Classifying level 1 Fock spaces of a certain type

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This note arose from discussion with Peter Tingley. We work with quantum algebra $U = U_q(\widehat{\mathfrak{sl}_n})$ for $n \geqslant 3$. We refer to the elements of the indexing set $\mathbb{Z}/n\mathbb{Z}$ as *residues*, and we say that residues i, j are *adjacent* if they're adjacent in the Dynkin diagram, i.e. $j = i \pm 1$.

An important U-module is the level 1 Fock space. This has a basis the set of all partitions; let's quickly revise some basic notions about partitions.

- A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers with finite sum.
- The *Young diagram* of a partition λ is the subset

  $$[\lambda] = \{(k, l) \mid l \leqslant \lambda_k\}$$

  of $\mathbb{N}^2$. We refer to the elements of $\mathbb{N}^2$ as *nodes*, and elements of $[\lambda]$ as nodes of λ.
- A node of partition λ is *removable* if it can be removed from $[\lambda]$ to leave the Young diagram of a partition, while a node not in $[\lambda]$ is an *addable node* of λ is it can be added to $[\lambda]$ to give the Young diagram of a partition. If a is a removable node of λ, we write λ_a for the partition (whose Young diagram is) obtained by removing a. We extend this notation and write partitions such as $\lambda_{a,b}$, with obvious meaning.
- The *residue* of the node (k, l) is $l - k + n\mathbb{Z}$; a node of residue $i \in \mathbb{Z}/n\mathbb{Z}$ is called an *i-node*.
- If λ and μ are partitions, and μ is obtained from λ by adding an addable i-node, we may write $\lambda \xrightarrow{i} \mu$.
- The node (k, l) is *above* the node (k', l') if and only if $k < k'$.
- The *axial distance* between nodes (k, l) and (k', l') is $|k - k' - l + l'|$. (This is a slightly unconventional definition; normally, the modulus sign is not included.)

Now we can describe the action of the generators f_i on the Fock space. Given a partitions λ, μ with $\lambda \xrightarrow{i} \mu$, we let a be the node added to λ to obtain μ, and set $N(\lambda, \mu)$ to be the number of addable

i-nodes of λ above a minus the number of removable i-nodes of λ above a. The action of f_i on the Fock space is then given by

$$f_i\lambda = \sum_{\lambda \xrightarrow{i} \mu} q^{N(\lambda,\mu)}\mu.$$

There is an alternative version of the Fock space, obtained by replacing 'above' with 'below' in the definition above. Each of these Fock spaces contains a summand isomorphic to the basic representation $V(\Lambda_0)$, and thereby yields a combinatorial model for the crystal $B(\Lambda_0)$, in which the crystal operators $\tilde{e}_i, \tilde{f}_i$ are realised by removing and adding certain nodes of residue i.

The paper [F] exhibits more models for $B(\Lambda_0)$ of this type, i.e. where the vertices are labelled by a certain class of partitions (with the highest-weight vertex labelled by the empty partition), and with the $\tilde{e}_i, \tilde{f}_i$ corresponding to removal or addition of i-nodes (with exactly the same notion of an i-node). It is reasonable to suppose that there might be variations on the definition of the Fock space above which give an algebraic explanation for these crystal models. Specifically, one is tempted to define a module which has the set of all partitions as a basis, and

$$f_i\lambda = \sum_{\lambda \xrightarrow{i} \mu} p_{\lambda\mu}\mu,$$

where $p_{\lambda\mu} \in \mathbb{Q}(q)$. A naïve approach would be to take each $p_{\lambda\mu}$ to be a power of q; in this note, we show that this can't produce anything new, i.e. the only modules satisfying these criteria are the two Fock spaces described above. Note that although we consider only level 1 here, the methods easily extend to higher levels.

Of course, one can produce a trivial variation of the Fock space just by re-scaling a basis vector λ by a power of q, and we want to ignore such things. The way we do this is to define a *square* to be a set of four partitions of the form

$$\{\lambda, \lambda_a, \lambda_b, \lambda_{a,b}\},$$

where λ is a partition and a, b are removable nodes of λ. We may write this square as (λ, a, b) for brevity. Suppose a has residue i and b has residue j, and that a is above b. Now, if we're given a version of the Fock space as above in which the $p_{\lambda\mu}$ are all powers of q, then let r, s, t, u be the integers such that

$$p_{\lambda_{m,n},\lambda_n} = q^r, \quad p_{\lambda_{m,n},\lambda_m} = q^s, \quad p_{\lambda_m,\lambda} = q^t, \quad p_{\lambda_n,\lambda} = q^u.$$

Then the integer $r - s - t + u$ is independent of a rescaling of any basis element; we'll call this integer the *value* of this square. Determining the value of every square determines the action of U^-, up to the rescaling of basis elements (this essentially says that every loop in Young's graph is built up from squares). Note that in the conventional Fock space, the square (λ, a, b) has value -2 if $i = j$, 1 if $i = j \pm 1$, and 0 otherwise.

So let's suppose we have assigned a value to each square, and consider the Serre relations. First of all, suppose we have a square whose nodes have residues i, j which are not equal and not adjacent. Then by the Serre relation $f_if_j = f_jf_i$, the value of this square must be 0. So we only need to care about squares in which the two residues are equal or adjacent; let's call these *live* squares.

Next let's consider a square (λ, a, b) for which a and b have the same residue i, and λ_a has a removable node c which is not a removable node of λ. Then the residue j of c must be adjacent to

i, so that we have the Serre relation

$$f_i^2 f_j + f_j f_i^2 = (q + q^{-1}) f_i f_j f_i.$$

Applying this and considering the coefficient of λ in $f_i^2 f_j(\lambda_{a,b,c})$, we find that the squares (λ, a, b) and (λ_a, b, c) have values ± 2 and ∓ 1 respectively, for some choice of sign. Let's say that these two squares are *adjacent*. There's a similar kind of adjacency, where λ, a, b, c are as above, but a, b, c have residues j, i, i respectively. Then the two squares (λ, a, b) and (λ_a, b, c) have values ± 1 and ∓ 2 respectively, for some choice of sign. It's not too hard to check that every live square is involved in an adjacency like this; hence every live square has value ± 1 if its residues are adjacent, or ± 2 if its residues are equal. Let's say that a live square is *positive* if its value is 2 or -1, and *negative* otherwise. Our aim is to show that all live squares have the same sign, and we've already seen that two adjacent live squares have the same sign.

Next, let's consider a partition λ with three removable nodes a, b, c (from high to low) whose residues are i, i, j in some order; consider the set of eight partitions which can be obtained from λ by removing some combination of the nodes a, b, c; let's call this an iij-cube (which we'll denote (λ, a, b, c)). Among these eight partitions, we've got six live squares (the faces of the cube). By considering the coefficient of λ in $f_i^2 f_j(\lambda_{a,b,c})$ and applying the Serre relation above, we find that the possible values of these six squares are as follows (the residues of a, b, c can be inferred in each case).

(λ_a, b, c)	(λ_b, a, c)	(λ_c, a, b)	(λ, a, b)	(λ, a, c)	(λ, b, c)	
± 1	± 1	∓ 2	∓ 2	± 1	± 1	†
± 1	∓ 1	∓ 2	± 2	± 1	∓ 1	∗
∓ 1	± 1	∓ 2	∓ 2	± 1	∓ 1	
∓ 1	∓ 1	∓ 2	∓ 2	∓ 1	∓ 1	
± 1	∓ 2	± 1	± 1	∓ 2	± 1	†
± 1	∓ 2	∓ 1	∓ 1	∓ 2	± 1	
∓ 1	∓ 2	± 1	± 1	∓ 2	∓ 1	
∓ 1	∓ 2	∓ 1	± 1	± 2	± 1	∗
∓ 2	± 1	± 1	± 1	± 1	∓ 2	†
∓ 2	± 1	∓ 1	∓ 1	± 1	∓ 2	
∓ 2	∓ 1	± 1	∓ 1	± 1	± 2	∗
∓ 2	∓ 1	∓ 1	∓ 1	∓ 1	∓ 2	

Notice that some of these possibilities (those marked ∗) yield live squares of opposite signs, which we want to avoid. Let's refer to these cubes as *ugly* cubes. Our next task is to show that ugly cubes can't exist.

Suppose (λ, a, b, c) is an iij-cube; let's suppose a has residue i (the other cases are similar). Then either

1. λ_a has a removable j-node d which is not a removable node of λ, or
2. λ has an addable j-node d which is not an addable node of λ_a.

Let's assume we're in the first case; the other case is similar. Consider the iij-cube (λ, a, b, c) and the jji-cube (λ_a, d, b, c). These two cubes have a common square (namely, (λ_a, b, c)), and their union contains four pairs of adjacent squares (for example, (λ, a, b) and (λ_a, d, b)). By considering the possible sets of values for the faces of the two cubes (from the table above) and the fact that adjacent squares have the same sign, it's easy to show that neither of the cubes can be ugly.

So there can be no ugly cubes at all. In particular, this means that two live squares which are opposite faces of some iij-cube must have the same sign; let's extend our notion of adjacent live squares to include such squares.

We want to extend adjacency on live squares further: consider a partition λ with three removable nodes a, b, c of residues i, i, k respectively, where i and k are neither equal nor adjacent. One can then easily show that the live squares (λ, a, b) and (λ_c, a, b) have the same sign; so we'll extend our notion of adjacency to include squares like this too. We extend adjacency transitively so that we get an equivalence relation on live squares. From what we've seen, we know that two live squares in the same equivalence class must have the same sign.

What do equivalence classes look like? In any live square, the axial distance between the two nodes is either a multiple of n, or one more or one less then a multiple of n (because the residues are either equal or adjacent). So it makes sense to define the *span* of a live square to be the closest multiple of n to the axial distance between the two nodes. Then it's not too had to show that two live squares lie in the same equivalence class if and only if they have the same span.

Now, how do we show that signs agree between equivalence classes? We prove that the live squares of span sn have the same sign as the live squares of span n by induction on s. Given $s > 1$, it's easy to find an iij-cube (λ, a, b, c) such that (λ, a, b) has span n and (λ, b, c) has span $(s-1)$ (and therefore (λ, a, c) has span sn). By induction the squares (λ, a, b) and (λ, b, c) have the same sign; now by consulting the above table (and noting that there are no ugly cubes), we see that we are in one of the cases labelled †, so the sign of the square (λ, a, c) agrees too.

References

[F] M. Fayers, 'Partition models for the crystal of the basic $U_q(\widehat{\mathfrak{sl}}_n)$-module', arXiv:0906.4129.