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# The ordinary quiver of a weight three block of the symmetric group is bipartite

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## Abstract

Suppose  $\mathbb{F}$  is a field of characteristic  $p \geq 5$ , and that  $B$  is a  $p$ -block of the symmetric group  $\mathfrak{S}_n$  of defect 3. We prove that the  $\text{Ext}^1$ -quiver of  $B$  is bipartite, with the bipartition being described in a simple way using the leg-lengths of  $p$ -hooks of partitions.

## 1 Introduction

Suppose  $\mathbb{F}$  is a field, and  $A$  is a finite-dimensional  $A$ -algebra. The  $\text{Ext}^1$ -quiver or *ordinary quiver* of  $A$  is a directed multi-graph with edges indexed by isomorphism classes of simple  $A$ -modules, and with the number of arrows from  $S$  to  $T$  being the  $\mathbb{F}$ -dimension of the space  $\text{Ext}_A^1(S, T)$ . The quiver is a useful tool for understanding the representation theory of  $A$  – indeed, Gabriel’s Theorem asserts that  $A$  is Morita equivalent to a certain quotient of the *quiver algebra* of  $A$ .

In this paper, we shall be concerned with modular representation theory of symmetric groups. So  $A$  will be a block of the group algebra  $\mathbb{F}\mathfrak{S}_n$ , where  $n$  is a non-negative integer and  $\mathbb{F}$  is a field of prime characteristic  $p$ . In the case of symmetric groups, all simple modules are self-dual, and so the quiver may be drawn as an undirected (multi-)graph, with an edge indicating an arrow in each direction. Many of these quivers have been calculated, and have been seen to enjoy certain properties. One property which remains conjectural in general (but which has important implications for radical filtrations of modules) is that if  $A$  is a block whose defect group is abelian, then the quiver of  $A$  is bipartite. This is one of a variety of nice properties which have been conjectured for symmetric group blocks of abelian defect. The purpose of this paper is to verify this property in the case of blocks of defect 3.

As is well known, the representations of the symmetric group  $\mathfrak{S}_n$  are indexed by partitions of  $n$ . To each partition is associated a non-negative integer, called the  $p$ -weight of the partition; this is a block invariant, and turns out to be an excellent measure of how complicated the representation theory of a block is. Indeed, if the weight of a block is less than  $p$ , then it coincides with the  $p$ -defect of the block. Much of the representation theory of the symmetric groups has concentrated on blocks of a given small weight. Blocks of weight 0 and 1 have been understood for some time – the former are simple, while the latter are described by the Brauer–Dade cyclic defect theory. Blocks of weight 2 were studied by several authors, including Richards, who gave a combinatorial description of their decomposition

numbers over a field of odd characteristic [21]. Chuang and the second author built on this work and were able to describe the quiver of an arbitrary weight 2 block in odd characteristic [2], verifying that all such quivers are bipartite. Martin and Russell [15] initiated the study of blocks of weight 3 with the construction of the  $\text{Ext}^1$ -quiver of the principal block of  $\mathfrak{S}_{3p}$  where  $p \geq 5$ . They also attempted to show that the decomposition numbers for a weight 3 block of abelian defect are at most 1 [16], but mistakes have been found in their proof. The complete proof was finally announced by the first author of the present paper in [9].

Among our most important tools are the so-called ‘Rouquier blocks’, which have played an increasingly prominent rôle in the representation theory of the symmetric groups in recent years. These are blocks defined for any weight and any characteristic which have certain nice properties. In particular, there exists an explicit description of their  $\text{Ext}^1$ -quivers (in the case where the defect group is abelian), due to Chuang and the second author [3]. This enables our main theorem to be verified immediately for Rouquier blocks, and we make significant use of this result in proving the main theorem in general. Our other important tools are the Mullineux map and Kleshchev’s modular branching rules.

As with any result in the representation theory of the symmetric groups, it is natural to ask whether our main theorem generalises to the Iwahori–Hecke algebra  $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ . It seems likely that this is the case (as long as we make the assumption  $q \neq -1$ ), and that our methods would carry over. The major obstruction is that we do not at present have a description of the  $\text{Ext}^1$ -quiver of the Rouquier blocks of the Iwahori–Hecke algebras in full generality. (We note however that if  $q$  is in the prime subfield of  $\mathbb{F}$ , then the Morita equivalence between the Rouquier block of a finite general linear groups and a wreath product of a ‘weight 1’ block of finite general linear group proved independently by Hida–Miyachi [12] and Turner [26] can be used to construct an analogous Morita equivalence for the Iwahori–Hecke algebras. One can then use this Morita equivalence in conjunction with the results of general wreath products developed by Chuang and the second author [4] to obtain the  $\text{Ext}^1$ -quiver of the Rouquier blocks of Iwahori–Hecke algebras in this case.)

We now indicate the layout of this paper. In the remainder of this introduction, we set out the background theory we shall require. In Section 2, we specialise to blocks of small weight, stating the results we shall need on blocks of weight 0, 1, 2 and 3, and outlining the proof of the main theorem, which is by induction on  $n$ . We concentrate on blocks of particular types in sections 3, 4 and 5, and conclude with the proof of the main theorem in section 6.

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## 1.1 Background theory

We now survey the background theory we shall require. An excellent introduction to the modular representation theory of the symmetric group may be found in James’s book [13].

We record here some notation that we use later for module structures. If  $N, N_1, \dots, N_r$  are modules, then we may write

$$N \sim \begin{matrix} N_1 \\ \vdots \\ N_r \end{matrix}$$

to indicate that  $N$  has a filtration in which the factors are  $N_1, \dots, N_r$  from top to bottom. If these factors are all isomorphic, to  $M$  say, then we may just write  $N \sim M^r$ .

### 1.1.1 Partitions, blocks and parity

Throughout this paper, we assume that the reader is familiar with the combinatorics of partitions, Young diagrams and rim hooks. A suitable introduction may be found in [13]. For any partition  $\lambda$  of  $n$  and any field  $\mathbb{F}$ , one defines a *Specht module*  $S^\lambda$  for  $\mathbb{F}\mathfrak{S}_n$ . If  $\mathbb{F}$  has infinite characteristic, then the Specht modules are irreducible, and afford all the irreducible representations of  $\mathfrak{S}_n$  as  $\lambda$  ranges over the partitions of  $n$ . If  $\mathbb{F}$  has finite characteristic  $p$ , then the Specht modules are not always irreducible. If  $\lambda$  is  $p$ -regular (i.e. if it does not have  $p$  or more equal non-zero parts), then  $S^\lambda$  has an irreducible self-dual cosocle  $D^\lambda$ , and the modules  $D^\lambda$  afford all the irreducible representations of  $\mathfrak{S}_n$  as  $\lambda$  ranges over the  $p$ -regular partitions of  $n$ .

Given a partition  $\lambda$  and a prime  $p$ , we form a new partition by repeatedly removing rim  $p$ -hooks from the Young diagram of  $\lambda$  until we can remove no more. Remarkably, the partition we end up with does not depend on which rim  $p$ -hook we choose to remove at each stage, and this partition is called the  $p$ -core of  $\lambda$ . The  $p$ -core of  $\lambda$  is a partition of  $n - wp$  for some non-negative integer  $w$ , which is called the  $p$ -weight of  $\lambda$ . We may talk simply of the core and weight of  $\lambda$  if the prime  $p$  is understood. Nakayama's 'Conjecture' tells us that if  $\mathbb{F}$  is a field of characteristic  $p$ , then two partitions  $\lambda$  and  $\mu$  of  $n$  lie in the same block of  $\mathbb{F}\mathfrak{S}_n$  (by which we mean that the Specht modules  $S^\lambda$  and  $S^\mu$  lie in the same block) if and only if  $\lambda$  and  $\mu$  have the same  $p$ -core. This automatically implies that they have the same  $p$ -weight, and accordingly we may speak of the  $p$ -core and  $p$ -weight of a block of  $\mathbb{F}\mathfrak{S}_n$ .

We now define the *parity* of a partition  $\lambda$ . When we remove rim  $p$ -hooks from  $[\lambda]$  to reach the  $p$ -core of  $\lambda$ , we may examine the leg-length of each hook. Morris and Olsson [19, Proposition 2.2 and Corollary 2.3] showed that if  $l$  is the sum of these leg-lengths, then  $(-1)^l$  equals the relative  $(p)$ -sign of  $\lambda$  defined by Farahat [5]. In particular, this shows that while these leg-lengths may depend on which rim hook we choose to remove at each stage, the parity of their sum does not. We refer to this parity as the parity of  $\lambda$ , which we write as  $\mathcal{P}\lambda$ . Now we may state the main theorem of this paper.

**Theorem 1.1.** *Suppose that  $\mathbb{F}$  is a field of characteristic at least 5, and that  $B$  is a weight 3 block of  $\mathbb{F}\mathfrak{S}_n$ . If  $\lambda$  and  $\mu$  are  $p$ -regular partitions in  $B$  with  $\mathcal{P}\lambda = \mathcal{P}\mu$ , then  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$ . In particular, the  $\text{Ext}^1$ -quiver of  $B$  is bipartite.*

Note that the assumption  $\text{char}(\mathbb{F}) \geq 5$  is essential; the  $\text{Ext}^1$ -quivers of weight 3 blocks of symmetric groups in characteristic 3 have been calculated by the authors in a series of papers [24, 25, 6, 7], and these are not all bipartite. In characteristic 2, the theorem fails more spectacularly; in particular, there are self-extensions of simple modules.

### 1.1.2 The abacus

A useful way to represent partitions, which makes it very clear when two partitions lie in the same block, is on an *abacus*. We take an abacus with  $p$  vertical runners, which we label  $1, \dots, p$  from left to right. We mark positions on each runner, and then label the positions on runner  $i$  with the integers  $i - 1, i + p - 1, i + 2p - 1, \dots$  from the top down, so that (if  $p \nmid j$ ) position  $j - 1$  lies directly to the left of position  $j$ . We shall frequently talk of moving beads on runner  $i$  of the abacus 'one space to the left', and we wish to include the possibility  $i = 1$  here, so moving a bead at position  $j$  one space to the left will simply mean moving it to position  $j - 1$ . Given a partition  $\lambda$ , we take an integer  $r$  which is at least the number of non-zero parts of  $\lambda$ , and then for  $i = 1, \dots, r$  we define the *beta-number*

$$\beta_i = \lambda_i + r - i.$$

Now we place a bead at position  $\beta_i$  for each  $i$ . The resulting configuration is called an *abacus display* for  $\lambda$ . The usefulness of the abacus display comes from the fact [14, §2.7] that removing a rim  $p$ -hook from the Young diagram of  $\lambda$  corresponds exactly to moving a bead up to an empty space immediately above it; therefore an abacus display for the core of  $\lambda$  may be obtained by sliding all the beads as far up their runners as they will go. So if partitions  $\lambda$  and  $\mu$  are displayed using abacuses with the same number of beads, then  $\lambda$  and  $\mu$  lie in the same block if and only if the numbers of beads on corresponding runners are the same, and we may specify the abacus for a block of  $\mathfrak{S}_n$  by specifying the number of beads on each runner, without specifying their positions.

### 1.1.3 The Jantzen–Schaper dominance order

The dominance order  $\triangleright$  is frequently used when working with partitions, and is particularly useful for representation theory. It will be useful for us to use a coarser version of this order, which depends on the prime  $p$ . If  $\lambda$  and  $\mu$  are partitions of  $n$ , then we say that  $\lambda$  *dominates*  $\mu$  in the *Jantzen–Schaper order* if  $\lambda \triangleright \mu$  and if the Young diagram of  $\mu$  may be obtained from that of  $\lambda$  by removing a rim hook of length divisible by  $p$  and then adding a rim hook of the same length. We extend this order to a partial order, which we write as  $\trianglerighteq$ . No confusion should occur with this notation, since we shall not use the usual dominance order from now on, and the prime  $p$  will always be clear from the context. The first use of this order is in the following basic theorem, which follows from [13, Corollary 12.2] together with the Jantzen–Schaper formula [22] – here, and hereafter,  $[S^\lambda : D^\mu]$  denotes the multiplicity of  $D^\mu$  as a composition factor of  $S^\lambda$ .

**Theorem 1.2.** *Suppose  $\mathbb{F}$  is a field of characteristic  $p$ , and that  $\lambda$  and  $\mu$  are partitions of  $n$ , with  $\mu$   $p$ -regular. Then  $[S^\mu : D^\mu] = 1$ , and if  $[S^\lambda : D^\mu] > 0$  then  $\mu \trianglerighteq \lambda$ .*

### 1.1.4 The Mullineux map

Let  $\text{sgn}$  denote the 1-dimensional signature representation of  $\mathfrak{S}_n$ . The functor  $- \otimes \text{sgn}$  gives a self-equivalence of the category of  $\mathbb{F}\mathfrak{S}_n$ -modules. If  $M$  lies in a block  $B$  of  $\mathfrak{S}_n$ , then  $M \otimes \text{sgn}$  lies in the *conjugate block* of  $B$ , which we denote  $B^\sharp$ . We wish to describe the effect of the functor  $- \otimes \text{sgn}$  on Specht modules and simple modules. Let  $\lambda'$  denote the partition conjugate to  $\lambda$ , and for a  $\mathbb{F}\mathfrak{S}_n$ -module  $M$ , let  $M^*$  denote the dual module.

**Theorem 1.3.** [13, Theorem 8.15] *For any partition  $\lambda$ ,*

$$S^\lambda \otimes \text{sgn} \cong (S^{\lambda'})^*.$$

We note one immediate consequence of this, which is that if  $B$  is a block with  $p$ -core  $\nu$ , then  $B^\sharp$  has  $p$ -core  $\nu'$ .

The corresponding result for simple modules is more complicated. If  $\mu$  is a  $p$ -regular partition, then of course  $D^\mu \otimes \text{sgn}$  is a simple module, and so there is an involutory bijection  $\diamond$  from the set of  $p$ -regular partitions of  $n$  to itself such that  $D^\mu \otimes \text{sgn} \cong D^{\mu^\diamond}$ . This bijection was given by Mullineux [20] (although a proof that Mullineux’s bijection is the correct one was not given until much later, by Ford and Kleshchev [11]). We shall frequently use this bijection, but in order to save space, we refer the reader to [20] for a description.

### 1.1.5 Induction and restriction

If  $\kappa$  is a positive integer, then the natural embedding  $\mathfrak{S}_{n-\kappa} < \mathfrak{S}_n$  gives rise to well-behaved induction and restriction functors between the module categories of  $\mathbb{F}\mathfrak{S}_{n-\kappa}$  and  $\mathbb{F}\mathfrak{S}_n$ . Given a module  $N$  for  $\mathbb{F}\mathfrak{S}_n$ , we write  $N \downarrow_{\mathfrak{S}_{n-\kappa}}$  for the restricted module, and if  $A$  is a block of  $\mathbb{F}\mathfrak{S}_{n-\kappa}$ , we write  $N \downarrow_A$  for the projection of  $N \downarrow_{\mathfrak{S}_{n-\kappa}}$  onto  $A$ . Similarly, if  $M$  is an  $\mathbb{F}\mathfrak{S}_{n-\kappa}$ -module and  $B$  is a block of  $\mathbb{F}\mathfrak{S}_n$ , we write  $M \uparrow^{\mathfrak{S}_n}$  and  $M \uparrow^B$  for the induced module and its projection onto  $B$ . We shall use the fact that these functors are exact without comment, and we shall frequently employ the Frobenius reciprocity theorem and the Eckmann–Shapiro relations. We also use the classical Branching Rule [13, Theorem 9.3 & Corollary 17.14] and Kleshchev’s ‘modular branching rules’ (see for example the discussion in [1, Section 2]).

### 1.1.6 Scopes equivalences

Some Morita equivalences between symmetric group blocks of the same weight were discovered by Scopes, and we shall make frequent use of these. Suppose  $B$  is a block of  $\mathfrak{S}_n$  of weight  $w$ , and that in some abacus display for  $B$  there are  $\kappa$  more beads on runner  $i$  than runner  $i-1$ . By interchanging runners  $i-1$  and  $i$ , we obtain an abacus for a weight  $w$  block  $A$  of  $\mathfrak{S}_{n-\kappa}$ . We say that  $A$  and  $B$  form a  $[w : \kappa]$ -pair. If we have such a pair with  $\kappa \geq w$ , then we define a function  $\Phi$  from the set of partitions in  $B$  to the set of partitions in  $A$  by interchanging runners  $i-1$  and  $i$  of the abacus.

**Theorem 1.4** ([23]). *Suppose that  $A$  and  $B$  form a  $[w : \kappa]$ -pair as above, with  $w \leq \kappa$ , and suppose  $\lambda$  and  $\mu$  are partitions in  $B$  with  $\mu$   $p$ -regular.*

1.  $\Phi$  is a bijection between the set of partitions in  $B$  and the set of partitions in  $A$ .

2.  $\Phi(\lambda)$  is  $p$ -regular if and only if  $\lambda$  is  $p$ -regular.

3.

$$S^\lambda \downarrow_A \sim (S^{\Phi(\lambda)})^{\kappa!}, \quad S^{\Phi(\lambda)} \uparrow^B \sim (S^\lambda)^{\kappa!}.$$

4.

$$D^\mu \downarrow_A \cong (D^{\Phi(\mu)})^{\oplus \kappa!}, \quad D^{\Phi(\mu)} \uparrow^B \cong (D^\mu)^{\oplus \kappa!}.$$

5.

$$[S^\lambda : D^\mu] = [S^{\Phi(\lambda)} : D^{\Phi(\mu)}].$$

6. The bijection  $\mu \leftrightarrow \Phi(\mu)$  is induced by a Morita equivalence between  $A$  and  $B$ .

### 1.1.7 Rouquier blocks

A certain class of symmetric group blocks has been shown to have particularly nice properties, and has proved particularly useful for studying the representation theory of  $\mathfrak{S}_n$  in general. Suppose  $B$  is a block of  $\mathfrak{S}_n$  of weight  $w$ , and take an abacus for  $B$ . We say that  $B$  is *Rouquier* if for each  $1 \leq i < j \leq p$  either

- there are at least  $w-1$  more beads on runner  $j$  than on runner  $i$ , or
- there are at least  $w$  more beads on runner  $i$  than on runner  $j$ .

It is a simple exercise to check that this property does not depend on the choice of abacus. For a given prime  $p$  and a given choice of  $w$ , the Rouquier blocks form a single equivalence class under the ‘Scopes equivalence’ described in the previous section. If  $w < p$ , then we have explicit closed formulae for the decomposition numbers for Rouquier blocks, and for the dimensions of the  $\text{Ext}^1$ -spaces between simple modules. It is the latter in which we are interested, and in particular the following corollary.

**Proposition 1.5.** *Suppose  $\mathbb{F}$  is a field of characteristic  $p$ , and that  $B$  is a Rouquier block of  $\mathbb{F}\mathfrak{S}_n$  of weight  $w < p$ . If  $\lambda$  and  $\mu$  are  $p$ -regular partitions in  $B$  with  $\mathcal{P}\lambda = \mathcal{P}\mu$ , then  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$ .*

**Proof.** This follows from [3, Theorem 6.3], in which the  $\text{Ext}^1$ -quiver of a Rouquier block is given. It is easy to calculate the parity of a partition in a Rouquier block and hence read off the result.  $\square$

## 2 Blocks of small weight

In this section, we outline some of the basic properties that we shall need for blocks of weight at most 3. We assume from now on that  $\mathbb{F}$  is a field of characteristic  $p \geq 5$ .

### 2.1 Blocks of weight 0

It is a well-known result that a block  $B$  of  $\mathfrak{S}_n$  has weight 0 if and only if it is simple. Therefore, a block of weight 0 contains a single partition  $\lambda$  (which is a  $p$ -core), and we have  $D^\lambda = S^\lambda = P(D^\lambda)$ .

### 2.2 Blocks of weight 1

Blocks of weight 1 have been understood for some time. Their properties may be summarised in the following theorem.

**Theorem 2.1.** *Suppose  $B$  is a block of  $\mathfrak{S}_n$  of weight 1. Then  $B$  contains exactly  $p$  partitions, which are totally ordered by the Jantzen–Schaper order:  $\lambda^{(1)} \triangleleft \dots \triangleleft \lambda^{(p)}$ . The partition  $\lambda^{(i)}$  is  $p$ -regular if and only if  $i \geq 2$ , and the decomposition number  $[S^{\lambda^{(i)}} : D^{\lambda^{(j)}}]$  equals 1 if  $j = i$  or  $i + 1$ , and 0 otherwise. We have  $\mathcal{P}\lambda^{(i)} = \mathcal{P}\lambda^{(j)}$  if and only if  $i$  and  $j$  have the same parity. If  $i \geq 2$ , then the projective cover of  $D^{\lambda^{(i)}}$  has radical length 3, with socle and cosocle both isomorphic to  $D^{\lambda^{(i)}}$ , and heart containing factors  $D^{\lambda^{(i-1)}}$  (if  $i \geq 3$ ) and  $D^{\lambda^{(i+1)}}$  (if  $i \leq p - 1$ ). Hence*

$$\text{Ext}_B^1(D^{\lambda^{(i)}}, D^{\lambda^{(j)}}) \cong \begin{cases} \mathbb{F} & (|i - j| = 1) \\ 0 & (\text{otherwise}). \end{cases}$$

*In particular, if  $\mathcal{P}\lambda^{(i)} = \mathcal{P}\lambda^{(j)}$ , then  $\text{Ext}_B^1(D^{\lambda^{(i)}}, D^{\lambda^{(j)}}) = 0$ .*

### 2.3 Blocks of weight 2

Blocks of weight 2 were systematically studied by Richards [21]. By developing the combinatorics of these blocks and studying the application of the Jantzen–Schaper formula, he was able to give a simple description of the decomposition numbers. Chuang and the second author [2] used this to give a description of the  $\text{Ext}^1$ -spaces between simple modules. The important consequence of this result for us is the following.

**Proposition 2.2.** *Suppose  $B$  is a block of  $\mathbb{F}\mathfrak{S}_n$  of weight 2, and that  $\lambda$  and  $\mu$  are  $p$ -regular partitions in  $B$ . If  $\mathcal{P}\lambda = \mathcal{P}\mu$ , then  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$ .*

**Proof.** This may be deduced easily from [2, Theorem 6.1]: note that the parity of a partition  $\lambda$  is the parity of the integer  $\partial\lambda$  defined by Richards.  $\square$

## 2.4 Notation for blocks of weight 3

Now we turn to blocks of weight 3, which are our main object of study. First we describe some notation for partitions in blocks of weight 3, following Martin and Russell [15]. Suppose  $B$  is a block of weight 3, and that we have an abacus display for the core of  $B$ . If  $\lambda$  is a partition in  $B$ , then an abacus display for  $\lambda$  is obtained by moving beads down their runners a total of three spaces. We write:

- $\lambda = \langle i \rangle$  if  $\lambda$  is obtained by moving the lowest bead on runner  $i$  down three spaces;
- $\lambda = \langle i, j \rangle$  if  $\lambda$  is obtained by moving the lowest bead on runner  $i$  down two spaces, and a bead on runner  $j$  down one space (where  $j$  may equal  $i$ );
- $\lambda = \langle i, j, k \rangle$  if  $\lambda$  is obtained by moving three beads, on runners  $i, j$  and  $k$ , down one space each (where  $i, j$  and  $k$  may coincide).

If the number of beads on runner  $i$  of the abacus is  $b_i$ , then we refer to this as the  $\langle b_1, \dots, b_n \rangle$  notation.

## 2.5 $[3 : \kappa]$ -pairs

In view of Scopes's theorem, the study of blocks of weight 3 centres around  $[3 : 1]$ - and  $[3 : 2]$ -pairs. In this section, we set up some notation for these pairs, and prove some basic results.

Suppose  $A$  and  $B$  are blocks forming a  $[3 : \kappa]$ -pair, and that the abacus for  $A$  is obtained from that for  $B$  by interchanging runners  $i-1$  and  $i$ . If  $\lambda$  is a partition in  $B$ , then say that  $\lambda$  is *exceptional* for this  $[3 : \kappa]$ -pair if in the abacus display for  $\lambda$  there are more than  $\kappa$  beads on runner  $i$  with no beads immediately to the left, and *non-exceptional* otherwise. If  $\lambda$  is  $p$ -regular, then we say that  $D^\lambda$  is exceptional if there are more than  $\kappa$  normal beads on runner  $i$ , and non-exceptional otherwise. We make similar definitions for partitions in  $A$ : a partition is exceptional if its abacus has more than  $\kappa$  beads on runner  $i-1$  with no bead immediately to the right, and a simple module is exceptional if the abacus display for the corresponding  $p$ -regular partition has more than  $\kappa$  conormal beads on runner  $i-1$ .

Let  $\lambda$  be a partition in  $B$ . The abacus display for  $\lambda$  has at least  $\kappa$  normal beads on runner  $i$ . We define the partition  $\Phi(\lambda)$  by moving the  $\kappa$  highest normal beads one place to the left. Note that if  $\lambda$  is non-exceptional, then  $\Phi(\lambda)$  is obtained simply by swapping runners  $i-1$  and  $i$  in the abacus display. If  $\kappa \geq 3$  then every partition is non-exceptional, and so the definition of  $\Phi$  agrees with the definition in §1.1.6. The following is standard theory for  $[3 : \kappa]$ -pairs.

**Proposition 2.3.** *Suppose  $A$  and  $B$  form a  $[3 : \kappa]$ -pair as above, and that  $\lambda$  and  $\mu$  are partitions in  $B$  with  $\mu$   $p$ -regular.*

1.  $\Phi$  is a bijection from the set of partitions in  $B$  to the set of partitions in  $A$ .
2.  $\lambda$  is  $p$ -regular if and only if  $\Phi(\lambda)$  is  $p$ -regular.



3.  $\lambda$  is non-exceptional if and only if  $\Phi(\lambda)$  is non-exceptional, and in this case

$$S^\lambda \downarrow_A \sim (S^{\Phi(\lambda)})^{\kappa!}, \quad S^{\Phi(\lambda)} \uparrow^B \sim (S^\lambda)^{\kappa!},$$

and

$$[S^\lambda : D^\mu] = [S^{\Phi(\lambda)} : D^{\Phi(\mu)}].$$

4.  $D^\mu$  is non-exceptional if and only if  $D^{\Phi(\mu)}$  is non-exceptional, and in this case

$$D^\mu \downarrow_A \cong (D^{\Phi(\mu)})^{\oplus \kappa!}, \quad D^{\Phi(\mu)} \uparrow^B \cong (D^\mu)^{\oplus \kappa!}.$$

We also wish to note that the map  $\Phi$  is compatible with the Mullineux map; the following is immediate from [11].

**Proposition 2.4.** *Suppose  $A$  and  $B$  form a  $[3 : \kappa]$ -pair, and that  $\lambda$  is a  $p$ -regular partition in  $B$ . Let  $\Phi$  be the map described above for this pair, and let  $\Phi^\sharp$  be the corresponding map for the  $[3 : \kappa]$ -pair  $(A^\sharp, B^\sharp)$ . Then  $\Phi^\sharp(\lambda^\circ) = \Phi(\lambda)^\circ$ .*

### 2.5.1 $[3 : 1]$ -pairs

We suppose now that  $\kappa = 1$ . Then there are  $3p$  exceptional partitions in each of  $A$  and  $B$ , which we label as follows (with  $j$  ranging over the set  $\{1, \dots, p\}$ ):

$$\begin{array}{cc} A & B \\ \bar{\alpha}_j = \begin{cases} \langle i \rangle & (j = i) \\ \langle i, i \rangle & (j = i - 1) \\ \langle i, j \rangle & (\text{otherwise}); \end{cases} & \alpha_j = \begin{cases} \langle i, i \rangle & (j = i) \\ \langle i, i, i \rangle & (j = i - 1) \\ \langle i, i, j \rangle & (\text{otherwise}); \end{cases} \\ \bar{\beta}_j = \begin{cases} \langle i - 1, i \rangle & (j = i) \\ \langle i - 1, i, i \rangle & (j = i - 1) \\ \langle i - 1, i, j \rangle & (\text{otherwise}); \end{cases} & \beta_j = \begin{cases} \langle i, i - 1 \rangle & (j = i) \\ \langle i - 1, i, j \rangle & (\text{otherwise}); \end{cases} \\ \bar{\gamma}_j = \begin{cases} \langle i - 1, i - 1 \rangle & (j = i) \\ \langle i - 1, i - 1, j \rangle & (\text{otherwise}); \end{cases} & \gamma_j = \begin{cases} \langle i - 1 \rangle & (j = i) \\ \langle i - 1, j \rangle & (\text{otherwise}). \end{cases} \end{array}$$

The map  $\Phi$  has the following effect on exceptional partitions:

$$\begin{aligned} \alpha_j &\mapsto \bar{\alpha}_j, \\ \beta_j &\mapsto \bar{\gamma}_j, \\ \gamma_j &\mapsto \bar{\beta}_j. \end{aligned}$$

The exceptional simple modules are  $D^{\bar{\alpha}_j}$  and  $D^{\alpha_j}$  for those  $j$  for which  $\alpha_j$  is  $p$ -regular.

The following result comes from the branching rules.

**Proposition 2.5.** *Suppose  $A$  and  $B$  form a  $[3 : 1]$ -pair as above. Then:*

1. for  $1 \leq j \leq p$ , we have

$$\begin{aligned} S^{\tilde{\alpha}_j} \uparrow^B &\sim \frac{S^{\alpha_j}}{S^{\beta_j}}, & S^{\tilde{\beta}_j} \uparrow^B &\sim \frac{S^{\alpha_j}}{S^{\gamma_j}}, & S^{\tilde{\gamma}_j} \uparrow^B &\sim \frac{S^{\beta_j}}{S^{\gamma_j}}, \\ S^{\alpha_j} \downarrow_A &\sim \frac{S^{\tilde{\alpha}_j}}{S^{\tilde{\beta}_j}}, & S^{\beta_j} \downarrow_A &\sim \frac{S^{\tilde{\alpha}_j}}{S^{\tilde{\gamma}_j}}, & S^{\gamma_j} \downarrow_A &\sim \frac{S^{\tilde{\beta}_j}}{S^{\tilde{\gamma}_j}}; \end{aligned}$$

2. if  $\alpha_j$  is  $p$ -regular, then  $D^{\tilde{\alpha}_j} \uparrow^B$  is an indecomposable self-dual module with socle isomorphic to  $D^{\alpha_j}$  and with  $[D^{\tilde{\alpha}_j} \uparrow^B : D^{\alpha_j}] = 2$ ;
3. if  $\alpha_j$  is  $p$ -regular, then  $D^{\alpha_j} \downarrow_A$  is an indecomposable self-dual module with socle isomorphic to  $D^{\tilde{\alpha}_j}$  and with  $[D^{\alpha_j} \downarrow_A : D^{\tilde{\alpha}_j}] = 2$ ;

The study of  $[3 : 1]$ -pairs is facilitated by looking at blocks of weight 1. Let  $\check{A}$  be the block of weight 1 whose abacus is obtained from that for  $A$  by moving a bead from runner  $i$  to runner  $i - 1$ , and denote the partitions in  $\check{A}$  as  $\check{\alpha}_1 \triangleleft \cdots \triangleleft \check{\alpha}_p$ . We let  $\hat{B}$  be the block of weight 1 whose abacus is obtained from that for  $B$  by moving a bead from runner  $i - 1$  to runner  $i$ . We denote the partitions in  $\hat{B}$  as  $\hat{\alpha}_1 \triangleleft \cdots \triangleleft \hat{\alpha}_p$ . The following result is also standard.

**Proposition 2.6.** *There is a permutation  $\pi \in \mathfrak{S}_p$  such that the following hold.*

1. If  $\lambda$  is a partition in  $A$ , then

$$S^{\lambda} \downarrow_{\check{A}} \cong \begin{cases} S^{\check{\alpha}_k} & (\text{if } \lambda \text{ equals } \tilde{\alpha}_{\pi(k)}, \tilde{\beta}_{\pi(k)} \text{ or } \tilde{\gamma}_{\pi(k)}) \\ 0 & (\text{if } \lambda \text{ is non-exceptional}). \end{cases}$$

If  $\lambda$  is  $p$ -regular, then

$$D^{\lambda} \downarrow_{\check{A}} \cong \begin{cases} D^{\check{\alpha}_k} & (\text{if } \lambda \text{ equals } \tilde{\alpha}_{\pi(k)}) \\ 0 & (\text{if } D^{\lambda} \text{ is non-exceptional}). \end{cases}$$

2.

$$S^{\check{\alpha}_k} \uparrow^A \sim \begin{matrix} S^{\tilde{\alpha}_{\pi(k)}} \\ S^{\tilde{\beta}_{\pi(k)}} \\ S^{\tilde{\gamma}_{\pi(k)}} \end{matrix},$$

and if  $k \geq 2$  then  $D^{\check{\alpha}_k} \uparrow^A$  is an indecomposable self-dual module with cosocle and socle both isomorphic to  $D^{\tilde{\alpha}_{\pi(k)}}$ .

3. If  $\lambda$  is a partition in  $B$ , then

$$S^{\lambda} \uparrow^{\hat{B}} \cong \begin{cases} S^{\hat{\alpha}_k} & (\text{if } \lambda \text{ equals } \alpha_{\pi(k)}, \beta_{\pi(k)} \text{ or } \gamma_{\pi(k)}) \\ 0 & (\text{if } \lambda \text{ is non-exceptional}). \end{cases}$$

If  $\lambda$  is  $p$ -regular, then

$$D^{\lambda} \uparrow^{\hat{B}} \cong \begin{cases} D^{\hat{\alpha}_k} & (\text{if } \lambda \text{ equals } \alpha_{\pi(k)}) \\ 0 & (\text{if } D^{\lambda} \text{ is non-exceptional}). \end{cases}$$

4.

$$S^{\hat{\alpha}_k} \downarrow_B \sim \begin{matrix} S^{\alpha_{\pi(k)}} \\ S^{\beta_{\pi(k)}} \\ S^{\gamma_{\pi(k)}} \end{matrix},$$

and if  $k \geq 2$  then  $D^{\hat{\alpha}_k} \downarrow_B$  is an indecomposable self-dual module with cosocle and socle both isomorphic to  $D^{\alpha_{\pi(k)}}$ .

We also need the following result on decomposition numbers.

**Proposition 2.7.** *The partitions  $\alpha_{\pi(j)}$  and  $\bar{\alpha}_{\pi(j)}$  are  $p$ -regular if and only if  $j \geq 2$ . In this case, we have*

$$[S^\lambda : D^{\alpha_{\pi(j)}}] = \begin{cases} 1 & (\text{if } \lambda \in \{\alpha_{\pi(j)}, \beta_{\pi(j)}, \gamma_{\pi(j)}, \alpha_{\pi(j-1)}, \beta_{\pi(j-1)}, \gamma_{\pi(j-1)}\}) \\ 0 & (\text{otherwise}), \end{cases}$$

and

$$[S^\lambda : D^{\bar{\alpha}_{\pi(j)}}] = \begin{cases} 1 & (\text{if } \lambda \in \{\bar{\alpha}_{\pi(j)}, \bar{\beta}_{\pi(j)}, \bar{\gamma}_{\pi(j)}, \bar{\alpha}_{\pi(j-1)}, \bar{\beta}_{\pi(j-1)}, \bar{\gamma}_{\pi(j-1)}\}) \\ 0 & (\text{otherwise}). \end{cases}$$

The following lemma follows from the Eckmann-Shapiro relations:

**Proposition 2.8** ([17, Lemma 4.11]). *Suppose  $A$  and  $B$  are as above,  $\mu$  is a  $p$ -regular partition in  $B$  and  $k \geq 2$ . If  $\text{Ext}_B^1(D^{\alpha_{\pi(k)}}, D^\mu) \neq 0$ , then exactly one of the following holds:*

- $\mu = \alpha_{\pi(k+1)}$ ;
- $\mu = \alpha_{\pi(k-1)}$ ;
- $D^\mu$  is non-exceptional and occurs in the second radical layer of  $D^{\hat{\alpha}_k} \downarrow_B$ .

**Corollary 2.9.** *Suppose  $A$  and  $B$  are as above,  $\mu$  is a  $p$ -regular partition in  $B$  and  $j \in \{1, \dots, p\} \setminus \{\pi(1)\}$ . If  $\mathcal{P}\mu = \mathcal{P}\alpha_j$  and  $\text{Ext}_B^1(D^{\alpha_j}, D^\mu) \neq 0$ , then  $\mu \supseteq \gamma_j$ .*

**Proof.** Let  $j = \pi(k)$ . It is straightforward to show (compare the proof of Lemma 2.21 below) that  $\mathcal{P}\alpha_{\pi(k+1)} \neq \mathcal{P}\alpha_{\pi(k)}$ . Thus by Proposition 2.8,  $D^\mu$  must appear as a composition factor of  $D^{\hat{\alpha}_k} \downarrow_B$ . But this is a quotient of  $S^{\hat{\alpha}_k} \downarrow_B$ , which is filtered by the Specht modules  $S^{\alpha_{\pi(k)}}$ ,  $S^{\beta_{\pi(k)}}$  and  $S^{\gamma_{\pi(k)}}$ . The result follows, since  $\alpha_{\pi(k)} \supset \beta_{\pi(k)} \supset \gamma_{\pi(k)}$ .  $\square$

We now prove some results which we shall need later.

**Lemma 2.10.** *The module  $S^{\bar{\gamma}_{\pi(k)}} \uparrow^B$  has a simple socle, isomorphic to  $D^{\alpha_{\pi(k+1)}}$  if  $k \neq p$  and to  $D^{\alpha_{\pi(p)}}$  if  $k = p$ . Furthermore, it has a simple cosocle  $D^{\beta_{\pi(k)}}$  if  $\beta_{\pi(k)}$  is  $p$ -regular.*

*Analogous statements hold for  $S^{\gamma_{\pi(k)}} \downarrow_A$ .*

**Proof.** If  $k < p$ , then  $\bar{\gamma}_{\pi(k)}$  is the least dominant partition such that  $[S^{\bar{\gamma}_{\pi(k)}} : D^{\bar{\alpha}_{\pi(k+1)}}] \neq 0$ ; thus,  $S^{\bar{\gamma}_{\pi(k)}}$  has socle  $D^{\bar{\alpha}_{\pi(k+1)}}$ . If  $k = p$ , then  $S^{\bar{\gamma}_{\pi(p)}}$  is a submodule of  $S^{\bar{\alpha}_p} \uparrow^A = D^{\bar{\alpha}_p} \uparrow^A$  and so has socle isomorphic to  $D^{\bar{\alpha}_p}$  by Proposition 2.6(2). The socle of  $S^{\bar{\gamma}_{\pi(k)}} \uparrow^B$  then follows as claimed by Frobenius reciprocity (note that  $[D^{\alpha_{\pi(l)}} \downarrow_A : D^{\bar{\alpha}_{\pi(k)}}] = 0$  whenever  $l \neq k$ ; see [17, Proposition 4.2(1)]).

Now suppose  $\beta_{\pi(k)}$  is  $p$ -regular, and  $D^\lambda$  occurs in the cosocle of  $S^{\bar{\gamma}_{\pi(k)}} \uparrow^B$ . If  $D^\lambda$  is non-exceptional, then  $\lambda = \beta_{\pi(k)}$  by Frobenius reciprocity – in fact,  $D^{\beta_{\pi(k)}}$  occurs exactly once in the cosocle. If  $D^\lambda$  is

exceptional, isomorphic to  $D^{\alpha_{\pi(l)}}$ , then Frobenius reciprocity implies that the socle  $D^{\tilde{\alpha}_{\pi(l)}}$  of  $D^{\alpha_{\pi(l)}} \downarrow_A$  is a composition factor of  $S^{\tilde{\gamma}_{\pi(k)}}$ , and that the cosocle  $D^{\tilde{\gamma}_{\pi(k)}}$  of  $S^{\tilde{\gamma}_{\pi(k)}}$  is a composition factor of  $D^{\alpha_{\pi(l)}} \downarrow_A$ . The former yields  $l = k$  or  $k + 1$ , while the latter implies that  $D^{\tilde{\gamma}_{\pi(k)}}$  is a composition factor of  $S^{\alpha_{\pi(l)}} \downarrow_A$ , and hence of either  $S^{\tilde{\alpha}_{\pi(l)}}$  or  $S^{\tilde{\beta}_{\pi(l)}}$ . Thus we get  $l = k$  or  $k + 1$ , and  $\tilde{\gamma}_{\pi(k)} \triangleright \tilde{\beta}_{\pi(l)}$ , which is impossible.

An entirely analogous argument applies to  $S^{\gamma_{\pi(k)}} \downarrow_A$ .  $\square$

**Lemma 2.11.** *Let*

$$0 \leq M \leq N \leq S^{\check{\alpha}_k} \uparrow^A$$

*be a filtration of  $(S^{\check{\alpha}_k} \uparrow^A)$  with  $(S^{\check{\alpha}_k} \uparrow^A)/N \cong S^{\tilde{\alpha}_{\pi(k)}}$ ,  $N/M \cong S^{\tilde{\beta}_{\pi(k)}}$  and  $M \cong S^{\tilde{\gamma}_{\pi(k)}}$ . If  $\tilde{\beta}_{\pi(k)}$  is  $p$ -regular, then  $N \cong S^{\gamma_{\pi(k)}} \downarrow_A$ ; in particular,  $\text{cosoc}(N) \cong D^{\tilde{\beta}_{\pi(k)}}$ .*

*An analogous statement holds for  $S^{\hat{\alpha}_k} \downarrow_B$ .*

**Proof.** We consider the cases  $k < p$  and  $k = p$  separately. First suppose that  $k < p$ . By Theorem 2.1 we have  $\text{soc}(S^{\check{\alpha}_k}) \cong D^{\check{\alpha}_{k+1}}$ ; by Frobenius reciprocity, we find that  $S^{\check{\alpha}_k} \uparrow^A$ , and hence  $N$ , has socle  $D^{\tilde{\alpha}_{\pi(k+1)}}$ . By Lemma 2.10, we also have  $\text{soc}(S^{\gamma_{\pi(k)}} \downarrow_A) \cong D^{\tilde{\alpha}_{\pi(k+1)}}$ . Regarding  $N$  and  $S^{\gamma_{\pi(k)}} \downarrow_A$  as submodules of the projective cover of  $D^{\tilde{\alpha}_{\pi(k+1)}}$ , the claim will follow once we show that  $N \geq S^{\gamma_{\pi(k)}} \downarrow_A$ , since both modules have the same composition factors with multiplicities.

The projective module  $P(D^{\tilde{\alpha}_{\pi(k+1)}})$  has a filtration by the Specht modules  $S^{\tilde{\alpha}_{\pi(k+1)}}$ ,  $S^{\tilde{\beta}_{\pi(k+1)}}$ ,  $S^{\tilde{\gamma}_{\pi(k+1)}}$ ,  $S^{\tilde{\alpha}_{\pi(k)}}$ ,  $S^{\tilde{\beta}_{\pi(k)}}$ ,  $S^{\tilde{\gamma}_{\pi(k)}}$ . Since  $\tilde{\beta}_{\pi(k)}$  does not dominate any of  $\tilde{\alpha}_{\pi(k+1)}$ ,  $\tilde{\beta}_{\pi(k+1)}$ ,  $\tilde{\gamma}_{\pi(k+1)}$  or  $\tilde{\alpha}_{\pi(k)}$ , we have

$$[P(D^{\tilde{\alpha}_{\pi(k+1)}}) : D^{\tilde{\beta}_{\pi(k)}}] = [N : D^{\tilde{\beta}_{\pi(k)}}] (= 2 = [S^{\gamma_{\pi(k)}} \downarrow_A : D^{\tilde{\beta}_{\pi(k)}}]).$$

So  $N \geq S^{\gamma_{\pi(k)}} \downarrow_A$  by [18, Lemma 2.1], since  $\text{cosoc}(S^{\gamma_{\pi(k)}} \downarrow_A) \cong D^{\tilde{\beta}_{\pi(k)}}$  by Lemma 2.10.

Now we suppose  $k = p$ . Again, we begin by noting that  $N$  and  $S^{\gamma_{\pi(p)}} \downarrow_A$  have isomorphic simple socles. This time we have  $\text{soc}(S^{\gamma_{\pi(p)}} \downarrow_A) \cong D^{\tilde{\alpha}_{\pi(p)}}$  by Lemma 2.10, while  $\text{soc}(S^{\check{\alpha}_p} \uparrow^A) \cong D^{\tilde{\alpha}_{\pi(p)}}$  (by Proposition 2.6(2) as  $S^{\check{\alpha}_p} = D^{\check{\alpha}_p}$ ) implies that  $\text{soc}(N) \cong D^{\tilde{\alpha}_{\pi(p)}}$ . Now we regard  $N$  and  $S^{\gamma_{\pi(p)}} \downarrow_A$  as submodules of  $P(D^{\tilde{\alpha}_{\pi(p)}})$ , and we need to show that  $N \geq S^{\gamma_{\pi(p)}} \downarrow_A$ .

Note first that  $P(D^{\tilde{\alpha}_{\pi(p)}})/(S^{\check{\alpha}_p} \uparrow^A) \cong (P(D^{\check{\alpha}_p})/S^{\check{\alpha}_p}) \uparrow^A$  has a simple socle  $D^{\tilde{\alpha}_{\pi(p-1)}}$  by Frobenius reciprocity. Since

$$\begin{aligned} \left[ \frac{S^{\check{\alpha}_p} \uparrow^A + S^{\gamma_{\pi(p)}} \downarrow_A}{S^{\check{\alpha}_p} \uparrow^A} : D^{\tilde{\alpha}_{\pi(p-1)}} \right] &= \left[ \frac{S^{\gamma_{\pi(p)}} \downarrow_A}{(S^{\gamma_{\pi(p)}} \downarrow_A \cap S^{\check{\alpha}_p} \uparrow^A)} : D^{\tilde{\alpha}_{\pi(p-1)}} \right] \\ &\leq [S^{\gamma_{\pi(p)}} \downarrow_A : D^{\tilde{\alpha}_{\pi(p-1)}}] = 0, \end{aligned}$$

we see that  $S^{\gamma_{\pi(p)}} \downarrow_A \leq S^{\check{\alpha}_p} \uparrow^A$ . Now we may apply [18, Lemma 2.1] again to get  $S^{\gamma_{\pi(p)}} \downarrow_A \leq N$ , since  $\text{cosoc}(S^{\gamma_{\pi(p)}} \downarrow_A) \cong D^{\tilde{\beta}_{\pi(p)}}$  and  $[N : D^{\tilde{\beta}_{\pi(p)}}] = [S^{\check{\alpha}_p} \uparrow^A : D^{\tilde{\beta}_{\pi(p)}}] (= 2)$ .

An entirely analogous argument applies to  $S^{\hat{\alpha}_k} \downarrow_B$ .  $\square$

**Corollary 2.12.** *Suppose  $A$  and  $B$  are as above, and  $k \geq 2$ , and  $\beta_{\pi(k)}$  is  $p$ -regular. If  $\text{Ext}_B^1(D^{\alpha_{\pi(k)}}, D^\mu) \neq 0$ , then exactly one of the following holds:*

- $\mu = \alpha_{\pi(k-1)}$ ;
- $\mu = \alpha_{\pi(k+1)}$ ;
- $\mu = \beta_{\pi(k)}$ ;

- $D^u$  is non-exceptional and occurs in the second radical layer of  $S^{\alpha_{\pi(k)}}$ .

In particular,  $\text{Ext}_B^1(D^{\alpha_{\pi(k)}}, D^{\alpha_{\pi(k)}}) = 0 = \text{Ext}_B^1(D^{\alpha_{\pi(k)}}, D^{\gamma_{\pi(k)}})$ .

**Proof.** This follows directly from Proposition 2.8 and Lemma 2.11.  $\square$

### 2.5.2 $[3 : 2]$ -pairs

Now we consider  $[3 : 2]$ -pairs. These are much easier to deal with, since there are fewer exceptional partitions and we need fewer basic results. If  $A$  and  $B$  form a  $[3 : 2]$ -pair with abacuses as above, then there are four exceptional partitions in each of  $A$  and  $B$ , which we label as follows:

$A$	$B$
$\bar{\alpha} = \langle i \rangle;$	$\alpha = \langle i, i, i \rangle;$
$\bar{\beta} = \langle i, i - 1 \rangle;$	$\beta = \langle i - 1, i, i \rangle;$
$\bar{\gamma} = \langle i, i - 1, i - 1 \rangle;$	$\gamma = \langle i - 1, i \rangle;$
$\bar{\delta} = \langle i - 1, i - 1, i - 1 \rangle;$	$\delta = \langle i - 1 \rangle.$

The exceptional simple modules are  $D^{\bar{\alpha}}$  and  $D^{\alpha}$ , and the effect of the map  $\Phi$  on exceptional partitions is

$$\begin{aligned} \alpha &\mapsto \bar{\alpha}, \\ \beta &\mapsto \bar{\delta}, \\ \gamma &\mapsto \bar{\gamma}, \\ \delta &\mapsto \bar{\beta}. \end{aligned}$$

The following is a consequence of the classical and modular branching rules.

#### Proposition 2.13.

1.

$$\begin{array}{cccc} \begin{array}{c} S^{\alpha} \\ S^{\alpha} \\ S^{\beta} \\ S^{\gamma} \\ S^{\gamma} \end{array} & \begin{array}{c} S^{\alpha} \\ S^{\alpha} \\ S^{\beta} \\ S^{\delta} \\ S^{\delta} \end{array} & \begin{array}{c} S^{\alpha} \\ S^{\alpha} \\ S^{\gamma} \\ S^{\delta} \\ S^{\delta} \end{array} & \begin{array}{c} S^{\beta} \\ S^{\beta} \\ S^{\gamma} \\ S^{\delta} \\ S^{\delta} \end{array} \\ S^{\bar{\alpha}} \uparrow^B \sim & S^{\bar{\beta}} \uparrow^B \sim & S^{\bar{\gamma}} \uparrow^B \sim & S^{\bar{\delta}} \uparrow^B \sim \\ \begin{array}{c} S^{\bar{\alpha}} \\ S^{\bar{\alpha}} \\ S^{\bar{\beta}} \\ S^{\bar{\gamma}} \\ S^{\bar{\gamma}} \end{array} & \begin{array}{c} S^{\bar{\alpha}} \\ S^{\bar{\alpha}} \\ S^{\bar{\beta}} \\ S^{\bar{\delta}} \\ S^{\bar{\delta}} \end{array} & \begin{array}{c} S^{\bar{\alpha}} \\ S^{\bar{\alpha}} \\ S^{\bar{\gamma}} \\ S^{\bar{\delta}} \\ S^{\bar{\delta}} \end{array} & \begin{array}{c} S^{\bar{\beta}} \\ S^{\bar{\beta}} \\ S^{\bar{\gamma}} \\ S^{\bar{\delta}} \\ S^{\bar{\delta}} \end{array} \\ S^{\alpha} \downarrow_A \sim & S^{\beta} \downarrow_A \sim & S^{\gamma} \downarrow_A \sim & S^{\delta} \downarrow_A \sim \end{array}$$

2.  $D^{\bar{\alpha}} \uparrow^B \cong N \oplus N$ , where  $N$  is a self-dual indecomposable module with socle isomorphic to  $D^{\alpha}$  and with  $[N : D^{\alpha}] = 3$ .

3.  $D^\alpha \downarrow_A \cong M \oplus M$ , where  $M$  is a self-dual indecomposable module with socle isomorphic to  $D^{\tilde{\alpha}}$  and with  $[M : D^{\tilde{\alpha}}] = 3$ .

We use two blocks of weight 0 to help us understand the pair  $(A, B)$ . Let  $\check{A}$  be the weight 0 block whose abacus is obtained from the abacus for  $A$  by moving a bead from runner  $i$  to runner  $i - 1$ , and let  $\hat{B}$  be the weight 0 block obtained from  $B$  by moving a bead from runner  $i - 1$  to runner  $i$ . Let  $\check{\alpha}$  denote the unique partition in  $\check{A}$ , and  $\hat{\alpha}$  the unique partition in  $\hat{B}$ .

**Proposition 2.14.**

1. If  $\lambda$  is a partition in  $A$ , then

$$S^\lambda \downarrow_{\check{A}} \cong \begin{cases} S^{\check{\alpha}} & (\text{if } \lambda \text{ equals } \bar{\alpha}, \bar{\beta}, \bar{\gamma} \text{ or } \bar{\delta}) \\ 0 & (\text{otherwise}). \end{cases}$$

If  $\lambda$  is  $p$ -regular, then

$$D^\lambda \downarrow_{\check{A}} \cong \begin{cases} D^{\check{\alpha}} & (\text{if } \lambda \text{ equals } \bar{\alpha}) \\ 0 & (\text{otherwise}). \end{cases}$$

- 2.

$$S^{\check{\alpha}} \uparrow^A \sim \begin{matrix} S^{\bar{\alpha}} \\ S^{\bar{\beta}} \\ S^{\bar{\gamma}} \\ S^{\bar{\delta}} \end{matrix},$$

and  $D^{\check{\alpha}} \uparrow^A$  is an indecomposable self-dual module with cosocle and socle both isomorphic to  $D^{\bar{\alpha}}$ .

3. If  $\lambda$  is a partition in  $B$ , then

$$S^\lambda \uparrow^{\hat{B}} \cong \begin{cases} S^{\hat{\alpha}} & (\text{if } \lambda \text{ equals } \alpha, \beta, \gamma \text{ or } \delta) \\ 0 & (\text{otherwise}). \end{cases}$$

If  $\lambda$  is  $p$ -regular, then

$$D^\lambda \uparrow^{\hat{B}} \cong \begin{cases} D^{\hat{\alpha}} & (\text{if } \lambda \text{ equals } \alpha) \\ 0 & (\text{otherwise}). \end{cases}$$

- 4.

$$S^{\hat{\alpha}} \downarrow_B \sim \begin{matrix} S^\alpha \\ S^\beta \\ S^\gamma \\ S^\delta \end{matrix},$$

and  $D^{\hat{\alpha}} \downarrow_B$  is an indecomposable self-dual module with cosocle and socle both isomorphic to  $D^\alpha$ .

**Lemma 2.15.** Suppose  $A$  and  $B$  are as above and that  $\mu$  is a  $p$ -regular partition in  $B$ . If  $[P(D^\alpha) : D^\mu] > 0$ , then  $\mu \triangleright \delta$ . In particular, we have  $\text{Ext}_B^1(D^\alpha, D^\mu) = 0$  unless  $\mu \triangleright \delta$ .

**Proof.** Since  $\hat{B}$  has weight 0, we have  $P(D^{\hat{\alpha}}) = S^{\hat{\alpha}}$ . By Frobenius reciprocity we have

$$P(D^{\alpha}) = P(D^{\hat{\alpha}}) \downarrow_{B \sim} \begin{matrix} S^{\alpha} \\ S^{\beta} \\ S^{\gamma} \\ S^{\delta} \end{matrix},$$

and so if  $[P(D^{\alpha}) : D^{\mu}] \neq 0$ , then  $\mu$  dominates one of  $\alpha, \beta, \gamma, \delta$ . But in fact  $\alpha \triangleright \beta \triangleright \gamma \triangleright \delta$ , so  $\mu \triangleright \delta$ .  $\square$

**Lemma 2.16** ([18, Lemma 5.4(2) and Corollary 5.5(3)]). *Let*

$$0 \leq L \leq M \leq N \leq S^{\check{\alpha}} \uparrow^A$$

*be a filtration of  $(S^{\check{\alpha}} \uparrow^A)$  with  $(S^{\check{\alpha}} \uparrow^A)/N \cong S^{\bar{\alpha}}$ ,  $N/M \cong S^{\bar{\beta}}$ ,  $M/L \cong S^{\bar{\gamma}}$  and  $L \cong S^{\bar{\delta}}$ . If  $\bar{\beta}$  is  $p$ -regular, then  $N^{\oplus 2} \cong S^{\bar{\delta}} \downarrow_A$ ; in particular,  $\text{cosoc}(N) \cong D^{\bar{\beta}}$ .*

*An analogous statement holds for  $S^{\hat{\alpha}} \downarrow_B$ .*

**Corollary 2.17.** *Suppose  $A$  and  $B$  are as above, and that  $\beta$  is  $p$ -regular. If  $\text{Ext}_B^1(D^{\alpha}, D^{\mu}) \neq 0$ , then either  $\mu = \beta$  or  $D^{\mu}$  lies in the second radical layer of  $S^{\alpha}$ .*

*In particular,  $\text{Ext}_B^1(D^{\alpha}, D^{\alpha}) = \text{Ext}_B^1(D^{\alpha}, D^{\gamma}) = \text{Ext}_B^1(D^{\alpha}, D^{\delta}) = 0$ .*

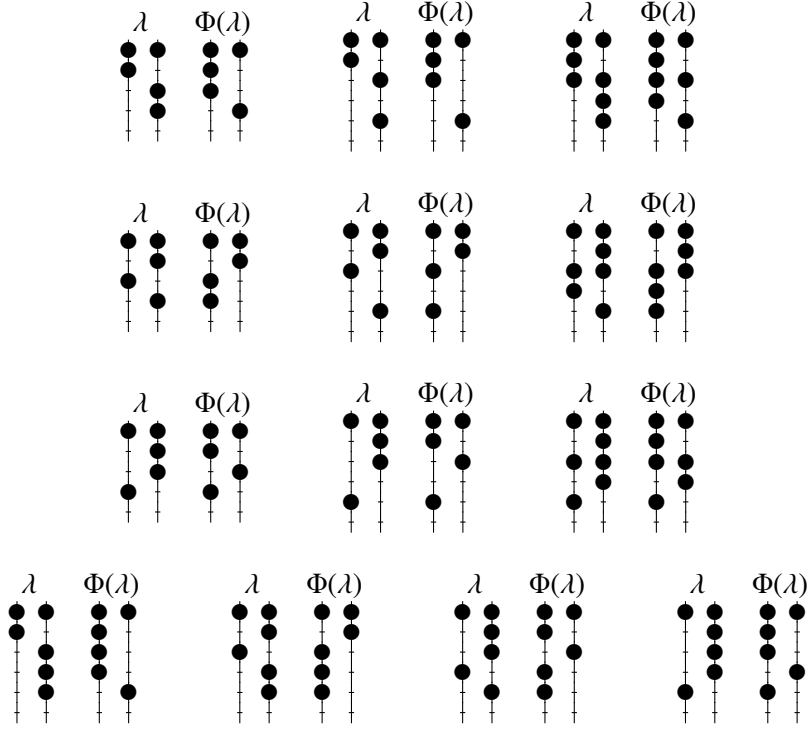
## 2.6 $[3 : \kappa]$ -pairs and parity

The next result is not standard, but is essential to this paper.

**Proposition 2.18.** *Suppose  $A$  and  $B$  form a  $[3 : \kappa]$ -pair as above, and that  $\lambda$  is a  $p$ -regular partition in  $B$ . If  $\kappa = 1$  and  $\lambda$  is a partition of the form  $\alpha_j$ , then  $\mathcal{P}\Phi(\lambda) \neq \mathcal{P}\lambda$ . Otherwise,  $\mathcal{P}\Phi(\lambda) = \mathcal{P}\lambda$ .*

**Proof.** This is really a matter of comparing abacus displays. When we remove a rim  $p$ -hook from  $[\lambda]$ , we move a bead in the abacus up one space, from position  $x$  to position  $x - p$ , say. The leg-length of the removed hook is the number of beads in positions  $x - 1, x - 2, \dots, x - p + 1$ . If position  $x$  does not lie on runner  $i - 1$  or runner  $i$ , then we may also remove a rim  $p$ -hook from  $\Phi(\lambda)$  by moving a bead from position  $x$  to position  $x - p$ , and the leg-length will be the same. So we may ignore these rim hooks from  $\lambda$  and  $\Phi(\lambda)$ , and concentrate only on the beads which may be moved up on runners  $i - 1$  and  $i$ . If  $\lambda$  is a non-exceptional partition, and we can remove a rim hook by sliding a bead up from position  $x$  to  $x - p$  on runner  $i$ , then we can remove a rim hook from  $\Phi(\lambda)$  by sliding a bead up from position  $x - 1$  to  $x - p - 1$ , and the leg-length will be just the same. A similar statement applies if we remove slide a bead up on runner  $i - 1$  for a non-exceptional partition  $\lambda$ , and so we suppose  $\lambda$  is exceptional. We now

examine all possible configurations of these two runners, and we may verify the proposition.



□

This enables us to prove the following important result.

**Proposition 2.19.** *For any  $p$ -regular partition  $\lambda$  of weight 3, we have  $\mathcal{P}\lambda^\diamond \neq \mathcal{P}\lambda$ .*

In order to prove this, we need to use induction to Rouquier blocks. We define a *Scopes sequence* to be a sequence  $B_0, \dots, B_r$  of weight 3 blocks such that  $B_{j-1}$  and  $B_j$  form a  $[3 : \kappa_j]$ -pair, for  $j = 1, \dots, r$ . Lemma 3.1 of [8] states that for any weight 3 block  $B$  there is a Scopes sequence  $B = B_0, \dots, B_r$  with  $B_r$  a Rouquier block.

**Proof of Proposition 2.19.** The result may be checked for Rouquier blocks using the results of [3]. Now suppose  $\lambda$  lies in a weight 3 block  $B$ , and take a Scopes sequence  $B = B_0, \dots, B_r$  such that  $B_r$  is Rouquier. The partition  $\lambda^\diamond$  lies in the block  $B^\sharp$ , and the blocks  $B^\sharp = B_0^\sharp, \dots, B_r^\sharp$  form a Scopes sequence, with  $B_r^\sharp$  Rouquier. The result now follows using Proposition 2.18 and Proposition 2.4 – note that  $\lambda$  is a partition of the form  $\alpha_j$  for the  $[3 : 1]$ -pair  $(A, B)$  if and only if  $\lambda^\diamond$  is of the form  $\alpha_k$  for the  $[3 : 1]$ -pair  $(A^\sharp, B^\sharp)$ . □

Of course, Proposition 2.19 implies that Theorem 1.1 holds for partitions  $\lambda$  and  $\mu$  if and only if it holds for  $\lambda^\diamond$  and  $\mu^\diamond$ . We shall make frequent use of this fact later in the paper. A version of this proposition holds for arbitrary weight  $w$ , where we replace ‘ $\neq$ ’ with ‘ $=$ ’ if  $w$  is even. This may be proved using the  $\nu$ -decomposition numbers defined by Lascoux, Leclerc and Thibon; we do not include details here.

## 2.7 Semi-simple induction

Suppose  $B_0, \dots, B_r$  is a Scopes sequence of weight 3 blocks. For  $i = 1, \dots, r$ , let  $\Phi_i$  be the map described above from the set of partitions in  $B_i$  to the set of partitions in  $B_{i-1}$ . If  $\lambda$  is a  $p$ -regular partition



in  $B_0$ , then we say that  $\lambda$  induces semi-simply to  $B_r$  (via  $B_0, \dots, B_r$ ) if there are  $p$ -regular partitions  $\lambda = \lambda^0, \lambda^1, \dots, \lambda^r$  lying in  $B_0, \dots, B_r$  respectively such that for each  $i$   $D^{\lambda^i}$  is a non-exceptional simple module for the pair  $(B_{i-1}, B_i)$ , with  $\Phi_i(\lambda^i) = \lambda^{i-1}$ .

In view of Proposition 1.5 and Proposition 2.18, we can see that if  $\lambda$  and  $\mu$  are  $p$ -regular partitions lying in a weight 3 block  $B$  and if there is a Scopes sequence  $B = B_0, \dots, B_r$  with  $B_r$  a Rouquier block such that  $\lambda$  and  $\mu$  both induce semi-simply to  $B_r$  via  $B_0, \dots, B_r$ , then Theorem 1.1 holds for  $\lambda$  and  $\mu$ . We now give a list of the weight 3 partitions which induce semi-simply to Rouquier blocks. In order to do this, we need to introduce some additional notation, which is a slight re-casting of Richards's ‘pyramid’ notation [21]. Suppose  $B$  is a block of  $\mathfrak{S}_n$ , and fix an abacus display for  $B$ . We define a total order  $\leq$  on  $\{1, \dots, p\}$  by putting  $i < j$  if and only if

- $i < j$  and there are at least as many beads on runner  $j$  as on runner  $i$ , or
- $i > j$  and there are more beads on runner  $j$  than runner  $i$ .

We extend this order to the whole of  $\mathbb{Z}$  by stipulating that  $i \leq j$  if and only if  $i \leq j$ , when  $i$  and  $j$  do not both lie in  $\{1, \dots, p\}$ . For any integer  $i$ , we define  $i^+$  to be the smallest (in the  $\leq$  order) integer such that  $i < i^+$ .

Now for every pair of integers  $i \leq j$ , we define an integer  ${}_i a_j$  as follows:

- if  $i = j$ , put  ${}_i a_j = 0$ ;
- if  $i < j$  and  $i$  and  $j$  do not both lie in  $\{1, \dots, p\}$ , put  ${}_i a_j = w - 1$ ;
- if  $1 \leq i < j \leq p$  with  $i < j$ , let  ${}_i a_j$  be the difference between the number of beads on runner  $i$  and runner  $j$ ;
- if  $1 \leq j < i \leq p$  with  $i < j$ , let  ${}_i a_j$  be the difference between the number of beads on runner  $i$  and runner  $j$  minus 1;

We use shorthand such as  ${}_i 0_j 1_k$  to indicate that  ${}_i a_j = 0$  and  ${}_j a_k = 1$ , and  ${}_i 1^+_k$  to indicate that  ${}_i a_k \geq 1$ . Now we can give a list of partitions of weight 3 which induce semi-simply to a Rouquier block.

**Proposition 2.20.** *If  $\lambda$  is a  $p$ -regular partition of weight 3 lying in a block  $B$  of  $\mathfrak{S}_n$ , then  $\lambda$  induces semi-simply to some Rouquier block if and only if  $\lambda$  and the pyramid for  $B$  satisfy one of the following sets of conditions.*

$\lambda$	Conditions on the pyramid for $B$
$\langle i \rangle$	${}_i 2^+_{i^+}$
$\langle i, i \rangle$	${}_i 1^+_{i^+}$
$\langle i, i, i \rangle$	—
$\langle i, i^+ \rangle$	${}_i 1^+_{i^{++}}$
$\langle i, j \rangle$ ( $i^+ < j$ )	${}_i 1^+_{i^+}, {}_i 2^+_j$
$\langle j, i \rangle$ ( $i < j$ )	${}_i 1^+_j 1^+_{j^+}$
$\langle i, i, i^+ \rangle$	—
$\langle i, i, j \rangle$ ( $i^+ < j$ )	${}_i 1^+_j$
$\langle i, j, j \rangle$ ( $i < j$ )	${}_i 2^+_j$ or $({}_i 1^+_j, {}_i 1^-_j, {}_i 0_j)$
$\langle i, i^+, i^{++} \rangle$	${}_i 0_{i^{++}}$ or ${}_i 1^+_{i^+}$ or ${}_i 1^+_{i^{++}}$
$\langle i, i^+, k \rangle$ ( $i^{++} < k$ )	${}_i 1^+_k$
$\langle i, j, j^+ \rangle$ ( $i^+ < j$ )	${}_i 1^+_j$
$\langle i, j, k \rangle$ ( $i^+ < j, j^+ < k$ )	${}_i 1^+_j 1^+_k$

**Proof.** This table in the proposition is a reformulation of that in [10, Proposition 3.4], and the result may be deduced from there. Alternatively, we may check it directly. Let  $\mathcal{S}$  denote the set of  $p$ -regular partitions described. It suffices to check the following two statements.

- (1) is easy to check, given that a weight 3 block  $B$  is Rouquier if and only if  $i a_j \geq 2$  for every  $i, j$  with  $i < j$ . (2) is laborious but straightforward, given the explicit description of the map  $\Phi$  described above.  $\square$

$$\langle 2, 4 \mid 3, 5, 4^2, 3 \rangle =$$
$$\langle 3 \mid 3, 4^2, 3, 5 \rangle =$$

Here we prove some results which will help us to show that certain  $\text{Ext}^1$ -spaces are zero by examining blocks of smaller weight. Suppose  $B$  is a weight 3 block of  $\mathfrak{S}_n$ , and  $\mu$  is a  $p$ -regular partition in  $B$ . If  $C$  is a block of  $\mathfrak{S}_{n-1}$  of weight less than 3 and if  $D^\mu \downarrow_C \neq 0$ , then we say that  $\mu$  is *lowerable* (to  $C$ ). In this case, it is easy to see from the modular branching rules that  $D^\mu \downarrow_C$  is a simple module, say  $D^{\mu^-}$ .

If  $\mu$  is lowerable with  $D^\mu \downarrow_{\mathbb{C}} \cong D^{\mu^-}$ , then  $S^{\mu^-} \uparrow^B$  has a filtration by Specht modules  $S^{\mu^{(1)}}, \dots, S^{\mu^{(t)}}$ , where  $\mu^{(1)}, \dots, \mu^{(t)}$  are given by the Branching Rule. The partitions  $\mu^{(1)}, \dots, \mu^{(t)}$  are totally ordered by

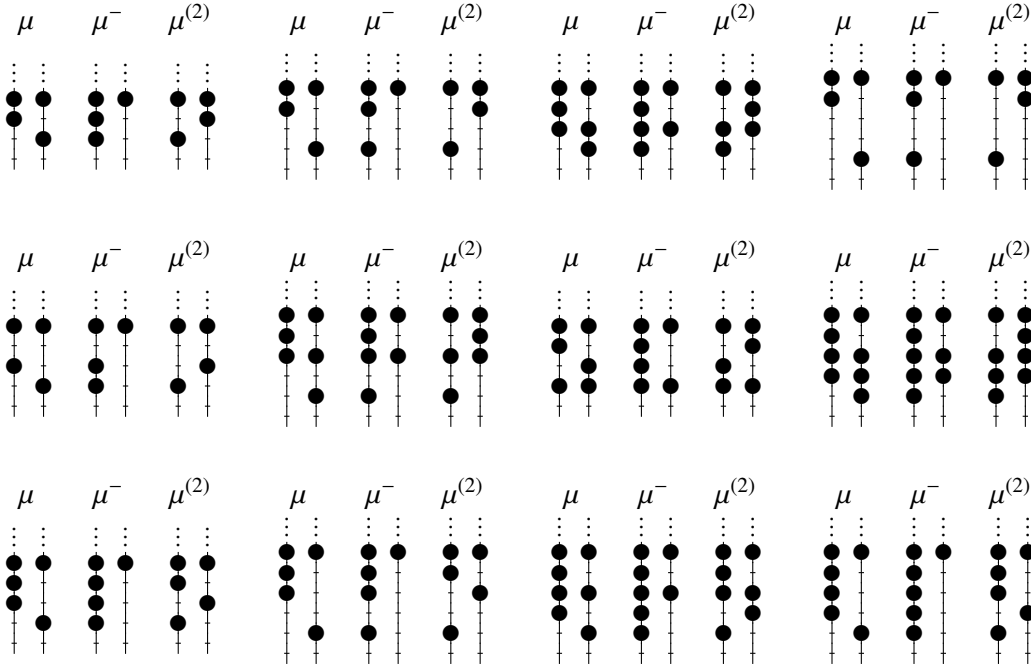
the Jantzen–Schaper order, and we suppose that  $\mu^{(1)} \triangleright \cdots \triangleright \mu^{(t)}$ , so that  $\mu^{(1)} = \mu$ . If  $w$  is the weight of  $C$ , then we have  $t = 4 - w \geq 2$ , and we shall be interested in the partition  $\mu^{(2)}$ . The next lemma relates the parities of  $\mu$ ,  $\mu^-$  and  $\mu^{(2)}$ .

**Lemma 2.21.** *Suppose  $\lambda$  and  $\mu$  are  $p$ -regular partitions lying in a weight 3 block of  $\mathfrak{S}_n$ , and that  $C$  is a block of  $\mathfrak{S}_{n-1}$  of weight less than 3.*

1. *If  $\lambda$  and  $\mu$  are both lowerable to  $C$ , then  $\mathcal{P}\lambda = \mathcal{P}\mu$  if and only if  $\mathcal{P}\lambda^- = \mathcal{P}\mu^-$ .*
2. *If  $\mu$  is lowerable to  $C$ , then  $\mathcal{P}\mu \neq \mathcal{P}\mu^{(2)}$ .*

**Proof.** Suppose the abacus display for  $C$  is obtained from that for  $B$  by moving a bead from runner  $i$  to runner  $i - 1$ . Suppose that in the abacus display for the core of  $B$ , the lowest bead on runner  $i - 1$  lies in position  $x$ , and the lowest bead on runner  $i$  lies in position  $y$ . We list all possible configurations of runners  $i - 1$  and  $i$  of  $\mu$ , when  $\mu$  is lowerable to  $C$ ; in each case, we illustrate  $\mu^-$  and  $\mu^{(2)}$  as well.

There are twelve possible configurations, as follows.



To prove (1), we must show that the condition  $\mathcal{P}\mu = \mathcal{P}\mu^-$  depends only on the abacus display for the core of  $B$ . If there is a bead on a runner other than  $i - 1$  or  $i$  of the abacus display for  $\mu$  which may be moved up one space, then the corresponding bead on the display for  $\mu^-$  may also be moved up one space, and the leg-lengths of the corresponding rim hooks will be the same. So we may move all such beads up in both abacus displays without affecting the difference between the parities of  $\mu$  and  $\mu^-$ . Then from the above diagrams, we see that  $\mathcal{P}\mu = \mathcal{P}\mu^-$  if and only if the number of beads in positions  $y + 1, y + 2, \dots, x + p - 1$  of the abacus display for the core of  $B$ , other than those lying on runners  $i - 1$  and  $i$ , is even.

For (2), we may again move beads up on runners other than  $i - 1$  and  $i$  of the abacus display for  $\mu$ , and move the corresponding beads up in the display for  $\mu^{(2)}$ , without affecting the difference between the parities of  $\mu$  and  $\mu^{(2)}$ . We may then verify that  $\mathcal{P}\mu \neq \mathcal{P}\mu^{(2)}$  directly from the above diagrams.  $\square$

For the remainder of this subsection, we assume that  $B$  is a weight 3 block of  $\mathfrak{S}_n$ , and  $\mu$  is a  $p$ -regular partition in  $B$  lowerable to a block  $C$  of  $\mathfrak{S}_{n-1}$ , with  $D^\mu \downarrow_C \cong D^{\mu^-}$ . Let  $\mu = \mu^{(1)} \triangleright \cdots \triangleright \mu^{(t)}$  be the partitions

in  $B$  that can be obtained from  $\mu^-$  by moving a bead one place to the right. Thus these partitions are precisely those labelling the Specht modules which appear as factors of the filtration of  $S^{\mu^-} \uparrow^B$  given by the Branching Rule.

**Proposition 2.22.** *Suppose  $\lambda$  is a  $p$ -regular partition in  $B$  with  $\mathcal{P}\lambda = \mathcal{P}\mu$ . Then  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$  unless  $D^\lambda$  lies in the second radical layer of  $D^{\mu^-} \uparrow^B$ .*

**Proof.** By Lemma 2.21(1), we have  $\text{Ext}_C^1(D^{\mu^-}, D^\lambda \downarrow_C) = 0$ , even if  $D^\lambda \downarrow_C$  is non-zero, since the  $\text{Ext}^1$ -quiver of  $C$  is bipartite. Hence  $\text{Ext}_B^1(D^{\mu^-} \uparrow^B, D^\lambda) = 0$ . Thus if  $\text{Ext}_B^1(D^\lambda, D^\mu) \neq 0$ , then  $D^\lambda$  lies in the second radical layer of  $D^{\mu^-} \uparrow^B$ .  $\square$

We say that  $\mu$  is *regular-lowerable* if the partition  $\mu^{(2)}$  is  $p$ -regular.

**Proposition 2.23.** *Suppose  $\mu$  is regular-lowerable. If  $D^\lambda$  is a simple module lying in the second radical layer of  $D^{\mu^-} \uparrow^B$ , then either  $\lambda = \mu^{(2)}$  or  $D^\lambda$  lies in the second radical layer of  $S^\mu$ .*

**Proof.** Suppose that

$$S^{\mu^-} \uparrow^B = F_0 \geq \dots \geq F_t = 0$$

is the filtration given by the Branching Rule, with  $F_{i-1}/F_i \cong S^{\mu^{(i)}}$  for  $i = 1, \dots, t$ . Since  $D^{\mu^-} \uparrow^B$  is a quotient of  $S^{\mu^-} \uparrow^B$ , it suffices to show that  $F_1$  has a simple cosocle  $D^{\mu^{(2)}}$ . If  $C$  has weight 2 (so that  $t = 2$ ), then this is trivial, since in this case  $F_1 = S^{\mu^{(2)}}$ .

Now suppose that  $C$  has weight 1, so that  $t = 3$ . The abacus display for  $B$  then has one more bead on runner  $i - 1$  than runner  $i$ ; if we let  $D$  be the weight 3 block obtained by swapping runners  $i - 1$  and  $i$ , then  $B$  and  $D$  form a  $[3 : 1]$ -pair. The fact that  $D^{\mu^-} \neq 0$  means that  $\mu$  is a partition of the form  $\bar{\alpha}_k$  for this pair, by Proposition 2.6, with  $\mu^- = \check{\alpha}_{\pi^{-1}(k)}$  and  $\mu^{(2)} = \bar{\beta}_k$ . By Lemma 2.11, we have  $F_1 \cong S^{\gamma_k} \downarrow_B$ , and thus have a simple cosocle  $D^{\bar{\beta}_k} = D^{\mu^{(2)}}$ .

The case where  $C$  has weight 0 (so that  $t = 4$ ) follows in a very similar way, using Lemma 2.16.  $\square$

**Corollary 2.24.** *Let  $\lambda$  be a  $p$ -regular partition in  $B$  with  $\mathcal{P}\lambda = \mathcal{P}\mu$ . If either of the following conditions holds, then  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$ :*

1.  $\lambda \not\supseteq \mu^{(i)}$ ;
2.  $\mu$  is regular-lowerable to  $C$ , and  $\lambda \not\supseteq \mu$ .

**Proof.** Suppose  $\text{Ext}_B^1(D^\lambda, D^\mu) \neq 0$ . Then by Proposition 2.22,  $D^\lambda$  lies in the second radical layer of  $D^{\mu^-} \uparrow^B$ . By the Branching Rule,  $S^{\mu^-} \uparrow^B$  has a filtration in which the factors are precisely  $S^{\mu^{(1)}}, S^{\mu^{(2)}}, \dots, S^{\mu^{(t)}}$ . Since  $D^{\mu^-} \uparrow^B$  is a quotient of  $S^{\mu^-} \uparrow^B$ , we see that  $[S^{\mu^{(i)}} : D^\lambda] \neq 0$  for some  $i$ , so that  $\lambda \supseteq \mu^{(i)} \supseteq \mu^{(t)}$ .

If  $\mu$  is regular-lowerable, then by Proposition 2.23 and Lemma 2.21(2), this implies that  $D^\lambda$  lies in the second radical layer of  $S^\mu$ , so that  $\lambda \supset \mu$ .  $\square$

### 3 Blocks forming two $[3 : \kappa]$ -pairs

In this section, we suppose that  $B$  is a weight 3 block of  $\mathfrak{S}_n$ , and that there are distinct blocks  $A_1$  and  $A_2$  of  $\mathfrak{S}_{n-\kappa_1}$  and  $\mathfrak{S}_{n-\kappa_2}$  respectively, such that  $A_r$  forms a  $[3 : \kappa_r]$ -pair with  $B$ , for  $r = 1, 2$ ; we suppose

that there is no other block  $A$  forming a  $[3 : \kappa]$ -pair with  $B$ . By induction, we assume that Theorem 1.1 holds for  $A_1$  and  $A_2$ .

Suppose the abacus for  $A_1$  is obtained from the abacus for  $B$  by interchanging runners  $i - 1$  and  $i$ , while the abacus for  $A_2$  is obtained by interchanging runners  $j - 1$  and  $j$ . Re-labelling  $A_1$  and  $A_2$  and adjusting the number of beads in the abacus display if necessary, we may assume that  $i < j$  and that there are at least as many beads on runner  $j - 1$  as on runner  $i - 1$ .

Let  $\lambda$  and  $\mu$  be  $p$ -regular partitions in  $B$  with  $\mathcal{P}\lambda = \mathcal{P}\mu$ . If neither  $D^\lambda$  nor  $D^\mu$  is exceptional for the  $[3 : \kappa_r]$ -pair  $(A_r, B)$ , then by induction, Proposition 2.18 and the Eckmann–Shapiro relations, we have  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$ . So we assume that one of  $D^\lambda$  and  $D^\mu$  is exceptional for the pair  $(A_1, B)$  and one is exceptional for the pair  $(A_2, B)$ ; in particular, we have  $\kappa_1, \kappa_2 \leq 2$ . Since a simple module in  $B$  cannot be exceptional for both of these pairs, we find that (without loss of generality)  $D^\lambda$  is exceptional for the pair  $(A_1, B)$  and  $D^\mu$  is exceptional for the pair  $(A_2, B)$ . We use some additional notation for exceptional simple modules: if  $(A, B)$  is a  $[3 : \kappa]$ -pair and  $D^\mu$  is an exceptional simple module in  $B$ , then we write

$$\mu^\downarrow = \begin{cases} \gamma_j & (\text{if } \kappa = 1 \text{ and } \mu = \alpha_j) \\ \delta & (\text{if } \kappa = 2 \text{ and } \mu = \alpha). \end{cases}$$

By Corollary 2.9 and Lemma 2.15, we must have both  $\lambda \triangleright \mu^\downarrow$  and  $\mu \triangleright \lambda^\downarrow$  if  $\text{Ext}_B^1(D^\lambda, D^\mu) \neq 0$ . We consider the possible values of  $\kappa_1$  and  $\kappa_2$ .

### 3.1 The case $K_1 = K_2 = 2$

This case is easily dealt with. The abacus for  $B$  can take one of three different forms:

1.

$$\langle 3^{a+1}, 5^{b+1}, 4^c, 3^{d+1}, 5^{e+1}, 4^f, 3^g \rangle =$$

2.

$$\langle 3^{a+1}, 5^{b+1}, 4^{c+1}, 6^{d+1}, 5^e, 4^f, 3^g \rangle =$$

3.

$$\langle 3^{a+1}, 5^{b+1}, 7^{c+1}, 6^d, 5^e, 4^f, 3^g \rangle =$$

In each of these abacus displays,  $a, b, c, d, e, f, g$  are non-negative integers. We have  $\lambda = \langle i, i, i \rangle$  and  $\mu = \langle j, j, j \rangle$ , where  $i = a + 2$  and

$$j = \begin{cases} a + b + c + d + 4 & (\text{in case (1)}) \\ a + b + c + 4 & (\text{in case (2)}) \\ a + b + 3 & (\text{in case (3)}). \end{cases}$$

In none of these cases do we have  $\lambda \supseteq \mu^\downarrow$ , and so  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$  by Lemma 2.15.

### 3.2 The case $\kappa_1 = 2, \kappa_2 = 1$

In this case the abacus takes one of following three possible forms:

$$\langle 3^{a+1}, 5^{b+1}, 4^c, 3^{d+1}, 4^{e+1}, 3^f \rangle; \langle 3^{a+1}, 5^{b+1}, 4^{c+1}, 5^{d+1}, 4^e, 3^f \rangle; \langle 3^{a+1}, 5^{b+1}, 6^{c+1}, 5^d, 4^e, 3^f \rangle.$$

Here  $a, b, c, d, e, f$  are non-negative integers, and we have  $\lambda = \langle i, i, i \rangle$  and  $\mu = \langle j, j \rangle$  or  $\langle j, j, l \rangle$  for some  $l \neq j - 1$ , where  $i$  and  $j$  are as in the previous section. If  $\text{Ext}_B^1(D^\lambda, D^\mu) \neq 0$ , then we have  $\lambda \supseteq \mu^\downarrow$  by Corollary 2.9, which implies that  $\mu = \langle j, j, l \rangle$  for  $1 \leq l \leq a + 1$ . But this implies that  $\mu \not\supseteq \lambda^\downarrow$ , and so  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$  by Lemma 2.15.

### 3.3 The case $\kappa_1 = 1, \kappa_2 = 2$

We consider  $B^\sharp, \lambda^\diamond, \mu^\diamond$ . We have  $\mathcal{P}\lambda^\diamond = \mathcal{P}\mu^\diamond$  by Proposition 2.19, so that  $\text{Ext}_{B^\sharp}^1(D^{\lambda^\diamond}, D^{\mu^\diamond}) = 0$  by appealing to the previous case. Thus,  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$ .

### 3.4 The case $\kappa_1 = \kappa_2 = 1$

Here the abacus for  $B$  takes one of two following forms:

1.  $\langle 3^{a+1}, 4^{b+1}, 3^{c+1}, 4^{d+1}, 5^e \rangle;$
2.  $\langle 3^{a+1}, 4^{b+1}, 5^{c+1}, 4^d, 3^e \rangle.$

Now  $\lambda = \langle i, i \rangle$  or  $\langle i, i, l \rangle$  for some  $l \neq i - 1$ , and  $\mu = \langle j, j \rangle$  or  $\langle j, j, m \rangle$  for some  $m \neq j - 1$ . The only way we can have  $\lambda \supseteq \mu^\downarrow$  and  $\mu \supseteq \lambda^\downarrow$  is in case (2) with  $b = 0$ , where we have

$$\lambda = \langle a + 2, a + 2, a + 3 \rangle \text{ or } \langle a + 2, a + 2 \rangle$$

and

$$\mu = \langle a + 3, a + 3, a + 1 \rangle \text{ or } \langle a + 3, a + 3, a + 3 \rangle.$$

Note that  $\mathcal{P}\langle a + 2, a + 2, a + 3 \rangle = \mathcal{P}\langle a + 3, a + 3, a + 1 \rangle \neq \mathcal{P}\langle a + 2, a + 2 \rangle = \mathcal{P}\langle a + 3, a + 3, a + 3 \rangle$ , so that it suffices to consider the cases

$$\lambda = \langle a + 2, a + 2, a + 3 \rangle, \quad \mu = \langle a + 3, a + 3, a + 1 \rangle$$

and

$$\lambda = \langle a + 2, a + 2 \rangle, \quad \mu = \langle a + 3, a + 3, a + 3 \rangle.$$

In the first case, we find that  $\lambda$  and  $\mu$  both induce semi-simply to a Rouquier block, by Proposition 2.20, and so  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$ . In the second case we have  $\lambda = \mu^\downarrow$ , and so  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$  by Corollary 2.12.

To summarise, we have proved the following in this section.

**Proposition 3.1.** *Suppose  $B$  is a weight 3 block of  $\mathfrak{S}_n$ , and that there are exactly two blocks  $A_1, A_2$  forming  $[3 : \kappa_i]$ -pairs with  $B$ . If Theorem 1.1 holds for  $A_1$  and  $A_2$ , then it holds for  $B$ .*

## 4 Blocks with rectangular cores

In this section, we suppose that  $B$  is a weight 3 block of  $\mathfrak{S}_n$ , and that there is exactly one block  $A$  forming a  $[3 : \kappa]$ -pair with  $B$ , with  $\kappa = 1$ . This means that  $B$  has a ‘rectangular’ core of the form  $(a^b)$  for some positive integers  $a$  and  $b$  with  $a + b \leq p$ . We put  $c = p - a - b$ , and represent partitions in  $B$  using the  $\langle 3^a, 4^b, 3^c \rangle$  abacus notation. An abacus display for  $A$  may be obtained by interchanging runners  $a$  and  $a + 1$ .

We suppose that  $\lambda$  and  $\mu$  are  $p$ -regular partitions in  $B$ , with  $\mathcal{P}\lambda = \mathcal{P}\mu$ . If neither  $D^\lambda$  nor  $D^\mu$  is exceptional for the  $[3 : 1]$ -pair  $(A, B)$ , then by induction (together with Proposition 2.18 and the Eckmann–Shapiro relations) Theorem 1.1 holds for  $\lambda$  and  $\mu$ . So we suppose that  $D^\lambda$  is exceptional, i.e.  $\lambda$  is of the form

$$\alpha_j = \begin{cases} \langle a + 1, a + 1 \rangle & (j = a + 1) \\ \langle a + 1, a + 1, j \rangle & (j \neq a + 1) \end{cases}$$

for some  $j \neq a$ . Note that  $D^{\alpha_j^\circ}$  is an exceptional simple module for the  $[3 : 1]$ -pair  $(A^\#, B^\#)$ .

By Corollary 2.9, we may assume that  $\mu \triangleright \lambda^\downarrow = \gamma_j$ .

### 4.1 Case 1: $\lambda \triangleright \mu$

We suppose in this subsection that  $\lambda \triangleright \mu$ . The condition  $\alpha_j \triangleright \mu \triangleright \gamma_j$  implies that  $\mu$  is one of  $\alpha_j$ ,  $\beta_j$  or  $\gamma_j$ . Since  $\mathcal{P}\beta_j \neq \mathcal{P}\alpha_j$ , we only need to consider the cases  $\mu = \alpha_j$  and  $\mu = \gamma_j$ . Furthermore, if  $\beta_j$  is  $p$ -regular, then  $\text{Ext}_B^1(D^{\alpha_j}, D^{\alpha_j}) = 0 = \text{Ext}_B^1(D^{\alpha_j}, D^{\gamma_j})$  by Corollary 2.12. Thus we may assume  $\beta_j$  is  $p$ -singular, for which we must have either both  $a = 2$  and  $j = 1$ , or both  $a = 1$  and  $2 \leq j \leq b + 1$ . We note that the case  $\mu = \alpha_j$  is dealt with in [16, Section 6], but in view of the general unease about that paper, we provide a separate proof here.

For  $\mu = \alpha_j$ , we may apply the Mullineux map and assume that  $\alpha_j^\circ$  has a similar form. As noted in [16], this implies that either  $a = 1$ ,  $b = 2$  and  $j = 2$ , or  $a = 2$ ,  $b = 1$  and  $j = 1$ ; these two cases correspond to each other under the Mullineux map, so it suffices to consider only the first. We apply the partial function  $\mathfrak{f} = \mathfrak{f}_p \mathfrak{f}_{p-1} \cdots \mathfrak{f}_3$ , and we find  $\mathfrak{f}(\alpha_j) = \langle 2, 2 \mid 5, 4, 3^{p-3}, 2 \rangle$ . This is regular-lowerable, and we may appeal to Corollary 2.24(2) to get  $\text{Ext}_B^1(D^{\mathfrak{f}(\alpha_j)}, D^{\mathfrak{f}(\alpha_j)}) = 0$ , and hence  $\text{Ext}_B^1(D^{\alpha_j}, D^{\alpha_j}) = 0$ .

It remains to consider the case  $\mu = \gamma_j$ , with either both  $a = 2$  and  $j = 1$ , or both  $a = 1$  and  $2 \leq j \leq b + 1$ . We shall show that in these cases we have  $\text{Ext}_{B^\#}^1(D^{\alpha_j^\circ}, D^{\gamma_j^\circ}) = 0$ . Applying the Mullineux map to  $\gamma_j$ , we find

$$\gamma_j^\circ = \begin{cases} \langle b + 2, b + 3 \rangle & (a = 2, j = 1, c \geq 1) \\ \langle b + 2, b + 1 \rangle & (a = 2, j = 1, c = 0) \\ \langle b + 2, 1 \rangle & (a = 1, j = 2, c \geq 1) \\ \langle 1 \rangle & (a = 1, j = 2, c = 0) \\ \langle b + 2, b + 3 - j \rangle & (a = 1, 3 \leq j \leq b + 1, c \geq 1) \\ \langle b + 3 - j \rangle & (a = 1, 3 \leq j \leq b + 1, c = 0), \end{cases}$$

where the partitions on the right are written with the  $\langle 3^b, 4^a, 3^c \rangle$  notation. First we note that  $\gamma_j^\circ$  is always regular-lowerable in these cases, and so in order to have non-zero  $\text{Ext}_{B^\#}^1(D^{\alpha_j^\circ}, D^{\gamma_j^\circ})$  we need  $[S^{\gamma_j^\circ} : D^{\alpha_j^\circ}] > 0$  by Propositions 2.22 and 2.23. But  $D^{\alpha_j^\circ}$  is an exceptional simple module for the pair  $(A^\#, B^\#)$ , and so by Proposition 2.7 we know in which Specht modules it lies. In particular, if  $\gamma_j^\circ$  is not an exceptional partition for the pair  $(A^\#, B^\#)$ , then we have  $[S^{\gamma_j^\circ} : D^{\alpha_j^\circ}] = 0$ , and hence  $\text{Ext}_{B^\#}^1(D^{\alpha_j^\circ}, D^{\gamma_j^\circ}) = 0$ .

Looking at the above list, we see that the only case where  $\gamma_j^\diamond$  is an exceptional partition is where  $a = 1$ ,  $j = 3$  and  $c = 0$ , where  $\gamma_j^\diamond = \langle b \rangle$ . In this case we apply the Mullineux map to  $\alpha_j$ , and find that  $\alpha_j^\diamond = \langle b + 1, b + 1 \rangle$ . But now we have  $\text{Ext}_{B^a}^1(D^{\alpha_j^\diamond}, D^{\gamma_j^\diamond}) = 0$  by Corollary 2.12, since the partition  $\langle b + 1, b + 1 \mid 3^b, 4 \rangle$  is  $p$ -regular.

## 4.2 Case 2: $\lambda \not\triangleright \mu$

Now we assume that  $\lambda \not\triangleright \mu$ . First we note that if  $\mu$  is a partition of the form  $\alpha_l$  for some  $l$ , then we may interchange  $\lambda$  and  $\mu$ ; by Proposition 2.7 and Theorem 1.2, the partitions  $\alpha_l$  are totally ordered by the dominance order, and so we then have  $\lambda \triangleright \mu$ , and we may appeal to the previous section. So we assume that  $\mu$  is not one of the partitions  $\alpha_l$ . If  $\mu$  is regular-lowerable, then we get  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$  by Corollary 2.24(2). If  $\mu$  is lowerable, but not regular-lowerable, then  $\mu$  is one of the partitions described in the following table.

$\mu$	Conditions on $a, b, c$	$\mu^{(2)}$
$\langle a + 1, 2 \rangle$	$a \geq 2$	$\langle a + 1, 1 \rangle$
$\langle a + 1, a + 2, 2 \rangle$	$a, b \geq 2$	$\langle a + 1, a + 2, 1 \rangle$

Here, if  $C$  is the block to which  $\mu$  is lowerable, with  $D^\mu \downarrow_C \cong D^{\mu^-}$ , then  $\mu^{(2)}$  is the other partition in  $B$  that can be obtained from  $\mu^-$  by moving a bead one place to the right. Observe that in these cases, if  $\lambda = \alpha_j \not\triangleright \mu$ , then  $\lambda \not\triangleright \mu^{(2)}$ , so that  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$  by Corollary 2.24(1).

So we turn to those partitions  $\mu$  which are not lowerable. These were listed in [9, Table 1], and are as follows.

	$\mu$	Conditions on $a, b, c$
A	$\langle a + 1 \rangle$	—
B	$\langle a + b + 1, a + 1 \rangle$	$b = 1, c \geq 1$
C	$\langle a + 1, a + 2 \rangle$	$b \geq 2$
D	$\langle a + 1, a + b + 1 \rangle$	$c \geq 1$
E	$\langle 1, a + 1 \rangle$	$a \geq 2, c = 0$
F	$\langle 1, a + 1, a + b + 1 \rangle$	$c \geq 1$
G	$\langle a + 1, a + 2, a + 3 \rangle$	$b \geq 3$
H	$\langle a + 1, a + 2, a + b + 1 \rangle$	$b \geq 2, c \geq 1$
I	$\langle a + 1, a + b + 1, a + b + 2 \rangle$	$c \geq 2$
J	$\langle 1, 2 \rangle$	$a \geq 3, c = 0$
K	$\langle 1, 2, a + b + 1 \rangle$	$a \geq 3, c \geq 1$
L	$\langle 1, a + b + 1, a + b + 2 \rangle$	$a \geq 2, c \geq 2$
M	$\langle a + b + 1, a + b + 2, a + b + 3 \rangle$	$c \geq 3$
N	$\langle 1, a + b + 1 \rangle$	$a \geq 2, c = 1$

Recalling that  $\lambda = \alpha_j$  and  $\mu \triangleright \gamma_j$  for some  $j \neq a$ , we may immediately discount cases J,K,L,M,N, since in these cases there is no such  $j$ . We deal with most of the other cases by inducing both  $D^\lambda$  and  $D^\mu$  through a Scopes sequence until  $D^\mu$  becomes regular-lowerable; the remaining cases are dealt with by ad hoc methods.



### 4.2.1 Induction 1

Consider the partial function  $\tilde{f} = \tilde{f}_{a-1}\tilde{f}_{a-2}\cdots\tilde{f}_1\tilde{f}_p$ . The effect of this is to move a bead (or two beads, if  $c = 0$ ) from runner  $p$  up to runner  $a$ . Applying this to  $\lambda = \alpha_j$ , we find that  $\tilde{f}(\lambda)$  is always defined; we get

$$\tilde{f}(\alpha_j) = \begin{cases} \langle a+1, a+1, p \rangle & (j = 1 < a) \\ \langle a+1, a+1, j-1 \rangle & (2 \leq j \leq a-1) \\ \langle a+1, a+1 \rangle & (j = a+1) \\ \langle a+1, a+1, j \rangle & (a+2 \leq j \leq p-1) \\ \langle a+1, a+1, a \rangle & (j = p > a+1), \end{cases}$$

where the partitions on the right are in the  $\langle 3^{a-1}, 4^{b+1}, 3^{c-1}, 2 \rangle$  notation if  $c > 0$ , or the  $\langle 3^{a-1}, 5, 4^{b-1}, 2 \rangle$  notation if  $c = 0$ .

Applying  $\tilde{f}$  to  $\mu$ , we find in several cases that  $\tilde{f}(\mu)$  is defined and regular-lowerable, with  $\tilde{f}(\lambda) \not\triangleright \tilde{f}(\mu)$ . This gives  $\text{Ext}^1(D^{\tilde{f}(\lambda)}, D^{\tilde{f}(\mu)}) = 0$  by Corollary 2.24(2), and hence  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$  by the Eckmann–Shapiro relations. We find that this works in cases A (except when  $b = 1, c = 0$ ), C (except when  $b = 2, c = 0$ ), D, and G (except when  $c = 0$ ). So in these cases we are done.

We now deal with cases E and F by applying the Mullineux map; if  $\mu$  is in case E, then by [9, Table 1],  $\mu^\diamond$  is in case A or case C for the block  $B^\sharp$ , while if  $\mu$  is in case F, then  $\mu^\diamond$  is in case D for  $B^\sharp$ . So by the cases we have already dealt with we have  $\text{Ext}_{B^\sharp}^1(D^{\lambda^\diamond}, D^{\mu^\diamond}) = 0$ , and so  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$ .

### 4.2.2 Induction 2

Now assume  $c > 0$ , and consider the partial function  $\tilde{f} = \tilde{f}_{a+1}\tilde{f}_{a+2}\cdots\tilde{f}_{a+b}$ . Again, we find that  $\tilde{f}(\lambda)$  is always defined, with

$$\tilde{f}(\alpha_j) = \begin{cases} \langle a+2, a+1, j \rangle & (1 \leq j \leq a-1 \text{ or } j \geq a+b+2) \\ \langle a+2, a+1 \rangle & (j = a+1) \\ \langle a+2, a+1, j+1 \rangle & (a+2 \leq j \leq a+b) \\ \langle a+2, a+2, a+1 \rangle & (j = a+b+1) \end{cases}$$

in the  $\langle 3^{a+1}, 4^b, 3^{c-1} \rangle$  notation. Let  $\tilde{f}(\lambda)^-$  be the partition obtained from  $\tilde{f}(\lambda)$  by moving the (unique movable) bead on runner  $a+1$  one place to its left, and let  $C$  be the weight 2 block in which  $\tilde{f}(\lambda)^-$  lies. Let  $\tilde{f}(\lambda)^{(2)}$  be the other partition besides  $\tilde{f}(\lambda)$  that can be obtained from  $\tilde{f}(\lambda)^-$  by moving a bead on runner  $a$  one place to its right. Since  $\tilde{f}(\lambda)$  is lowerable to the block  $C$ , we see that if  $\tilde{f}(\mu)$  is defined and  $\text{Ext}^1(D^{\tilde{f}(\lambda)}, D^{\tilde{f}(\mu)}) \neq 0$ , then  $\tilde{f}(\mu) \triangleright \tilde{f}(\lambda)^{(2)}$  by Corollary 2.24(1). For cases B, H and I, we find that  $\tilde{f}(\mu)$  is always defined, with

$$\tilde{f}(\mu) = \begin{cases} \langle a+1, a+2 \rangle, & \text{if } \mu \text{ is in Case B or H} \\ \langle a+1, a+b+2 \rangle & \text{if } \mu \text{ is in Case I.} \end{cases}$$

Furthermore,  $\tilde{f}(\mu)$  is always regular-lowerable to the block  $C$ . Thus, if  $\text{Ext}^1(D^{\tilde{f}(\lambda)}, D^{\tilde{f}(\mu)}) \neq 0$ , we must also have  $\tilde{f}(\lambda) \triangleright \tilde{f}(\mu)$  by Corollary 2.24(2). By checking the above lists, we see that it is impossible to have  $\tilde{f}(\lambda) \triangleright \tilde{f}(\mu) \triangleright \tilde{f}(\lambda)^{(2)}$ , so that  $\text{Ext}^1(D^{\tilde{f}(\lambda)}, D^{\tilde{f}(\mu)}) = 0$ , and hence  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$  by the Eckmann–Shapiro relations.

### 4.2.3 Exceptional cases

Now we deal with the remaining cases. First suppose we are in case G, so that  $\mu = \langle a+1, a+2, a+3 \rangle$  and  $b \geq 3$ , and we further assume that  $c = 0$ . Since  $\mu \triangleright \gamma_j$ , we have  $j \leq a+3$  and  $j \neq a+1$ . We can also bound  $j$  below by applying the Mullineux map:  $D^{\alpha_j}$  is an exceptional simple module for the pair  $(A^\sharp, B^\sharp)$ , and so if  $\text{Ext}_{B^\sharp}^1(D^{\lambda^\diamond}, D^{\mu^\diamond}) \neq 0$ , then we have  $\mu^\diamond \triangleright \lambda^{\diamond\downarrow}$ . We have

$$\mu^\diamond = \begin{cases} \langle b+1, b+2, b+3 \rangle & (a \geq 3) \\ \langle b+1, b+2 \rangle & (a = 2) \\ \langle b+1 \rangle & (a = 1), \end{cases} \quad \lambda^\diamond = \begin{cases} \langle b+1, b+1 \rangle & (j = 1, a \geq 2) \\ \langle b+1, b+1, b+2 \rangle & (j = a+2, a \geq 2) \\ \langle b+1, b+1 \rangle & (j = a+2, a = 1) \\ \langle b+1, b+1, 1 \rangle & (j = a+1) \\ \langle b+1, b+1, p+2-j \rangle & (\text{otherwise}), \end{cases}$$

so that

$$\lambda^{\diamond\downarrow} = \begin{cases} \langle b \rangle & (j = 1, a \geq 2) \\ \langle b, b+2 \rangle & (j = a+2, a \geq 2) \\ \langle b \rangle & (j = a+2, a = 1) \\ \langle b, 1 \rangle & (j = a+1) \\ \langle b, p+2-j \rangle & (\text{otherwise}), \end{cases}$$

with all partitions written in the  $\langle 3^b, 4^a \rangle$  notation. The condition  $\mu^\diamond \triangleright \lambda^{\diamond\downarrow}$  implies that  $j \geq a-1$ .

So we assume that  $j$  is one of  $a+3$ ,  $a+2$  or  $a-1$  (if  $a \geq 2$ ). Examining parities, we see that if  $j = a+3$  or  $a-1$  then  $\mathcal{P}\lambda \neq \mathcal{P}\mu$ . So we may assume that  $j = a+2$ , i.e.  $\lambda = \langle a+1, a+1, a+2 \rangle$ . But now  $\lambda$  and  $\mu$  both induce semi-simply to a Rouquier block by Proposition 2.20, and so we may appeal to Proposition 1.5 and the Eckmann–Shapiro relations.

By applying the Mullineux map, we also deal with the exceptions in cases A and C. We conclude the following.

**Proposition 4.1.** *Suppose  $B$  is a weight 3 block of  $\mathfrak{S}_n$ , and that there is exactly one block  $A$  forming a  $[3 : \kappa]$ -pair with  $B$ , with  $\kappa = 1$ . If Theorem 1.1 holds for  $A$ , then it holds for  $B$ .*

## 5 Blocks with birectangular cores

In this section, we suppose  $B$  is a weight 3 block of  $\mathfrak{S}_n$  and that there is exactly one block  $A$  forming a  $[3 : \kappa]$ -pair with  $B$ , and that  $\kappa = 2$ . This means that  $B$  has a ‘birectangular’ core  $((2a+d)^b, a^{b+c})$  for some non-negative integers  $a, b, c, d$  summing to  $p$ , with  $a, b \geq 1$ . We represent the partitions in  $B$  on an abacus with the  $\langle 3^a, 5^b, 4^c, 3^d \rangle$  notation, and we use the notation  $\alpha, \beta, \gamma, \delta$  for the exceptional partitions, as described in §2.5.2.

We suppose  $\lambda$  and  $\mu$  are  $p$ -regular partitions in  $B$  with  $\mathcal{P}\lambda = \mathcal{P}\mu$ . If neither  $D^\lambda$  nor  $D^\mu$  is exceptional for the pair  $(A, B)$ , then by induction, Proposition 2.18 and the Eckmann–Shapiro relations, we have  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$ . So we suppose that  $\lambda = \alpha = \langle a+1, a+1, a+1 \rangle$ . By Lemma 2.15, we can also assume that  $\mu \triangleright \lambda^\downarrow = \delta = \langle a \rangle$ . We note that by Proposition 2.20,  $\lambda = \alpha$  induces semi-simply to a Rouquier block.

### 5.1 Case 1: $\lambda \triangleright \mu$

If  $\lambda \triangleright \mu \triangleright \delta$ , then  $\mu$  must be one of  $\alpha, \beta, \gamma$  or  $\delta$ . But  $\mathcal{P}\beta = \mathcal{P}\delta \neq \mathcal{P}\alpha$ , so in fact we have  $\mu = \alpha$  or  $\gamma$ . The case  $\mu = \alpha$  is easy to deal with – since  $\alpha$  induces semi-simply to a Rouquier block, we have  $\text{Ext}_B^1(D^\alpha, D^\alpha) = 0$  by Proposition 1.5 and the Eckmann–Shapiro relations. So we are left with  $\mu = \gamma$ . If  $\beta$  is  $p$ -regular, then by Corollary 2.17 we have  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$ ; so we assume that  $\beta$  is  $p$ -singular, which happens if and only if  $a = 1$ . The assumption that  $\mu = \gamma$  is  $p$ -regular means that  $d \geq 1$ . We now apply the Mullineux map, and we find

$$\gamma^\diamond = \langle b+2, b+2 \rangle, \quad \alpha^\diamond = \langle b+1, b+1, b+1 \rangle$$

in the  $\langle 3^b, 5, 4^d, 3^c \rangle$  notation. So  $\gamma^\diamond$  is regular-lowerable with  $\alpha^\diamond \not\triangleright \gamma^\diamond$ , and we have  $\text{Ext}_{B^\alpha}^1(D^{\lambda^\diamond}, D^{\mu^\diamond}) = 0$  by Corollary 2.24(2), and hence  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$ .

### 5.2 Case 2: $\lambda \not\triangleright \mu$

In this section, we suppose that  $\lambda \not\triangleright \mu \triangleright \lambda^\downarrow$ . If  $\mu$  is regular-lowerable, then we have  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$  by Corollary 2.24(2). If  $\mu$  is lowerable, but not regular-lowerable, then  $\mu$  is one of the partitions described in the following table.

$\mu$	Conditions on $a, b, c$	$\mu^{(2)}$
$\langle a+1, 2 \rangle$	$a = 2$	$\langle a+1, 1 \rangle$
$\langle a+1, a+2, 2 \rangle$	$a = 2, b \geq 2$	$\langle a+1, a+2, 1 \rangle$
$\langle a+1, a+b+1, 2 \rangle$	$a = 2, c \geq 1$	$\langle a+1, a+b+1, 1 \rangle$

Here, if  $C$  is the block to which  $\mu$  is lowerable, with  $D^\mu \downarrow_C \cong D^{\mu^{(2)}}$ , then  $\mu^{(2)}$  is the other partition in  $B$  that can be obtained from  $\mu^-$  by moving a bead one place to the right. Observe that in these cases,  $\lambda \not\triangleright \mu^{(2)}$  so that  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$  by Corollary 2.24(1).

Thus we may assume that  $\mu$  is not lowerable. The possibilities for such  $\mu$  are listed in the following table.

	$\mu$	Conditions on $a, b, c, d$
A	$\langle a+1 \rangle$	—
B	$\langle a+1, a+1 \rangle$	—
C	$\langle a+1, a+2 \rangle$	$b \geq 2$
D	$\langle a+1, a+b+c+1 \rangle$	$d \geq 1$
E	$\langle a+1, a+b+1 \rangle$	$c \geq 1$
F	$\langle a+b+1, a+1 \rangle$	$b = 1, c \geq 1$
G	$\langle a+1, a+2, a+b+1 \rangle$	$b \geq 2, c \geq 1$
H	$\langle a+1, a+b+1, a+b+1 \rangle$	$c \geq 1$
I	$\langle a+1, a+b+1, a+b+2 \rangle$	$c \geq 2$
J	$\langle a+1, a+b+1, a+b+c+1 \rangle$	$c \geq 1, d \geq 1$
K	$\langle a+1, a+1, a+b+c+1 \rangle$	$d \geq 1$
L	$\langle a+b+c+1, a+1 \rangle$	$c = 0, d \geq 1$
M	$\langle a+1, a+1, a+2 \rangle$	—
N	$\langle a+1, a+1, a+b+1 \rangle$	$c \geq 1$
O	$\langle a+1, a+2, a+3 \rangle$	$b \geq 3$
P	$\langle a+1, a+2, a+b+c+1 \rangle$	$b \geq 2, d \geq 1$
Q	$\langle a+1, a+b+c+1, a+b+c+2 \rangle$	$d \geq 2$

### 5.2.1 Induction 1

Suppose  $b \geq 2$ , and consider the partial function  $\tilde{f} = \tilde{f}_{a-1}\tilde{f}_{a-2} \cdots \tilde{f}_1\tilde{f}_p\tilde{f}_{p-1} \cdots \tilde{f}_{a+b}$ . The effect of this is to move a bead from runner  $a + b$  up to runner  $a + b + c$ , then two beads from runner  $a + b + c$  to runner  $p$ , and then three beads from runner  $p$  up to runner  $a$ . We find that  $\tilde{f}(\alpha)$  is defined, with

$$\tilde{f}(\alpha) = \langle a + 1, a + 1, a + 1 \mid 3^{a-1}, 6, 5^{b-1}, 4^c, 3^d, 2 \rangle.$$

In cases A, B, C (provided  $b \geq 3$ ) and D, we find that  $\tilde{f}(\mu)$  is defined and regular-lowerable, with  $\tilde{f}(\alpha) \not\leq \tilde{f}(\mu)$ . This means that  $\text{Ext}^1(D^{\tilde{f}(\lambda)}, D^{\tilde{f}(\mu)}) = 0$  by Corollary 2.24(2), so that  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$ .

### 5.2.2 Induction 2

Now we suppose that  $c \geq 1$ , and consider  $\tilde{f} = \tilde{f}_{a-1}\tilde{f}_{a-2} \cdots \tilde{f}_1\tilde{f}_p\tilde{f}_{p-1} \cdots \tilde{f}_{a+b+c}$ . The effect of this is to move a bead from runner  $a + b + c$  to runner  $p$ , and then to move two beads from runner  $p$  to runner  $a$ . We find that

$$\tilde{f}(\alpha) = \langle a + 1, a + 1, a + 1 \mid 3^{a-1}, 5^{b+1}, 4^{c-1}, 3^d, 2 \rangle,$$

and that in the following cases  $\tilde{f}(\mu)$  is defined and regular-lowerable, with  $\tilde{f}(\alpha) \not\leq \tilde{f}(\mu)$ : cases E, F (provided  $c \geq 2$ ), G (provided  $c \geq 2$ ), H (provided  $c \geq 2$ ), I (provided  $c \geq 3$ ), and J. So we have  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$  for these cases.

### 5.2.3 Induction 3

Next we consider cases K and L; in these cases,  $d \geq 1$ . Let  $\tilde{f} = \tilde{f}_{a+1}\tilde{f}_{a+2} \cdots \tilde{f}_{a+b+c}$ . We find that

$$\begin{aligned} \tilde{f}(\lambda) &= \langle a + 2, a + 2, a + 1 \mid 3^{a+1}, 5^b, 4^c, 3^{d-1} \rangle, \\ \tilde{f}(\mu) &= \begin{cases} \langle a + 1, a + 2 \mid 3^{a+1}, 5^b, 4^c, 3^{d-1} \rangle, & \text{if } \mu \text{ is in case K} \\ \langle a + 1 \mid 3^{a+1}, 5^b, 4^c, 3^{d-1} \rangle, & \text{if } \mu \text{ is in case L.} \end{cases} \end{aligned}$$

Let  $\tilde{f}(\lambda)^-$  be the partition obtained from  $\tilde{f}(\lambda)$  by moving the (unique removable) bead on runner  $a + 1$  one place to its left, and let  $C$  be the weight 2 block in which  $\tilde{f}(\lambda)^-$  lies. Let  $\tilde{f}(\lambda)^{(2)} = \langle a + 2, a + 2, a \mid 3^{a+1}, 5^b, 4^c, 3^{d-1} \rangle$ ; then  $\tilde{f}(\lambda)^{(2)}$  is the other partition that can be obtained from  $\tilde{f}(\lambda)^-$  by moving a bead on runner  $a$  one place to its right. Since  $\lambda$  is lowerable to  $C$ , and  $\tilde{f}(\mu) \not\leq \tilde{f}(\lambda)^{(2)}$ , we see that  $\text{Ext}^1(D^{\tilde{f}(\lambda)}, D^{\tilde{f}(\mu)}) = 0$  by Corollary 2.24(1), and thus  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$ .

### 5.2.4 Inducing to a Rouquier block

As noted above, the partition  $\alpha$  induces semi-simply to a Rouquier block. We also find that  $\mu$  induces semi-simply to a Rouquier block in all the remaining cases, so that  $\text{Ext}_B^1(D^\lambda, D^\mu) = 0$  by the Eckmann–Shapiro relations.

We conclude the following.

**Proposition 5.1.** *Suppose  $B$  is a weight 3 block of  $\mathfrak{S}_n$ , and that there is exactly one block  $A$  forming a  $[3 : \kappa]$ -pair with  $B$ , with  $\kappa = 2$ . If Theorem 1.1 holds for  $A$ , then it holds for  $B$ .*

## 6 Proof of Theorem 1.1

We are now in a position to prove Theorem 1.1. We do this by induction on  $n$ , with the initial case being the principal block of  $\mathfrak{S}_{3p}$ , with empty core. The  $\text{Ext}^1$ -quiver of this block was found explicitly by Martin and Russell [15, Theorem 5.1]. By checking through their explicit description, it is easy to see that Theorem 1.1 holds for this block.

Now we suppose that  $B$  is a weight 3 block other than the principal block of  $\mathfrak{S}_{3p}$ , and that  $\lambda$  and  $\mu$  are  $p$ -regular partitions in  $B$ . There is at least one block  $A$  such that  $A$  and  $B$  form a  $[3 : \kappa]$ -pair. If neither  $D^\lambda$  nor  $D^\mu$  is exceptional for this  $[3 : \kappa]$ -pair, then Theorem 1.1 holds for  $\lambda$  and  $\mu$  by induction, using the Eckmann–Shapiro relations and Proposition 2.18. So we may suppose that for every such  $A$ , at least one of  $D^\lambda$  and  $D^\mu$  is exceptional for the pair  $(A, B)$ . A given exceptional simple module can be exceptional for at most one such pair, and so we find that there are at most two blocks forming  $[3 : \kappa]$ -pairs with  $B$ , and that for each such block we have  $\kappa \leq 2$ . The case where there are two blocks is dealt with in Section 3; the cases where there is only one are dealt with in Sections 4 (for  $\kappa = 1$ ) and 5 (for  $\kappa = 2$ ). The theorem follows by induction.

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