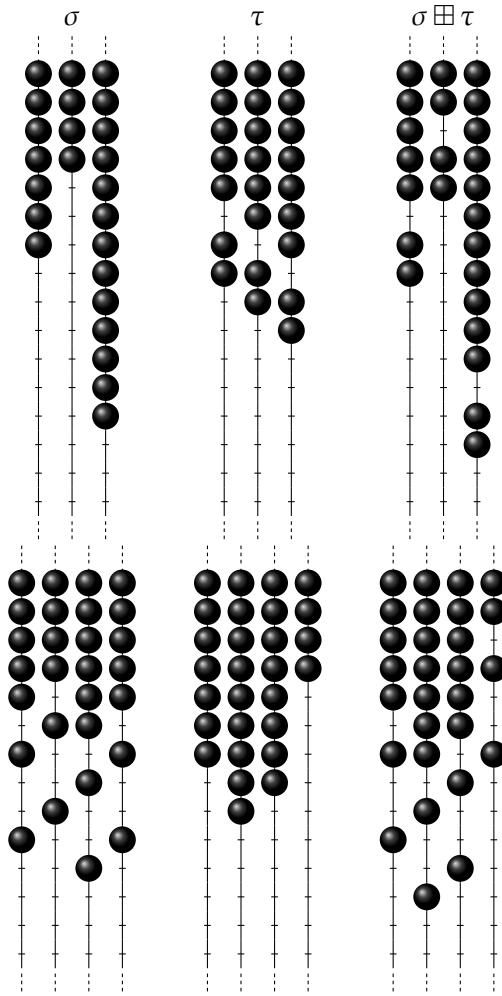
Proposition 6.3. *Suppose σ is an s -core and τ a t -core with $\text{cor}_t \sigma = \text{cor}_s \tau = \kappa_{s,t}$. Then $\sigma \boxplus \tau$ is the partition with s -core σ and the same s -quotient as τ .*

Proof. $\sigma \boxplus \tau$ has s -core σ by definition. By Proposition 4.6 τ and $\sigma \boxplus \tau$ lie in the same level t orbit of W_s , and so by Lemma 3.1(2) have the same s -quotient up to re-ordering. Since τ has s -quotient $(\lambda, \dots, \lambda)$ for some λ , $\sigma \boxplus \tau$ does too. $\square$

Example. Take $(s, t) = (3, 4)$, so that $\kappa_{s,t} = (3, 1^2)$. Then $\sigma = (15, 13, 11, 9, 7, 5, 3, 2^2, 1^2)$ is a 3-core with 4-core $(3, 1^2)$, while $\tau = (6, 4^2, 2^3, 1^3)$ is a 4-core with 3-core $(3, 1^2)$. Hence $\sigma \boxplus \tau$ is the partition with 3-core σ and the same 3-quotient as τ , or equivalently the partition with 4-core τ and the same 4-quotient as σ . We have

$$\text{quo}_4 \sigma = ((3, 1), (3, 1), (3, 1), (3, 1)), \quad \text{quo}_3 \tau = ((1^2), (1^2), (1^2)),$$

and by either route we find that $\sigma \boxplus \tau = (18, 16, 11, 9, 7, 5, 4^2, 3^2, 1^7)$. The 3- and 4-runner abacus displays for these partitions are as follows.



Now we give the converse to Proposition 6.2.

Proposition 6.4. Suppose v is a t -core, and let τ be the partition with s -core $\kappa_{s,t}$ and s -quotient $(v, \dots, v)$. Then τ is a t -core.

Proof. We consider the beta-set $\mathcal{B}_{sot}^\tau$. By Lemma 2.2, we must show that $c - t \in \mathcal{B}_{sot}^\tau$ for every $c \in \mathcal{B}_{sot}^\tau$. In other words, we must show that for each $b \in \mathbb{Z}/s\mathbb{Z}$ we have

$$\{c - t \mid c \in \mathcal{B}_{sot}^\tau \cap b\} \subseteq \mathcal{B}_{sot}^\tau \cap (b - t).$$

Since s and t are coprime, we can write $b = at + s\mathbb{Z}$ for some $a \in \{0, \dots, s-1\}$. Then by construction we have

$$\mathcal{B}_{\text{sof}}^\tau \cap b = \{at + s(v_i - i) \mid i \in \mathbb{N}\}.$$

If $a > 0$, then we also have

$$\mathcal{B}_{\text{sof}}^\tau \cap (b - t) = \{(a-1)t + s(v_i - i) \mid i \in \mathbb{N}\},$$

and therefore

$$\{c - t \mid c \in \mathcal{B}_{\text{sof}}^\tau \cap b\} = \mathcal{B}_{\text{sof}}^\tau \cap (b - t).$$

So it remains to consider the case $a = 0$. Now we have

$$\mathcal{B}_{\text{sof}}^\tau \cap b = \{s(v_i - i) \mid i \in \mathbb{N}\}, \quad \mathcal{B}_{\text{sof}}^\tau \cap (b - t) = \{(s-1)t + s(v_i - i) \mid i \in \mathbb{N}\}.$$

If the left-hand side is not contained in the right-hand side, then for some j we have

$$s(v_j - j) - t \notin \{(s-1)t + s(v_i - i) \mid i \in \mathbb{N}\},$$

which gives $v_j - j - t \notin \{v_i - i \mid i \in \mathbb{N}\}$. But by Lemma 2.2, this contradicts the fact that v is a t -core. $\square$

Now we can give a concrete description of $C_{s:t}^\kappa$, which says that as a $W_s \times W_t$ -set, $C_{s:t}^\kappa$ is isomorphic to the product of C_s and C_t (with W_s and W_t acting at level 1 on these factors).

Proposition 6.5.

1. *There is a bijection*

$$\begin{aligned} C_{s:t}^\kappa &\longrightarrow C_s \times C_t \\ \lambda &\longmapsto (\lambda^{(t\mathbb{Z})}, \lambda^{(s\mathbb{Z})}). \end{aligned}$$

2. *Given $\lambda \in C_{s:t}^\kappa$ and $w \in W_s$, we have*

$$(w\lambda)^{(t\mathbb{Z})} = w(\lambda^{(t\mathbb{Z})}),$$

where w acts at level t on $C_{s:t}^\kappa$, and at level 1 on C_s .

Proof.

1. Let Θ denote the given map, and suppose $\lambda \in C_{s:t}^\kappa$. Since λ is an st -core, every component of its s -quotient is a t -core, and every component of its t -quotient is an s -core, by Corollary 2.4; so Θ really does map to $C_s \times C_t$. To show that Θ is a bijection, we construct an inverse. If $(\rho, v) \in C_s \times C_t$, let τ be the partition with s -core $\kappa_{s,t}$ and s -quotient $(v, \dots, v)$. Then τ is a t -core by Proposition 6.4. Similarly the partition σ with t -core $\kappa_{s,t}$ and t -quotient $(\rho, \dots, \rho)$ is an s -core, and $\text{cor}_t \sigma = \text{cor}_s \tau$, so we can define a partition $\sigma \boxplus \tau$, which will lie in $C_{s:t}^\kappa$. We define $\Xi(\rho, v) = \sigma \boxplus \tau$, and we have a function $\Xi : C_s \times C_t \rightarrow C_{s:t}^\kappa$.

Now we show that Θ and Ξ are mutual inverses. Suppose $\lambda \in C_{s:t}^\kappa$, write $\Theta(\lambda) = (\rho, v)$ and let $\tau = \text{cor}_t \lambda$. Then τ is a t -core with s -core $\kappa_{s,t}$, so by Proposition 6.2 all the components of the s -quotient of τ are equal. But λ and τ lie in the same level t orbit of W_s , so have the same s -quotient up to re-ordering; since $v = \lambda^{(s\mathbb{Z})}$, this means that both λ and τ have s -quotient $(v, \dots, v)$. So τ is the (unique) partition with s -core $\kappa_{s,t}$ and s -quotient $(v, \dots, v)$. Similarly the s -core σ of λ is the partition with t -core $\kappa_{s,t}$ and t -quotient $(\rho, \dots, \rho)$. So $\Xi(\Theta(\lambda)) = \Xi(\rho, v) = \sigma \boxplus \tau$, which is the unique partition in $C_{s:t}^\kappa$ with s -core σ and t -core τ , i.e. λ .

Now take $(\rho, v) \in C_s \times C_t$. Let τ be the partition with s -core $\kappa_{s,t}$ and s -quotient $(v, \dots, v)$, and σ the partition with t -core $\kappa_{s,t}$ and t -quotient $(\rho, \dots, \rho)$, so that $\Xi(\rho, v)$ is by definition $\sigma \boxplus \tau$. This partition has the same t -quotient as σ since it lies in the same level s orbit of W_t , and in particular each component of its t -quotient is ρ . Similarly each component of the s -quotient of $\sigma \boxplus \tau$ is v , and so $\Theta(\Xi(\rho, v)) = \Theta(\sigma \boxplus \tau) = (\rho, v)$.

2. It suffices to consider the case where $w = w_i$, for $i \in \mathbb{Z}/s\mathbb{Z}$. Write $j = it - sot$, and let $\sigma = \lambda^{(t\mathbb{Z})}$. Then by the definition of t -quotient,

$$\mathcal{B}^\lambda \cap t\mathbb{Z} = \{bt + \Gamma_{t\mathbb{Z}}\lambda \mid b \in \mathcal{B}^\sigma\}.$$

Now the definition of the level t action of W_s gives

$$\mathcal{B}^{w_i\lambda} = \{c + t \mid c \in \mathcal{B}^\lambda \cap (j - t)\} \cup \{c - t \mid c \in \mathcal{B}^\lambda \cap j\} \cup (\mathcal{B}^\lambda \setminus (j - t \cup j)).$$

From Lemma 6.1 (with s and t interchanged) we have $\Gamma_{t\mathbb{Z}}\lambda \equiv -sot \pmod{s}$. Hence for $b \in \mathbb{Z}$ we have $bt + \Gamma_{t\mathbb{Z}}\lambda \in j - t$ if and only if $b \in i - 1$, while $bt + \Gamma_{t\mathbb{Z}}\lambda \in j$ if and only if $b \in i$. So

$$\mathcal{B}^{w_i\lambda} = \{(w_ib)t + \Gamma_{t\mathbb{Z}}\lambda \mid b \in \mathcal{B}^\sigma\} = \{bt + \Gamma_{t\mathbb{Z}}\lambda \mid b \in \mathcal{B}^{w_i\sigma}\},$$

where W_s acts at level t in the first term, and at level 1 in the other two terms. Hence

$$(w_i\lambda)^{(t\mathbb{Z})} = w_i\sigma,$$

as required. □

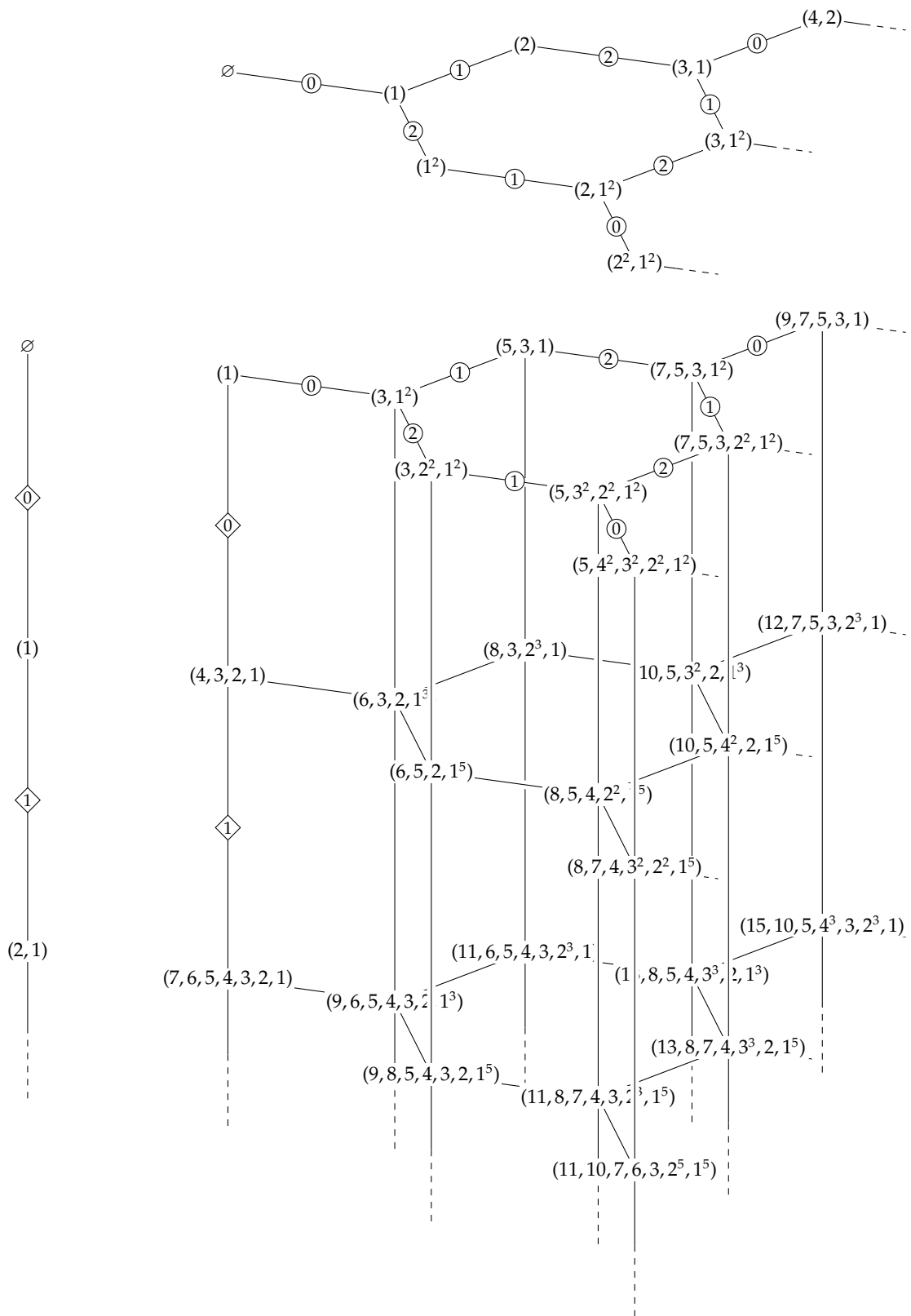
Of course, part (2) of the proposition also holds with s and t interchanged, yielding the desired statement about the action of $W_s \times W_t$.

In Figure 1, we illustrate Proposition 6.5 in the case $(s, t) = (2, 3)$. At the top of the diagram, we have drawn a portion of C_3 as a labelled graph, with edges indicating the actions of the generators $w_{3\mathbb{Z}}, w_{1+3\mathbb{Z}}, w_{2+3\mathbb{Z}}$ in the level 1 action of W_3 . On the left, we have drawn a portion of C_2 , with edges representing the actions of $w_{2\mathbb{Z}}, w_{1+2\mathbb{Z}}$ in the level 1 action of W_2 . The main part of the diagram shows a portion of $C_{2,3}^\kappa$, which (we hope) makes the bijection $C_2 \times C_3 \rightarrow C_{2,3}^\kappa$ clear. Here the edges represent the actions of the generators of W_3 in the level 2 action, and of W_2 in the level 3 action.

7 Final remarks

7.1 The non-coprime case

Throughout this paper we have assumed for simplicity that the integers s and t are coprime. In fact, this assumption is unnecessary for many of our results. Theorem 4.1 was generalised to the non-coprime case by Nath [N, Theorem 1.1] and (with a different proof) by Gramain and Nath [GN, Theorem 2.1]; the idea in the latter proof is to consider the g -quotient of a partition, where g is the greatest common divisor of s and t . Applying Theorem 4.1 to the components of this quotient and using results on quotients such as Lemma 2.5, one obtains the general result. This technique can be applied to many of our results, too, and Theorem 4.2, Propositions 4.3 and 5.1 and Corollaries 4.4 and 5.2 all hold without modification in the case where s and t are not coprime, while Lemma 4.8 requires minor modification. The results concerning the level t action of W_s do not generalise so readily: one must consider the action of a group consisting of a direct product of g copies of $W_{s/g}$. Then the results we have proved can be made to work, but the level of complication soon outweighs the reward; we leave the interested reader to work out the details. The results in Section 6 seem to have no analogue in the non-coprime case, where there are infinitely many (s, t) -cores.

Figure 1: The bijection between $C_2 \times C_3$ and $C_{2;3}^\kappa$

7.2 $[s:t:u]$ -cores

A natural extension of the results in this paper would be to try to extend from two integers s, t to three (or more): is there a suitable definition of an $[s:t:u]$ -core? The author has not been able to find the appropriate generalisation of our initial definition of an $[s:t]$ -core. However, Corollary 5.2 suggests a possibility: assuming $s, t, u > 1$, we could define an $[s:t:u]$ -core to be a partition which is uniquely determined by its size and its s -, t - and u -cores. We hope to be able to say something about such partitions in the future.

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