The n-bar-core of an m-bar-core

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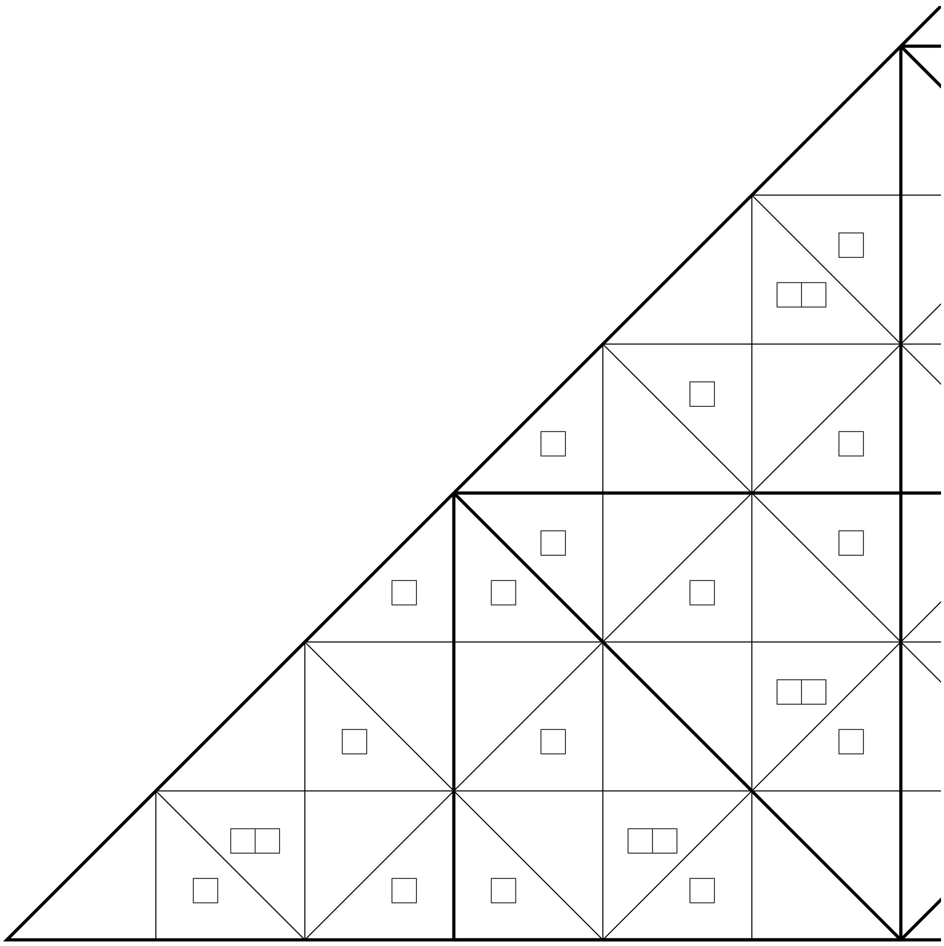
Abstract

This paper is an analogue of the author's earlier paper [F] on core partitions; here we prove corresponding results for bar-core partitions. The structure of [F] is largely retained, and this paper is not intended for publication.

We consider the n-bar-core of an m-bar-core partition, when m and n are coprime odd integers. Olsson has shown that the n-bar-core of an m-bar-core is again an m-bar-core, and we examine certain actions of the affine Coxeter group of type C on m-bar-cores which preserve the n-bar-core of an m-bar-core. Along the way, we give a new proof of Olsson's result.



1 Introduction

In this paper, we define a *bar partition* to be a finite set of positive integers. If m is an odd positive integer, then a bar partition is an *m-bar-core* if it is not possible to remove an m-bar from it; this concept is defined below. Any bar partition has an m-bar-core, which is the m-bar-core obtained by repeatedly removing m-bars.

m-bar-cores were introduced in the study of projective representations of the symmetric group: when m is a prime, the spin m-blocks of a given defect of the double covers of the symmetric groups of are indexed by the m-bar-cores, and the relationships between these blocks are controlled by the combinatorics of m-bar-cores. This representation-theoretic work uncovers a relationship between the set of m-bar-cores and the alcove geometry for the Coxeter group of type $\tilde{C}_{m'}$ (where we write $m = 2m' + 1$). Specifically, m-bar-cores are in bijection with alcoves in the dominant region of the weight space, which in turn are in bijection with cosets of the finite Coxeter group (of type $C_{m'}$) in its affine counterpart. Furthermore, the action of these groups of the set of alcoves can interpreted in terms of the relationships between m-bar-cores.

A recent trend in the study of cores has been to compare s-cores and t-cores, for different integers s, t. Much of this work has been carried over to the study of bar-cores, comparing m-bar-cores and n-bar-cores for different odd integers m, n. For $m \geqslant 3$ there are infinitely many m-bar-cores, but if m and n are coprime, there are only finitely many bar partitions which are simultaneously m-bar-cores and n-bar-cores. The exact number was found by Bessenrodt and Olsson [BO], who go on to study these '(m, n)-bar-cores' in more detail. In particular, they show that there is an (m, n)-bar-core which 'contains' all the others, in the sense that its Young diagram contains the Young diagram of any (m, n)-bar-core. Using the results in this paper, it is possible to give a different proof of this result, following the proof of the analogous result for cores in [F].

Another aspect of the comparison of m- and n-bar-cores is a result of Olsson [O], which says that if m and n are odd and coprime and one takes the n-bar-core of an m-bar-core, then the resulting partition is still an m-bar-core. The main focus of this paper is to ask which (m, n)-bar-core one obtains by taking the n-bar-core of an m-bar-core. We explore how the symmetry of the set of m-bar-cores is manifested when one replaces each m-bar-core with its n-bar-core. One by-product of this is a new proof of Olsson's result. We remark here that the hypothesis that m and n are coprime in Olsson's result is unnecessary, as observed by Gramain and Nath [GN].

We now summarise the layout of this paper. In Section 2, we give a brief account of m-bar-cores and abacus displays. In Section 3 we discuss alcove geometry and the affine Weyl group in type B. We go into more detail here, since the conventions we use for alcoves are slightly unusual. In Section 4 we connect m-bar-cores with alcove geometry and prove our main results on the symmetry inherent in taking the n-bar-core of an m-bar-core. Finally in Section 5 we comment briefly on the largest (m, n)-bar-core mentioned above.

2 Bar partitions

2.1 Partitions and m-bar-cores

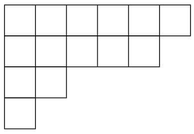
In this paper, a *bar partition* is a finite set of positive integers. When writing a bar partition, we usually write the integers in decreasing order. A bar partition λ is often represented by its

Young diagram, which is the set

$$[\lambda] = \left\{(i, j) \in \mathbb{N}^2 \;\middle|\; j \leqslant \lambda_i\right\},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the elements of λ written in decreasing order.

We draw the Young diagram as an array of boxes in the plane; for example, the array



represents the bar partition $\{6, 5, 2, 1\}$. (It is usual to use a symbol such as $\varnothing$ in place of the Young diagram for the empty partition, but in this paper we shall just use an empty diagram.)

Now fix an odd integer $m \geqslant 3$. If λ is a bar partition, then *removing an m-bar* from λ means one of two things:

removing a non-split m-bar: for some $a \geqslant m$ such that $a \in \lambda$ but $a - m \notin \lambda$, removing a from λ, and adding $a - m$ if $a > m$;

removing a split m-bar: for some $a, b \in \lambda$ such that $a + b = m$, removing both a and b from λ.

λ is an *m-bar-core* if it is not possible to remove an m-bar from λ. Any bar partition has an m-bar-core, which is obtained by repeatedly removing m-bars until it is not possible to remove any more; it is easy to see that this is independent of the choice of m-bars removed at each stage.

The definitions imply the following simple lemma, which will be useful later.

Lemma 2.1. *Suppose λ and ν are bar partitions, and that ν is obtained from λ by*

1. *replacing an element a by $a - km$ for some $k \in \mathbb{Z}$,*
2. *removing two elements summing to km, for some $k \in \mathbb{Z}$, or*
3. *removing a subset forming an arithmetic progression with average divisible by m.*

Then λ and ν have the same m-bar-core.

The notion of an m-bar-core derives from the theory of representations of the double cover of the symmetric group: for each bar partition whose sum is r, there is a corresponding spin representation of a double cover of the symmetric group $\mathfrak{S}_r$. If m is an odd prime and λ, μ are two bar partitions each of which has sum r, the corresponding spin representations lie in the same m-block if and only if λ and μ have the same m-bar-core. So the results in this paper can be interpreted as comparing projective representations of the symmetric group for two different odd primes. But from a combinatorial point of view, there is no need to assume that m is prime.

2.2 The abacus

Now we define the abacus display for a bar partition; this is a slight variation on the version introduced (as far as the author can tell) by Bessenrodt, Morris and Olsson [BMO]. Given an odd positive integer m, the *m-runner abacus* is an abacus with m vertical runners, numbered

$1,\dots,m$ from left to right; for each j, runner j has marked positions labelled by the positive integers congruent to j modulo m increasing down the runner. For example, the 5-runner abacus is as follows.

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ | & | & | & | & | \\ 6 & 7 & 8 & 9 & 10 \\ | & | & | & | & | \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

The *m-runner abacus display* for a bar partition λ is obtained by placing a bead on the abacus at position a for each $a \in \lambda$. From this display it is easy to see whether λ is an m-bar-core: this is the case if and only if

- every bead below the first row has a bead immediately above it,
- there are no beads on runner m, and
- for each $i = 1,\dots,m-1$, either there are no beads on runner i, or there are no beads on runner $m-i$.

Moreover, the procedure of removing m-bars is easy to see on the abacus: removing an m-bar consists of

- moving a bead up into an empty space immediately above it,
- removing a bead from position m, or
- removing beads from positions i and $m-i$ for some $i \in \{1,\dots,m-1\}$.

3 Alcoves and the affine Weyl group

In this section we introduce alcoves and the affine Coxeter group of type C. This material will be very familiar to many readers, but we give a detailed account here because the conventions we use are slightly unusual.

3.1 Alcoves and m'-points

As before, we assume m is an odd integer; in fact, we assume from now on that $m \geqslant 3$, and we write $m = 2m'+1$. Our alcove geometry will take place inside $\mathbb{R}^{m'}$. Define the *dominant region* to be the subset of $\mathbb{R}^{m'}$ consisting of points p for which $p_1 \geqslant \cdots \geqslant p_{m'} \geqslant \frac{m}{2}$. (This rather unusual convention will turn out to be useful later on.)

For each integer k, define the hyperplanes

$$\begin{aligned} H_i^k &= \left\{ p \in \mathbb{R}^{m'} \;\middle|\; p_i = \tfrac{1}{2}km \right\} && \text{for each } 1 \leqslant i \leqslant m', \\ H_{ij}^{k+} &= \left\{ p \in \mathbb{R}^{m'} \;\middle|\; p_i + p_j = km \right\} && \text{for each } 1 \leqslant i < j \leqslant m', \text{ and} \\ H_{ij}^{k-} &= \left\{ p \in \mathbb{R}^{m'} \;\middle|\; p_j - p_i = km \right\} && \text{for each } 1 \leqslant i < j \leqslant m'. \end{aligned}$$

Let

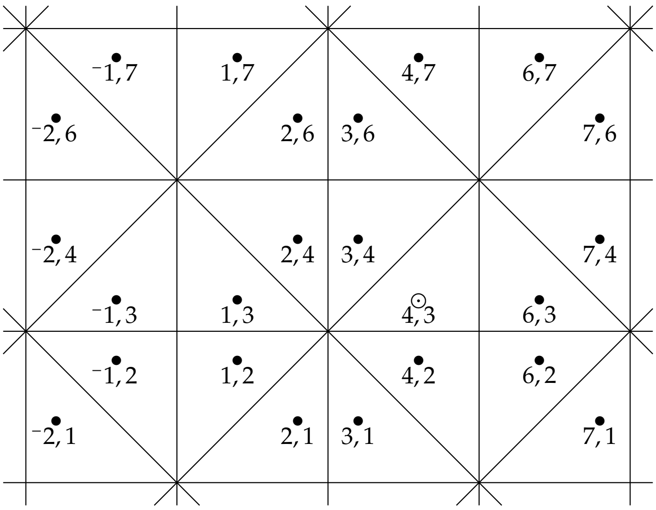
$$\mathcal{H} = \left\{ H_i^k \,\middle|\, 1 \leqslant i \leqslant m',\ k \in \mathbb{Z} \right\} \cup \left\{ H_{ij}^{k+} \,\middle|\, 1 \leqslant i < j \leqslant m',\ k \in \mathbb{Z} \right\} \cup \left\{ H_{ij}^{k-} \,\middle|\, 1 \leqslant i < j \leqslant m',\ k \in \mathbb{Z} \right\}.$$

The connected components of the complement in $\mathbb{R}^{m'}$ of the union of the hyperplanes in $\mathcal{H}$ are called *alcoves*. We will use the symbol $\odot$ to denote the point $(m-1, m-2, \dots, m'+1)$. The alcove $\mathtt{A}$ containing this point is called the *fundamental alcove*, and is bounded by the hyperplanes $H_{i(i+1)}^{0-}$ (for $1 \leqslant i < m'$), H_1^2 and $H_{m'}^1$; thus $\mathtt{A}$ is given by

$$\mathtt{A} = \left\{ p \,\middle|\, m \geqslant p_1 \geqslant \dots \geqslant p_{m'} \geqslant \frac{m}{2} \right\}.$$

Now define an *m'-point* to be a point $p = (p_1, \dots, p_{m'}) \in \mathbb{Z}^{m'}$ with the property that for each $1 \leqslant i \leqslant j \leqslant m'$, we have $p_i \not\equiv \pm p_j \pmod{m}$. Obviously each m'-point is contained in some alcove, and as we shall see below, each alcove contains a unique m'-point.

Example. In the case $m = 5$, we can draw a picture of part of $\mathbb{R}^2$ with 2-points and hyperplanes marked as follows.



Let $r_i^k, r_{ij}^{k\pm}$ denote the orthogonal (with respect to the usual inner product on $\mathbb{R}^{m'}$) reflections in the hyperplanes $H_i^k, H_{ij}^{k\pm}$ respectively; these reflections are given by

$$r_i^k : p \longmapsto p - (2p_i - km)\mathbf{e}_i$$
$$r_{ij}^{k\pm} : p \longmapsto p - (p_j \pm p_i - km)(\mathbf{e}_j \pm \mathbf{e}_i),$$

where $\mathbf{e}_1, \dots, \mathbf{e}_{m'}$ are the standard basis vectors. It is straightforward but tedious to check that these reflections all preserve the set of hyperplanes $\mathcal{H}$. Hence the group generated by all the r_i^k and $r_{ij}^{k\pm}$ preserves the set of alcoves. It also preserves the set of m'-points, and we can regard it as acting on alcoves or m'-points, as appropriate.

3.2 The Weyl group of type $\tilde{C}_{m'}$

Now we introduce the Coxeter group which will act on alcoves and m-bar-cores. This is the group $\mathfrak{W}'_m$ with generators $\tau_0, \dots, \tau_{m'}$, and relations

$$\begin{aligned}
\tau_i^2 &= 1 && \text{for } 0 \leqslant i \leqslant m',\\
\tau_i\tau_j &= \tau_j\tau_i && \text{when } 0 \leqslant i < j-1 \leqslant m'-1,\\
\tau_i\tau_{i+1}\tau_i &= \tau_{i+1}\tau_i\tau_{i+1} && \text{for } 1 \leqslant i \leqslant m'-2,\\
\tau_0\tau_1\tau_0\tau_1 &= \tau_1\tau_0\tau_1\tau_0 && \text{if } m' > 1,\\
\tau_{m'-1}\tau_{m'}\tau_{m'-1}\tau_{m'} &= \tau_{m'}\tau_{m'-1}\tau_{m'}\tau_{m'-1} && \text{if } m' > 1.
\end{aligned}$$

A level n action of $\mathfrak{W}'_m$

There is an action of $\mathfrak{W}'_m$ on $\mathbb{R}^{m'}$ given by mapping the generators $\tau_0, \dots, \tau_{m'}$ to the reflections in the walls of the fundamental alcove. In fact, we give a more general version of this action. For any positive integer n, define the *level n action* ψ_n of $\mathfrak{W}'_m$ by

$$\begin{aligned}
\tau_0 &\longmapsto r_1^{n+1},\\
\tau_i &\longmapsto r_{i(i+1)}^{0-} \qquad \text{for } i = 1, \dots, m'-1,\\
\tau_{m'} &\longmapsto r_{m'}^1.
\end{aligned}$$

Given $\tau \in \mathfrak{W}'_m$, we shall write $\check{\tau}$ for the image of τ under ψ_n, if n is understood. We may view ψ_n as an action on the set of alcoves, or on the set of m'-points, as appropriate. It is worth while to write down explicitly the action of the generators $\tau_0, \dots, \tau_{m'}$:

$$\begin{aligned}
\check{\tau}_0 : (p_1, \dots, p_{m'}) &\longmapsto ((n+1)m - p_1, p_2, \dots, p_{m'});\\
\check{\tau}_i : (p_1, \dots, p_{m'}) &\longmapsto (p_1, \dots, p_{i-1}, p_{i+1}, p_i, p_{i+2}, \dots, p_{m'}) \qquad \text{for } i = 1, \dots, m'-1;\\
\check{\tau}_{m'} : (p_1, \dots, p_{m'}) &\longmapsto (p_1, \dots, p_{m'-1}, m - p_{m'}).
\end{aligned}$$

From this, it is easy to check that ψ_n really is an action of $\mathfrak{W}'_m$. The next lemma concerns the case $n = 1$.

Lemma 3.1. *The image of the action ψ_1 includes all the reflections r_i^k and $r_{ij}^{k\pm}$, and is transitive on the set of alcoves.*

Proof. This is very well known, so we give a brief sketch. Let G denote the image of ψ_1. First we note that $r_i^2 \in G$ for each i by induction on i, using the relation $r_{(i-1)i}^{0-} r_{i-1}^2 r_{(i-1)i}^{0-} = r_i^2$. In a similar way, starting from the fact that $r_{m'}^1 \in G$, we get $r_i^1 \in G$ for all i. By repeatedly composing r_i^1 and r_i^2, we get $r_i^k \in G$ for all i.

Next, we observe that $r_{ij}^{0-} = r_{ji}^{0-} \in G$ for each $i < j$ by induction on $j - i$, using the relation $r_{i(i+1)}^{0-} r_{(i+1)j}^{0-} r_{i(i+1)}^{0-} = r_{ij}^{0-}$. Following this, we can show that $r_{ij}^{k+} \in G$ for all i, j via the relation $r_i^k r_{ij}^{0-} r_i^k = r_{ij}^{k+}$, and finally we show that $r_{ij}^{k-} \in G$ for all i, j using the relation $r_j^k r_{ij}^{0+} r_j^k = r_{ij}^{k-}$.

To see that this action is transitive on alcoves, we note that we can get from any alcove B to any other alcove C by crossing some finite sequence of hyperplanes in $\mathcal{H}$. Applying the reflections in each of these hyperplanes in turn takes B to C. □

Since the fundamental alcove A clearly contains a unique m'-point (namely the point $\odot$), we see that each alcove contains exactly one m'-point. Hence we may identify m'-points with alcoves.

A second level n action of $\mathfrak{W}'_m$

Now we assume that n is odd and m, n are coprime, and consider another level n action of $\mathfrak{W}'_m$ on the set of m'-points. Suppose p is an m'-point, and $i \in \{0, \dots, m'\}$. We consider the different values of i separately.

$i = 0$: Let $j \in \{1, \dots, m'\}$ be such that $p_j = am \pm n$ for $a \in \mathbb{Z}$. Define $\tilde{\tau}_0(p)$ by replacing p_j with $am \mp n$.

$1 \leqslant i \leqslant m' - 1$: Let j be such that $p_j = am \pm in$, and replace p_j with $am \pm (i+1)n$. At the same time, let k be such that $p_k = bm \pm (i+1)n$, and replace p_k with $bm \pm in$. Let $\tilde{\tau}_i(p)$ be the resulting m'-point.

$i = m'$: Let j be such that $p_j = am \pm m'n$ for $a \in \mathbb{Z}$, and define $\tilde{\tau}_{m'}(p)$ by replacing p_j with $am \pm (m'+1)n$.

It is clear that $\tilde{\tau}_i(p)$ is an m'-point in each case, and it is routine to verify that the map

$$\chi_n : \tau_i \longmapsto \tilde{\tau}_i$$

extends to give an action of $\mathfrak{W}'_m$ on the set of m'-points. Given any $\tau \in \mathfrak{W}'_m$, we write $\tilde{\tau}$ for the image of τ under χ_n, if n is understood.

Example. Suppose $m = 7$ and $p = (3, {}^-1, 2)$. Then p is a 3-point; for $n = 1$, we have

$$\tilde{\tau}_0(p) = (3, 1, 2), \qquad \tilde{\tau}_1(p) = (3, {}^-2, 1), \qquad \tilde{\tau}_2(p) = (2, {}^-1, 3), \quad \tilde{\tau}_3(p) = (4, {}^-1, 2),$$

while for $n = 5$ we have

$$\tilde{\tau}_0(p) = (3, {}^-1, 12), \quad \tilde{\tau}_1(p) = ({}^-2, {}^-1, {}^-3), \quad \tilde{\tau}_2(p) = (8, 4, 2), \qquad \tilde{\tau}_3(p) = (3, {}^-6, 2).$$

Note that $\tilde{\tau}$ is not an isometry, and there is no natural way to extend $\tilde{\tau}$ to the whole of $\mathbb{R}^{m'}$. However, given the correspondence between alcoves and m'-points, we may abuse notation and regard χ_t as an action of $\mathfrak{W}'_m$ on the set of alcoves. Recalling that $\mathtt{A}$ denotes the alcove containing the point $\odot = (m-1, m-2, \dots, m'+1)$, we have the following lemma, which is easy to check.

Lemma 3.2.

1. *If n is any odd integer coprime to m, then the actions ψ_n and χ_n on the set of alcoves commute.*
2. *If $n = 1$ and $i \in \{0, \dots, m'\}$, then $\check{\tau}_i(\mathtt{A}) = \tilde{\tau}_i(\mathtt{A})$.*

Now say that two alcoves are *adjacent* if there is only one hyperplane in $\mathcal{H}$ separating them.

Corollary 3.3. *Suppose $\mathtt{B}$ is an alcove, and $i \in \{0, \dots, m'\}$, and define $\tilde{\tau}_i$ using the level 1 action χ_1. Then $\tilde{\tau}_i(\mathtt{B})$ is adjacent to $\mathtt{B}$.*

Proof. Write $\check{\tau}$ for the image of $\tau \in \mathfrak{W}'_m$ under the level 1 action ψ_1. Since this action is transitive on the set of alcoves, we can write $\mathtt{B} = \check{\tau}(\mathtt{A})$ for some τ. Hence

$$\begin{aligned}
\tilde{\tau}_i(\mathtt{B}) &= \tilde{\tau}_i(\check{\tau}(\mathtt{A})) & \\
&= \check{\tau}(\tilde{\tau}_i(\mathtt{A})) & \text{by Lemma 3.2(1)} \\
&= \check{\tau}(\check{\tau}_i(\mathtt{A})). & \text{by Lemma 3.2(2)}
\end{aligned}$$

Clearly $\mathtt{A}$ and $\check{\tau}_i(\mathtt{A})$ are adjacent, and since $\check{\tau}$ is an affine transformation of $\mathbb{R}^{m'}$, it preserves adjacency of alcoves. □

Using a very similar argument, one can show that if p is an m'-point and $\mathtt{B}$ the alcove containing it, then each alcove adjacent to $\mathtt{B}$ contains the point $\tilde{\tau}_i(p)$ for some i.

3.3 m'-sets

Define an *m'-set* to be a set $\{p_1, \dots, p_{m'}\}$ of m' integers such that $p_i > \frac{m}{2}$ for each i, and for any i, j we have $p_i \not\equiv \pm p_j \pmod{m}$. There is a $2^{m'}m'!$-to-1 map from m'-points to m'-sets, given by

$$(p_1, \dots, p_{m'}) \longmapsto \{\max\{p_1, m - p_1\}, \dots, \max\{p_{m'}, m - p_{m'}\}\}.$$

If we restrict attention to m'-points in the dominant region, this map becomes a bijection. Given the correspondence between m'-points and alcoves, we have an $2^{m'}m'!$-to-1 map from the set of alcoves to the set of alcoves in the dominant region; this is given by 'folding' $\mathbb{R}^{m'}$ along the hyperplanes H_i^1 and H_{ij}^{0-} for all i, j. This folding will be useful in understanding symmetry later.

Note that our second action χ_n of $\mathfrak{W}'_m$ on the set of m'-points descends to an action on m'-sets (although the action ψ_n does not). We use the same notation χ_n (and $\tilde{\tau}$) for this action on m'-sets without fear of confusion.

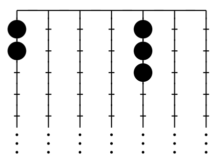
4 The n-bar-core of an m-bar-core

Now we come to the main part of the paper. We suppose m, n are coprime odd integers with $m \geqslant 3$, and we compare the n-bar-cores of different m-bar-cores. By representing m-bar-cores as m'-points, we use the geometric symmetry of the last section to see the symmetry of n-bar-cores of m-bar-cores.

4.1 m-bar-cores and m'-sets

Suppose λ is an m-bar-core, and consider the m-runner abacus display for λ. For each $i = 1, \dots, m'$, let b_i be the number of the first unoccupied position on runner i, and c_i the first unoccupied position on runner $m - i$. Let $a_i = \max\{b_i, c_i\}$. Then a_i is an integer greater than $\frac{m}{2}$. Moreover, for any i, j we have $a_i \not\equiv \pm a_j \pmod{m}$; hence the set $\{a_1, \dots, a_{m'}\}$ is an m'-set. We let $Q(\lambda)$ denote this m'-set, and we let p_λ be the corresponding dominant m'-point, i.e. the point whose coordinates are obtained by arranging the elements of $Q(\lambda)$ is descending order.

For example, take $m = 7$. The bar partition $\lambda = \{19, 12, 8, 5, 1\}$ has abacus display



and so is a 7-bar-core. We have $Q(\lambda) = \{15, 26, 4\}$, and so $p_\lambda = (26, 15, 4)$.

It is easy to check that any m'-set is obtained from a unique m-bar-core in this way: given an m'-set Q, construct an abacus display in which there is a bead at position b if and only if there is an element of Q below b on the same runner. From the conditions in §2.2 which describe when an abacus display is the display of an m-bar-core, we see that this display gives the unique m-bar-core λ for which $Q(\lambda) = Q$. Hence we have a natural bijection between m-bar-cores and

m'-sets, and therefore between m-bar-cores and alcoves in the dominant region. Using this bijection, we may regard the action χ_n of the group $\mathfrak{W}'_m$ on m'-sets as an action on the set of m-bar-cores.

Example. In Figure 1 we illustrate the bijection between 5-bar-cores and alcoves in the dominant region of $\mathbb{R}^2$, by drawing the Young diagram of a 5-bar-core inside the corresponding alcove.

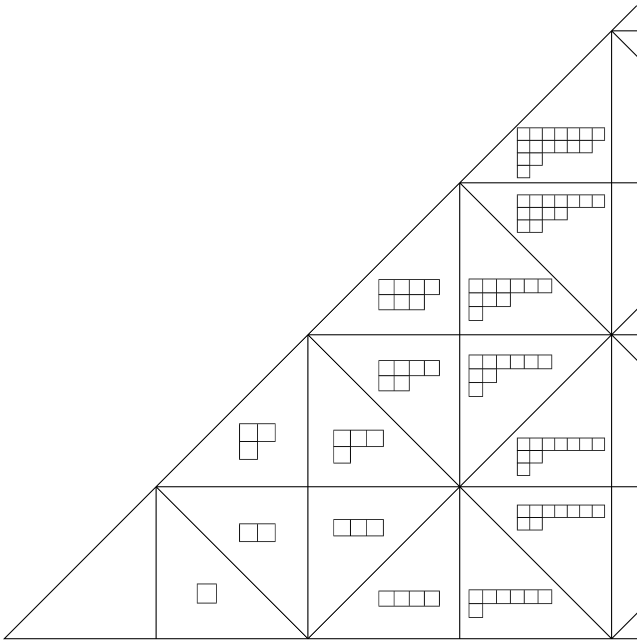


Figure 1: the correspondence between 5-bar-cores and dominant alcoves in $\mathbb{R}^2$

The aim of this paper is to compare the n-bar-cores of different m-bar-cores, when m and n are coprime odd integers. If we take $n = 3$, and expand and redraw Figure 1 with each 5-bar-core replaced by its 3-bar-core, we get the diagram on the first page of this paper.

4.2 m-bar-cores having the same n-bar-core

In comparing the n-bar-cores of different m-bar-cores, the following proposition will be crucial.

Proposition 4.1. *Suppose m, n are coprime odd positive integers. Suppose λ, μ are m-bar-cores, and that there is a bijection $\phi : \mathcal{Q}(\lambda) \to \mathcal{Q}(\mu)$ such that*

$$2\phi(a) - m \equiv \pm(2a - m) \pmod{n}$$

for every a. Then λ and μ have the same n-bar-core.

In order to prove this proposition, we introduce the *extended m'-set* of an m-bar-core: if λ is an m-bar-core, then we define

$$\mathcal{X}(\lambda) = \mathcal{Q}(\lambda) \cup \{\, m - a \mid a \in \mathcal{Q}(\lambda)\,\}\,.$$

Note that $\mathcal{X}(\lambda)$ is a set of $m-1$ integers, such that for every $i = 1, \dots, m-1$ there is exactly one element of $\mathcal{X}(\lambda)$ congruent to i modulo m. If n is prime to m, then the hypothesis of Proposition 4.1 is the same as saying that there is a bijection $\psi : \mathcal{X}(\lambda) \to \mathcal{X}(\mu)$ such that $\psi(a) \equiv a \pmod{n}$ for every a. If we write the elements of $\mathcal{X}(\lambda)$ as $q_1, \dots, q_{m-1}$ in such a way that $q_i \equiv in \pmod{m}$ for each i, then we have the following simple statements, which follow easily from the definitions:

- for each $i \in \{1, \dots, m-1\}$ we have $q_i = m - q_{m-i}$;
- if $a \in \lambda$ and $a \equiv in \pmod{m}$ for $i \in \{1, \dots, m-1\}$, then $q_i \geqslant a + m$;
- if $0 < a \notin \lambda$ and $a \equiv in \pmod{m}$ for $i \in \{1, \dots, m-1\}$, then $q_i \leqslant a$.

The proof of Proposition 4.1 uses the following lemma.

Lemma 4.2. *Suppose m, n are coprime odd positive integers. Suppose λ is an m-bar-core, and write the elements of $\mathcal{X}(\lambda)$ as $q_1, \dots, q_{m-1}$ with $q_i \equiv in \pmod{m}$ for each i. Then λ is an n-bar-core if and only if $q_1 \leqslant n$ and $q_{i+1} \leqslant q_i + n$ for each $i = 1, \dots, m-2$.*

Proof.

($\Rightarrow$) If $q_1 > n$, then $n \in \lambda$, so λ is not an n-bar-core. Now suppose $q_{i+1} > q_i + n$ for some $1 \leqslant i \leqslant m-2$. Then in fact $q_{i+1} \geqslant q_i + n + m$. Note that this also implies $q_{m-i} \geqslant q_{m-i-1} + n + m$.

If $q_{i+1} > m + n$, then $q_{i+1} - m \in \lambda$, while $0 < q_{i+1} - m - n \notin \lambda$, so λ fails to be an n-bar-core. So we can assume $q_{i+1} < m + n$. Similarly, we assume $q_{m-i} < m + n$. Now we have $q_i, q_{m-i-1} < 0$, and so $q_{i+1}, q_{m-i} > m$. So $q_{i+1} - m$ and $q_{m-i} - m$ are positive, so are elements of λ. Now we can find $a, b \in \lambda$ such that $a + b = n$ as follows. To start with, set $a = q_{i+1} - m$, $b = q_{m-i} - m$. Then we have $a + b \equiv n \pmod{m}$, and from the inequalities we have so far we can calculate $n \leqslant a + b < 2n$. If $a + b > n$, then either replace a with $a - m$ (which lies in λ if $a > m$) or replace b with $b - m$. We repeat this until we have $a + b = n$. The only way this might fail is if we had a and b both less than m while $a + b > n$. But then we would have $a + b \geqslant m + n$, $a + b < 2m$ and $a + b < 2n$, which gives a contradiction. So we can find $a, b \in \lambda$ such that $a + b = n$, so λ is not an n-bar-core.

($\Leftarrow$) Now suppose λ is not an n-bar-core. There are two cases to consider.

- Suppose $a \in \lambda$ and $0 \leqslant a - n \notin \lambda$. Let $i \in \{0, \dots, m-2\}$ be such that $a \equiv (i+1)n \pmod{m}$. If $i = 0$, then we have $q_1 > a \geqslant n$, as required. So suppose $i > 0$. Then we have $q_{i+1} > a$, while $q_i \leqslant a - n$, and so $q_{i+1} > q_i + n$.
- Suppose for some $0 < a < n$ we have $a, n-a \in \lambda$. Let i be such that $a \equiv (i+1)n \pmod{m}$; since neither a nor $n - a$ is divisible by m, we have $1 \leqslant i \leqslant m-2$. Now $n - a \equiv (m-i)n \pmod{m}$, so $q_{m-i} > n - a$. Hence $q_i < a - n + m$; on the other hand, $q_{i+1} \geqslant a + m$, and so $q_{i+1} > q_i + n$. □

Proof of Proposition 4.1. Write $\lambda \leftrightarrow \mu$ to denote the hypothesis of the proposition, i.e. the existence of the bijection ϕ. We proceed by induction on the size (by which we mean the sum of the elements) of λ. For the inductive step, we suppose λ is not an n-bar-core, and show that there is a smaller m-bar-core ν with the same n-bar-core as λ, such that $\lambda \leftrightarrow \nu$. (Since $\leftrightarrow$ is obviously an equivalence relation, we then have $\nu \leftrightarrow \mu$, so that ν, μ have the same n-bar-core by induction.)

As above, write the elements of $\mathcal{X}(\lambda)$ as $q_1, \dots, q_{m-1}$ with $q_i \equiv in \pmod{m}$ for each i. By Lemma 4.2, the assumption that λ is not an n-bar-core means that either $q_1 > n$ or $q_{i+1} > q_i + n$ for some i. We consider three cases.

- Suppose $q_1 > n$. Let j be the residue of n modulo m; then runner j of the abacus for λ contains beads in positions $j, j+m, \dots, q_1-m$, while runner $m-j$ is empty. Now consider two cases.
  - Suppose $q_1 - m > 2n$. Construct ν by replacing $q_1-m, q_1-2m, \dots, m-j+2n$ with $q_1-m-2n, q_1-2m-2n, \dots, m-j$, and then removing all of $j, j+m, \dots, 2n-j$. By repeated applications of Lemma 2.1, ν has the same n-bar-core as λ.

    In terms of the abacus, ν is obtained from λ by removing all the beads from runner j, and adding beads on runner $m-j$ at positions $m-j, 2m-j, \dots, a-2n$. So ν is an m-bar-core, and we have

    $$\mathcal{X}(\nu) = \mathcal{X}(\lambda) \setminus \{q_1, m-q_1\} \cup \{q_1-2n, m-q_1+2n\}; \tag{1}$$

    so $\lambda \leftrightarrow \nu$.
  - Now suppose $q_1 - m < 2n$. In this case, we construct ν simply by removing $q_1 - m, q_1-2m, \dots, 2n-q_1+m$. Again by Lemma 2.1, ν has the same n-bar-core as λ. It is also clear that ν is an m-bar-core, and in fact (1) still holds. So again $\lambda \leftrightarrow \nu$.
- Next suppose $q_{m'+1} > q_{m'} + n$ (where we write $m = 2m'+1$ as before). Then $q_{m'+1} \geqslant q_{m'} + m + n$; since $q_{m'} + q_{m'+1} = m$, this means that $q_{m'+1} - m > 0$. Let j be the residue of $(m'+1)n$ modulo m. Then in the abacus display for λ, there are beads at positions $j, j+m, \dots, q_{m'+1}-m$, and no beads on runner $m-j$. Again, we consider two cases.
  - Suppose $q_{m'+1} - m > n$. Construct ν by replacing $q_{m'+1}-m, q_{m'+1}-2m, \dots, m-j+n$ with $q_{m'+1}-m-n, q_{m'+1}-2m-n, \dots, m-j$, and removing $j, j+m, \dots, n-j$. (In other words, each element of λ congruent to j modulo m is reduced by n if it is greater than n, and deleted otherwise.) It is clear that ν is an m-bar-core, and by Lemma 2.1 ν has the same n-bar-core as λ. Moreover, we find

    $$\mathcal{X}(\nu) = \mathcal{X}(\lambda) \setminus \{q_{m'+1}, m-q_{m'+1}\} \cup \{q_{m'+1}-n, m-q_{m'+1}+n\}; \tag{2}$$

    so $\lambda \leftrightarrow \nu$.
  - Now suppose $q_{m'+1} - m < n$. We construct ν by removing $q_{m'+1}-m, q_{m'+1}-2m, \dots, n-q_{m'+1}+m$. Then ν is an m-bar-core with the same n-bar-core as λ, and in fact (2) holds in this case too.
- Now suppose $q_{i+1} > q_i + n$ for some $i \neq 0, m'$. Then we also have $q_{m-i} > q_{m-i-1} + n$. We assume that $q_{i+1} > m$; if not, then we have $q_{m-i} = m - q_i \geqslant m - (q_{i+1} - m - n) > m$, and we can replace i with $m-i-1$. Let j, k be the residues of $(i+1)n$ and in modulo m, respectively. We consider five cases.
  - Suppose $q_i > 0$. Then there are no beads on runners $m-j$ or $m-k$ of the abacus for λ, and the lowest bead on runner j lies at position $q_{i+1} - m > n$. We construct ν in this case by replacing $q_{i+1}-m, q_{i+1}-2m, \dots, q_i+n$ with $q_{i+1}-m-n, q_{i+1}-2m-n, \dots, q_i$. Then ν is an m-bar-core (since runners $m-j$ and $m-k$ of the abacus are still empty) and has the same n-bar-core as λ. Moreover, we have

    $$\mathcal{X}(\nu) = \mathcal{X}(\lambda) \setminus \{q_{i+1}, m-q_{i+1}, q_i, m-q_i\} \cup \{q_i+n, m-q_i-n, q_{i+1}-n, m-q_{i+1}+n\}, \tag{3}$$

    so $\lambda \leftrightarrow \nu$.

- Now suppose $q_{i+1} - m, q_{m-i} - m > n$. In this case runners k and $m - j$ of the abacus are empty, while runner j has beads down to $q_{i+1} - m$ and runner $m - k$ has beads down to $q_{m-i} - m$. We construct ν by replacing $q_{i+1} - m, q_{i+1} - 2m, \dots, k + n$ with $q_{i+1} - m - n, q_{i+1} - 2m - n, \dots, k$, replacing $q_{m-i} - m, q_{m-i} - 2m, \dots, m - j + n$ with $q_{m-i} - m - n, q_{m-i} - 2m - n, \dots, m - j$, and removing $k + n - m, k + n - 2m, \dots, j$ and $n - j, n - j - m, \dots, m - k$. (In other words, every element of λ congruent to j or $-k$ modulo m is reduced by n if it is greater than n, and is deleted otherwise.)

  ν is an m-bar-core because in its abacus display runners j and $m - k$ are empty while runners k and $m - j$ have beads in the topmost positions; furthermore, (3) holds, so $\lambda \leftrightarrow \nu$.
- Next suppose $q_{i+1} - m > n$, while $0 < q_{m-i} - m < n$. In this case, we construct ν by replacing $q_{i+1} - m, q_{i+1} - 2m, \dots, k + n$ with $q_{i+1} - m - n, q_{i+1} - 2m - n, \dots, k$ and removing $k + n - m, k + n - 2m, \dots, n - q_{m-i} + m$ and $q_{m-i} - m, q_{m-i} - 2m, \dots, m - k$. ν is again an m-bar-core, and (3) holds, so $\lambda \leftrightarrow \nu$.
- Next, suppose $0 < q_{i+1} - m < n$, while $q_{m-i} - m > n$. In this case, we can replace $i + 1$ with $m - i$ and appeal to the previous case.
- Finally, suppose $0 < q_{i+1} - m < n$ and $0 < q_{m-i} - m < n$. In this case we construct ν by removing $q_{i+1} - m, q_{i+1} - 2m, \dots, n - q_{m-i} + m$ and $q_{m-i} - m, q_{m-i} - 2m, \dots n - q_{i+1} - m$. Again ν is an m-bar-core and (3) holds, so $\lambda \leftrightarrow \nu$.

We have seen that whenever λ is not an n-bar-core, we can replace it with a smaller m-bar-core ν such that $\lambda \leftrightarrow \nu$ and use induction. So we assume λ is an n-bar-core. Symmetrically, we can assume μ is an n-bar-core, and we must show that $\lambda = \mu$. In other words, we must show that a bar partition λ which is both an m- and an n-bar-core is uniquely determined by the integers

$$g_i = \left|\left\{ a \in \mathcal{X}(\lambda) \;\middle|\; a \equiv i \pmod{n} \right\}\right|.$$

With $q_1, \dots, q_{m-1}$ as above, consider the following sequence $\mathcal{S}$ of $m + n - 1$ integers:

$$\begin{array}{rrrcl}
n, & n - m, & n - 2m, & \dots, & q_1, \\
q_1 + n, & q_1 + n - m, & q_1 + n - 2m, & \dots, & q_2, \\
q_2 + n, & q_2 + n - m, & q_2 + n - 2m, & \dots, & q_3, \\
 & & \vdots & & \\
q_{m-2} + n, & q_{m-2} + n - m, & q_{m-2} + n - 2m, & \dots, & q_{m-1}, \\
q_{m-1} + n, & q_{m-1} + n - m, & q_{m-1} + n - 2m, & \dots, & m.
\end{array}$$

Note that the definition of $\mathcal{S}$ makes sense, because the fact that λ is an n-bar-core means that $q_1 \leqslant n$ (and hence $q_{m-1} + n \geqslant m$) and $q_{i+1} \leqslant q_i + n$ for each i. The differences between consecutive terms of $\mathcal{S}$ are either $-m$ ($n - 1$ times) or $+n$ ($m - 1$ times). Hence modulo n, the steps are either 0 or $-m$. Since m and n are coprime and there are only $n - 1$ steps equal to $-m$, all the terms of $\mathcal{S}$ in a given congruence class modulo n must be consecutive in $\mathcal{S}$; so the set of terms of $\mathcal{S}$ in a given congruence class modulo n must take one of the following forms:

- $\{q_i, q_{i+1}, \dots, q_j, q_j + n\}$ for some $1 \leqslant i < j \leqslant m - 1$, where either $j = m - 1$ or $q_{j+1} < q_j + n$;

- $\{q_i + n - km\}$ for some $0 \leqslant i \leqslant m-1$ and some $1 \leqslant k \leqslant \frac{q_i - q_{i+1} + n - m}{m}$ (where we read $q_0 = m$ and $q_m = 0$).

In each case, exactly one element fails to lie in $\mathcal{X}(\lambda)$; hence in $\mathcal{S}$ the number of terms congruent to i modulo n is $g_i + 1$ for each i.

So $\mathcal{S}$ is determined by the integers g_i: it begins with $g_0 + 1$ terms divisible by n, starting from n; then there is a jump of $-m$ followed by $g_{-m} + 1$ terms congruent to $-m$ modulo n, then another jump of $-m$, and so on. The integers $q_1, \dots, q_{m-1}$ are easily recovered from $\mathcal{S}$, and so λ is too. □

Using Proposition 4.1, we see that the action χ_n of $\mathfrak{W}'_m$ on the set of m-bar-cores preserves the n-bar-core of an m-bar-core. Recall that we write $\tilde{\tau}_i$ for the image of τ_i under the action χ_n.

Proposition 4.3. *Suppose m and n are coprime odd positive integers with $m \geqslant 3$. Suppose λ is an m-bar-core and $i \in \{0, \dots, m'\}$, and define $\tilde{\tau}_i(\lambda)$ as above. Then λ and $\tau_i(\lambda)$ have the same n-bar-core.*

Proof. This is immediate from Proposition 4.1 and the definition of $\tilde{\tau}_i$. □

We shall refer to an orbit in the set of m-bar-cores under the action χ_n as a *level n orbit*. From Proposition 4.3, we see that two m-bar-cores have the same n-bar-core if they lie in the same level n orbit. We shall prove the converse of this statement; the way we do this is to show that each level n orbit contains a n-bar-core. Before we do this, it will be helpful for later to introduce some more notation: for m, n coprime odd integers, define $\mathcal{R}^m_n$ to be the *level n parallelepiped*

$$\mathcal{R}^m_n = \left\{ p \in \mathbb{R}^{m'} \;\middle|\; 1 \leqslant p_i - p_{i+1} \leqslant n \text{ for } i = 1, \dots, m'-1, \text{ and } \frac{m+1}{2} \leqslant p_{m'} \leqslant \max\left\{n, \frac{m+n}{2}\right\} \right\}.$$

For example, consider the case $m = 5$. If $n = 1$, then $\mathcal{R}^5_n$ just consists of the point $(4, 3)$. For $n > 1$, $\mathcal{R}^5_n$ is the parallelogram with vertices

$$\begin{cases} (4,3), (6,3), (5,4), (7,4) & (n = 3) \\ (4,3), (n+3,3), (n+1,n), (2n,n) & (n > 5). \end{cases}$$

Proposition 4.4. *Suppose m and n are coprime odd integers with $m \geqslant 3$, and that O is a level n orbit. Let ν be an element of O for which the sum $\sum_{k \in Q(\nu)} (k - \frac{m}{2})^2$ is minimised. Then ν is an n-bar-core, and p_ν lies in $\mathcal{R}^m_n$.*

For this proposition, we again use the extended beta-set of an m-bar-core. It is quite easy to write down the effect of the level n action χ_n on m-bar-cores in terms of their extended beta-sets: write the elements of $\mathcal{X}(\nu)$ as $q_1, \dots, q_{m-1}$ with $q_i \equiv in \pmod{m}$ for each i. Then for $i \in \{0, \dots, m'\}$ the set $\mathcal{X}(\tilde{\tau}_i(\nu))$ is obtained from $\mathcal{X}(\nu)$ as follows.

$i = 0:$ Replace q_1, q_{m-1} with $q_1 - 2n, q_{m-1} + 2n$.

$1 \leqslant i \leqslant m'-1:$ Replace $q_i, q_{i+1}, q_{m-i-1}, q_{m-i}$ with $q_i + n, q_{i+1} - n, q_{m-i-1} + n, q_{m-i} - n$.

$i = m':$ Replace $q_{m'}, q_{m'+1}$ with $q_{m'} + n, q_{m'+1} - n$.

Proof of Proposition 4.4. Note that we have

$$\sum_{k\in\mathcal{X}(\nu)}\left(k-\frac{m}{2}\right)^2=2\sum_{k\in Q(\nu)}\left(k-\frac{m}{2}\right)^2;$$

so ν also minimises $\sum_{k\in\mathcal{X}(\nu)}(k-\frac{m}{2})^2$ across O. As above, write the elements of $\mathcal{X}(\nu)$ as $q_1,\dots,q_{m-1}$ such that $q_i\equiv in \pmod m$ for each i.

Claim 1. $q_1\leqslant n$.

Proof. We have $q_1=am+n$ for some $a\in\mathbb{Z}$. When we apply $\tilde{\tau}_0$, we replace $am+n$ and $(1-a)m-n$ with $am-n$ and $(1-a)m+n$. This changes $\sum_{k\in\mathcal{X}(\nu)}(k-\frac{m}{2})^2$ by $4(1-2a)mn$, which is negative if $a>0$, contradicting the choice of ν. So $a\leqslant 0$.

Claim 2. For $i=1,\dots,m-2$ we have $q_{i+1}-q_i\leqslant n$.

Proof. Suppose not. Since $q_{i+1}-q_i=q_{m-i}-q_{m-i-1}$, we can assume $i\leqslant m'$. Now applying $\tilde{\tau}_i$ yields a contradiction to the choice of ν, as in Claim 1.

These two claims imply immediately that ν is an n-bar-core, by Lemma 4.2. Now suppose $p_\nu\notin\mathcal{R}^m_n$. This means that either $p_j-p_{j+1}>n$ for some $j\in\{1,\dots,m'-1\}$, or $p_{m'}>\max\left\{n,\frac{m+n}{2}\right\}$.

Suppose $p_j-p_{j+1}>n$. Since by assumption $p_1>\dots>p_{m'}$, this means that there no p_k between p_j and p_{j+1} (and so there is no q_k between p_j and p_{j+1}). Since $p_j>n$, we must therefore have $q_1\leqslant p_{j+1}$, by Claim 1. But now there must be some i such that $q_i\leqslant p_{j+1}$ and $q_{i+1}\geqslant p_j$, and this contradicts Claim 2.

On the other hand, suppose $p_{m'}>\max\left\{n,\frac{m+n}{2}\right\}$. Then we have $p_{m'}-(m-p_{m'})>n$. The assumption that $p_1>\dots>p_{m'}$ means that there is no q_k between $m-p_{m'}$ and $p_{m'}$. Furthermore, by Claim 1 we have $q_1<p_{m'}$, so $q_1\leqslant(m-p_{m'})$. But now as in the previous case there must be some i such that $q_i\leqslant m-p_{m'}$ and $q_{i+1}\geqslant p_{m'}$, and again this contradicts Claim 2. □

As a consequence, we see that the element $\nu\in O$ is uniquely defined, since by Proposition 4.3 O cannot contain more than one n-bar-core. Another consequence is a new proof of the following result of Olsson [O, Theorem 4].

Theorem 4.5. *Suppose m and n are coprime odd positive integers, and λ is an m-bar-core. Then the n-bar-core of λ is also an m-bar-core.*

Proof. The case $m=1$ is trivial, so we may assume $m>1$. Then by Proposition 4.4, the level n orbit O containing λ also contains a n-bar-core ν. By Proposition 4.3 λ and ν have the same n-bar-core, i.e. ν is the n-bar-core of λ. Since $\nu\in O$, ν is an m-bar-core. □

Another consequence of Proposition 4.4 is that two m-bar-cores have the same n-bar-core only if they lie in the same level n orbit.

Corollary 4.6. *Suppose m and n are coprime odd positive integers, and that λ and μ are m-bar-cores which have the same n-bar-core. Then λ and μ lie in the same level n orbit.*

Proof. Let ν be the n-bar-core of λ and μ. Then ν lies in both the level n orbit containing λ and the level n orbit containing μ; so these orbits coincide. □

4.3 Symmetry

Now we consider the symmetry in the diagram on the first page. We have seen that under the action χ_n on m-bar-cores, the n-bar-core of an m-bar-core is preserved. However, this symmetry is obscured in the diagram on the first page because of the replacement of m'-points by m'-sets, or equivalently alcoves by dominant alcoves.

To show the symmetry corresponding to χ_n, we consider the whole of the space $\mathbb{R}^{m'}$. In our examples, we continue to take $m = 5$ and $n = 3$. Figure 2 shows part of the $\mathbb{R}^2$, with hyperplanes drawn. The marked 2-points are those in the level 3 orbit containing $\odot$.

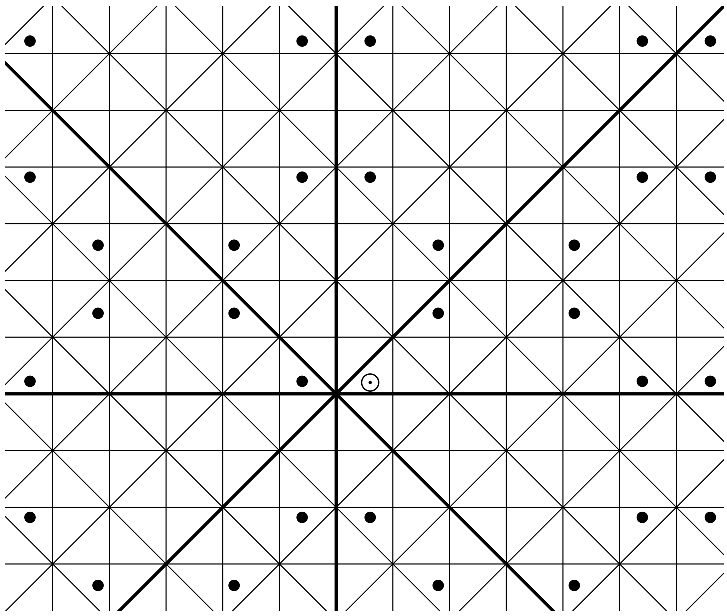


Figure 2: the level 3 orbit of $\odot$ in $\mathbb{R}^2$

To see the corresponding orbit on 2-sets, we fold the diagram in Figure 2 along the bold lines (which represent the hyperplanes $H_{12}^{0-}, H_{12}^{1+}, H_1^1, H_2^1$). We obtain the diagram in Figure 3, which shows just the alcoves in the dominant region. Comparing this with the diagram on the first page, we see that this orbit corresponds to the set of 5-bar-cores whose 3-bar-core is empty.

Now we consider the additional symmetry in the diagram on the first page: the reader will observe that the cores in the diagram are invariant under reflection in the bold lines. These reflections (extended to the whole plane) are the reflections contained in the image of the action ψ_3. The next proposition shows that this symmetry holds in general.

Proposition 4.7. *Suppose m, n are coprime odd positive integers, and p, q are m'-points which lie in the same orbit under the level n action ψ_n. Then the m-bar-cores corresponding to p, q have the same n-bar-core.*

Proof. This is immediate from Proposition 4.1 and the formulæ for $\check{\tau}_0, \dots, \check{\tau}_{m'}$ in §3.2. □

Note that, unlike the orbits for the action χ_n, different orbits under ψ_n can yield the same

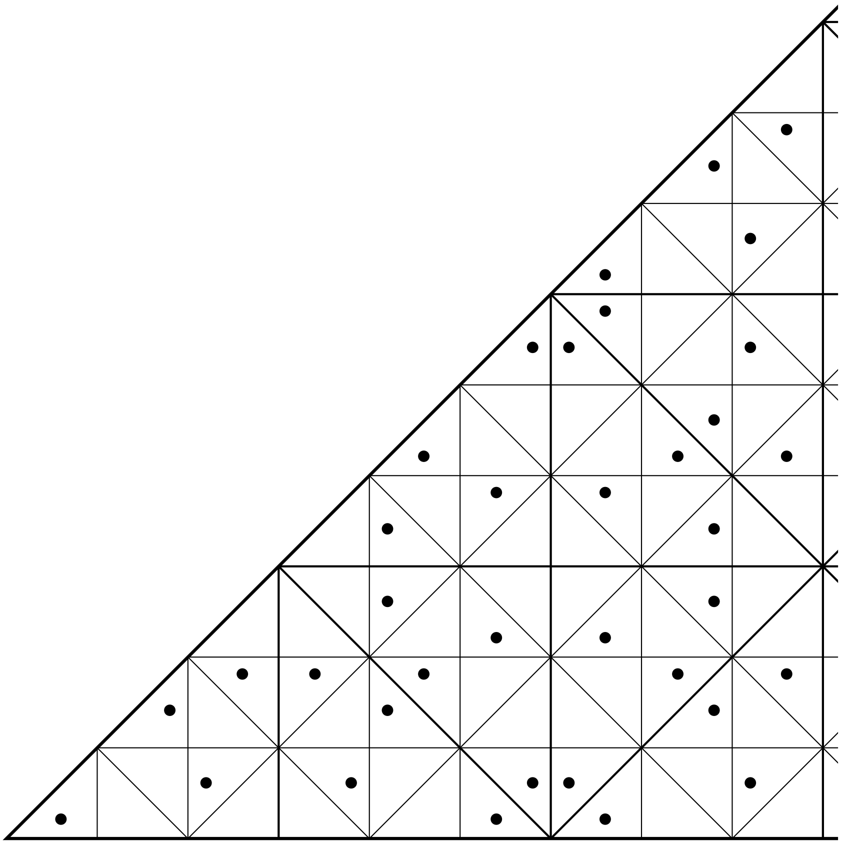


Figure 3: the level 3 orbit of $\odot$ in the dominant region of $\mathbb{R}^2$

n-bar-core; this can be seen in the diagram on the first page, where there are four different orbits yielding the empty partition.

5 The largest (m, n)-bar-core

In this section we briefly comment on bar partitions which are both m- and n-bar-cores, when m and n are coprime odd positive integers; we call such partitions *(m, n)-bar-cores*. It can be inferred from the results of previous sections that there are only finitely many (m, n)-bar-cores. In fact, Bessenrodt and Olsson have shown [BO, Theorem 3.2] that the exact number (writing $m = 2m' + 1$, $n = 2n' + 1$) is $\binom{m'+n'}{m'}$. Moreover, one of these (m, n)-bar-cores (the *Yin partition*) is maximal in the sense that its Young diagram contains the Young diagram of any other (m, n)-bar-core [BO, Theorem 3.6]. We can describe the Yin partition in terms of our set-up; as before, we assume $m \geqslant 3$ and write $m = 2m' + 1$. It follows from Propositions 4.3 and 4.4 that if ν is an (m, n)-bar-core, then the corresponding m'-point lies in the level n parallelpiped $\mathcal{R}^m_n$. The Yin partition is the (m, n)-bar-core corresponding to the vertex of $\mathcal{R}^m_n$ opposite the origin;

this is the point

$$\begin{cases}(m'n, (m'-1)n, \dots, 2n, n) & (n > m)\\ \left(\frac{m+(m-2)n}{2}, \frac{m+(m-4)n}{2}, \dots, \frac{m+3n}{2}, \frac{m+n}{2}\right) & (m > n).\end{cases}$$

It is possible to use this set-up to give a new proof that the Yin partition is maximal; this is entirely analogous to the proof in [F, §5] that there is a maximal (m, n)-bar-core when s, t are coprime positive integers. Since this result is relatively easy to prove by other means, we omit the details.

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