A conjecture on simultaneous cores

There has been extensive recent literature counting core partitions with various constraints. Recall that an integer partition is an *s-core* if it has no hooks of length s. Given a list (finite or infinite) of integers $s_1, s_2, \dots$, an $(s_1, s_2, \dots)$-core means a partition which is an s_i-core for every i. It is well-known that the number of such partitions is finite if and only if the integers $s_1, s_2, \dots$ are coprime, and there are several results giving an exact enumeration is certain cases. One of these is a recent paper [CHS] of Cho, Huh and Sohn, which considers the set of $(s, s+d, \dots, s+pd)$-cores for given positive integers s, d, p with s and d coprime. They show that these cores are in bijection with rational Motzkin paths satisfying certain conditions, and use this to give a (quite complicated) formula [CHS, Theorem 1.5] for the number of $(s, s+d, \dots, s+pd)$-cores.

Here we consider the limit as $p \to \infty$; that is, we consider the number of $(s, s+d, s+2d, \dots)$-cores. We conjecture a particular form for the number of such partitions, as follows.

Conjecture 0.1. *Suppose $d \geqslant 1$. Then there is a monic polynomial f_d of degree $d-1$ with non-negative integer coefficients such that for any $s \geqslant 1$ coprime to d the number of $(s, s+d, s+2d, \dots)$-cores is*

$$\frac{2^{s-d} f_d(s)}{d!}.$$

The constant term of f_d is $(2^d-1)(d-1)!$, and if $d \geqslant 2$ then $f_d(s)$ is divisible by $s+d+(-1)^d$.

If this conjecture is true, then the first few f_ds are as follows.

$$\begin{aligned}
f_1(s) &= 1,\\
f_2(s) &= s+3,\\
f_3(s) &= (s+2)(s+7),\\
f_4(s) &= (s+5)(s^2+13s+18),\\
f_5(s) &= (s+4)(s^3+26s^2+171s+186),\\
f_6(s) &= (s+7)(s^4+38s^3+419s^2+1342s+1080),\\
f_7(s) &= (s+6)(s+3)(s^4+54s^3+931s^2+5454s+5080),\\
f_8(s) &= (s+9)(s^6+75s^5+1999s^4+23169s^3+115768s^2+232284s+142800),\\
f_9(s) &= (s+8)(s^7+100s^6+3778s^5+68056s^4\\
&\qquad +606961s^3+2543284s^2+4524300s+2575440),\\
f_{10}(s) &= (s+11)(s^8+124s^7+5986s^6+143944s^5+1836529s^4\\
&\qquad +12358156s^3+42005484s^2+64730736s+33747840,\\
f_{11}(s) &= (s+10)(s^9+155s^8+9670s^7+314150s^6+5748073s^5\\
&\qquad +59986235s^4+347628480s^3+1057047300s^2+1502341776s+742815360).
\end{aligned}$$

Please let me know if you manage to prove this!

References

[CHS] H. Cho, J. Huh & J. Sohn, 'The $(s, s+d, \dots, s+pd)$-core partitions and the rational Motzkin paths', arXiv:2001.06651. [1]