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Adjustment matrices for weight three blocks of Iwahori–Hecke algebras

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Abstract

We compute the 'adjustment matrices' for weight 3 blocks of Iwahori–Hecke algebras of type A, in characteristic 2 or 3. This enables all the decomposition numbers for weight 3 blocks to be calculated.

Contents

1 Introduction

Let $\mathbb{F}$ be any field, and q an invertible element of $\mathbb{F}$. Given a non-negative integer n, let $\mathcal{H}_n = \mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ denote the *Iwahori–Hecke* algebra of the symmetric group $\mathfrak{S}_n$. This has generators $T_1, \ldots, T_{n-1}$ and relations

$$\begin{aligned}(T_i - q)(T_i + 1) &= 0 && (1 \leqslant i \leqslant n-1)\\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && (1 \leqslant i \leqslant n-2)\\ T_iT_j &= T_jT_i && (1 \leqslant i \leqslant j-2 \leqslant n-3).\end{aligned}$$

Of course, if $q = 1$ then $\mathcal{H}_n$ is simply the group algebra of $\mathfrak{S}_n$.

The algebra $\mathcal{H}_n$ arises in the study of groups with BN-pairs, and its representation theory has been extensively studied. If q is not a root of unity in $\mathbb{F}$, then $\mathcal{H}_n$ is semi-simple; if q is a root of unity, then the representation theory is very similar to the representation theory of $\mathfrak{S}_n$ in characteristic p, with the prime p replaced by the integer e which is defined to be the least integer such that $1 + q + \cdots + q^{e-1} = 0$ in $\mathbb{F}$. One of the most important problems in the representation theory of $\mathcal{H}_n$ is to determine the *decomposition numbers*, i.e. the composition multiplicities of the irreducible modules in the so-called Specht modules; in the case $q = 1$, these are the decomposition numbers in the usual representation-theoretic sense. This problem remains open in general, but has been solved in some special cases. If $\mathbb{F}$ has infinite characteristic (throughout this paper we use the convention that the *characteristic* of $\mathbb{F}$ is the order of the prime subfield of $\mathbb{F}$) then there is a recursive method – the *LLT algorithm* – for calculating the decomposition numbers. If $\mathbb{F}$ has finite characteristic, then we can still use the infinite characteristic result: if we take a primitive eth root of unity q' in $\mathbb{C}$, then the decomposition matrix for $\mathcal{H}_n$ may be obtained from the decomposition matrix for $\mathcal{H}_{\mathbb{C},q'}(\mathfrak{S}_n)$ by post-multiplying by a certain square matrix with non-negative integer entries called the *adjustment matrix* for $\mathcal{H}_n$. So in effect the problem of calculating the decomposition matrix of $\mathcal{H}_n$ is equivalent to calculating its adjustment matrix.

Another approach which enables us to determine some decomposition numbers for $\mathcal{H}_n$ is to look at the blocks of $\mathcal{H}_n$ individually, and to concentrate on blocks of small weight. To each block of $\mathcal{H}_n$ is associated a non-negative integer called the weight of the block, and the complexity of the representation theory of blocks of weight w increases with w. Blocks of weight 0 are simple (and indeed all simple blocks have weight 0); and blocks of weight 1 are of finite representation type, and are well understood. Blocks of weight two were addressed by Richards [14] who gave a combinatorial description of their decomposition numbers (assuming that $\mathrm{char}(\mathbb{F}) \neq 2$). Using the Jantzen–Schaper formula, he showed that the decomposition numbers for weight 2 blocks are all at most 1 and that the adjustment matrices for these blocks are trivial; this is in accord with James's Conjecture, which suggests that the adjustment matrix for a block of $\mathcal{H}_n$ of weight w should be trivial if $w < \mathrm{char}(\mathbb{F})$. The first author [4] extended Richards's work to characteristic 2 and computed the adjustment matrices in this case. Blocks of weight 3 have been studied by several authors in the case where $\mathrm{char}(\mathbb{F}) \geqslant 5$, and the first author finally showed [5] that the decomposition numbers for such blocks are all at most 1, and verified James's Conjecture for weight 3 blocks.

The purpose of the present paper is to calculate the adjustment matrices for weight 3 blocks of $\mathcal{H}_n$ in the case where the characteristic of $\mathbb{F}$ is 2 or 3. The adjustment matrices remain mysterious in general, and it is difficult to observe general phenomena, except in blocks whose weight is less than $\mathrm{char}(\mathbb{F})$ (where James's Conjecture predicts the adjustment matrix) and in so-called 'Rouquier blocks', where the adjustment matrices are related to the decomposition matrices of q-Schur algebras (although this relationship is only conjectural when $q \neq 1$). So the information in the present paper may be helpful to those trying to spot patterns in adjustment matrices. However, the main result of this paper is awkward even to state without introducing additional terminology.

In the remainder of this introduction we introduce the background theory we shall require, and in Section 2, we review the known results for blocks of weight 1, 2 and 3. In Section 3, we introduce the machinery and notation required to state our main theorem, and we prove some general properties of adjustment matrices which facilitate an inductive proof of the main theorem. In the remaining sections, we prove the main theorem in certain special cases which suffice to allow a general proof by induction: we deal with the 'principal' block of $\mathcal{H}_{3e}$ in Section 4, blocks with rectangular cores in Section 5 and blocks with birectangular cores in Section 6.

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1.1 Background and notation

Mathas's book [12] is now the standard introduction to the representation theory of $\mathcal{H}_n$, and we take much of our notation from there. Note, however, that we use the Specht modules defined by Dipper and James [2].

As stated above, q is an invertible element of the field $\mathbb{F}$, and e is the least integer such that $1 + q + \cdots + q^{e-1} = 0$ in $\mathbb{F}$; we are assuming that such an integer exists.

We record here two items of notation we use for modules. If M is a module and n a non-negative integer, then $M^{\oplus n}$ denotes a direct sum of n isomorphic copies of M, and we write $N \sim M^n$ to indicate that N has a filtration with n factors all isomorphic to M.

1.1.1 Partitions, Specht modules and the abacus

As usual, a *partition* of n is defined to be a decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers whose sum is n. For each partition λ of n one defines a *Specht module* S^λ for $\mathcal{H}_n$. If λ is e-regular (that is, if it does not have e equal positive parts), then S^λ has an irreducible cosocle D^λ, and the modules D^λ give a complete set of irreducible $\mathcal{H}_n$-modules as λ ranges over the set of e-regular partitions of n.

A useful way to represent partitions of n is on the *abacus*. We take an abacus with e vertical runners, and we mark positions on these runners from the top down; we then number the positions with non-negative integers, so that the numbers $0, \dots, e-1$ appear across the top of the abacus from left to right, the numbers $e, \dots, 2e-1$ appear from left to right below these, and so on. For example, if $e = 3$, the numbering is as follows:

$$\begin{array}{ccc} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \\ \vdots & \vdots & \vdots \end{array}.$$

Given a partition λ, we choose an integer r greater than the number of non-zero parts of λ, and define the *beta-numbers* $\beta_1, \ldots, \beta_r$ by

$$\beta_i = \lambda_i + r - i.$$

Now we place a bead on the abacus at position β_i for each i. The resulting configuration is called an abacus display for λ.

One of the most useful features of the abacus is that it tells us in which block of $\mathcal{H}_n$ a partition lies (we abuse notation throughout this paper by saying that a partition lies in a block B to mean that the Specht module S^λ lies in B). Given an abacus display for λ, we slide all the beads as far up their runners as they will go. The partition whose abacus display we obtain in this way is called the e-core of λ. This is a partition of $n - ew$ for some non-negative integer w, which is referred to as the *weight* of λ. Nakayama's 'Conjecture' states that two partitions lie in the same block of $\mathcal{H}_n$ if and only if they have the same e-core. This automatically implies that they have the same weight, and so we may speak of the (e)-core and the weight of a block. We also define the abacus display for a block by specifying the number of beads on each runner, without specifying their positions.

In this paper we shall frequently talk of moving a bead on the abacus one place to the left. We wish to include the case where the bead lies on the leftmost runner of the abacus, and we abuse notation by saying 'move a bead one place to the left' to mean 'move a bead from position j to position $j-1$', even if $j \equiv 0 \pmod{e}$.

1.1.2 Decomposition matrices and adjustment matrices

The central problem in the representation theory of $\mathcal{H}_n$ is to determine the composition multiplicities $[S^\lambda : D^\mu]$ for partitions λ, μ of n with μ e-regular. These are usually recorded in the *decomposition matrix*, which has rows indexed by partitions of n and columns by e-regular partitions of n, with the (λ, μ)-entry of the matrix being $[S^\lambda : D^\mu]$. By restricting attention to the partitions lying in a particular block B of $\mathcal{H}_n$, one may speak of the decomposition matrix for B.

The following simple but very useful result on decomposition numbers follows from the fact that $\mathcal{H}_n$ is a cellular algebra. Recall the usual dominance order $\trianglerighteq$ on partitions of n.

Lemma 1.1. [12, Corollary 4.17] *Suppose λ and μ are partitions of n with μ e-regular. Then $[S^\mu : D^\mu] = 1$, while $[S^\lambda : D^\mu] = 0$ unless $\mu \trianglerighteq \lambda$.*

The main object of study in this paper is the *adjustment matrix* for $\mathcal{H}_n$, which relates the decomposition matrix for $\mathcal{H}_n$ to that for an Iwahori–Hecke algebra over a field of infinite characteristic. Recalling the integer e above, we fix a primitive eth root of unity q' in $\mathbb{C}$. Given a block B of $\mathcal{H}_n$, let γ be the e-core of B, and let B^0 be the block of $\mathcal{H}_{\mathbb{C},q'}(\mathfrak{S}_n)$ with e-core γ; we say that B^0 is *the block of $\mathcal{H}_{\mathbb{C},q'}(\mathfrak{S}_n)$ corresponding to B*. Then we have the following, which is proved using a form of modular reduction.

Theorem 1.2. [12, Theorem 6.35] *Suppose B and B^0 are as above. Let $\mathtt{D}$ and $\mathtt{D}_0$ be the decomposition matrices of B and B^0 respectively, with rows indexed by partitions of n with e-core γ, and columns indexed by e-regular partitions of n with e-core γ. Then there exists a square matrix $\mathtt{A}$ with non-negative integer entries and with rows and columns both indexed by e-regular partitions of n with e-core γ, such that $\mathtt{D} = \mathtt{D}_0\mathtt{A}$.*

The matrix $\mathtt{A}$ in Theorem 1.2 is known as the *adjustment matrix* for B. Given e-regular partitions λ and μ in B, we write $a_{\lambda\mu}$ for the (λ, μ)-entry of the adjustment matrix. Adjustment matrices were introduced by James in [7]; James's Conjecture asserts that if $\mathrm{char}(\mathbb{F}) > w$, then the adjustment matrix for a block of $\mathcal{H}_n$ of weight w is the identity matrix.

1.1.3 The Mullineux map

Let $T_1, \dots, T_{n-1}$ be the standard generators of $\mathcal{H}_n$. Let $\sharp : \mathcal{H}_n \to \mathcal{H}_n$ be the involutory automorphism sending T_i to $q - 1 - T_i$, and let $* : \mathcal{H}_n \to \mathcal{H}_n$ be the anti-automorphism sending T_i to T_i. Given a module M for $\mathcal{H}_n$, define $M^\sharp$ to be the module with the same underlying vector space and with action

$$h \cdot m = h^\sharp m,$$

and define M^* to be the module with underlying vector space dual to M and with $\mathcal{H}_n$-action

$$h \cdot f(m) = f(h^* m).$$

(Note that in the symmetric group case $q = 1$, $M^\sharp$ is simply $M \otimes \mathrm{sgn}$, where sgn is the one-dimensional signature representation, while M^* is the usual dual module to M.)

Of course, the functor $M \mapsto M^\sharp$ respects the block structure of $\mathcal{H}_n$; that is, if M and N lie in the same block of $\mathcal{H}_n$, then $M^\sharp$ and $N^\sharp$ lie in the same block. If a module M lies in a block B, then we write $B^\sharp$ for the block in which $M^\sharp$ lies, and we call $B^\sharp$ the *conjugate block* of B.

Now we look at the effect of these functors on Specht modules; let λ' denote the partition conjugate to λ.

Lemma 1.3. [12, Exercise 3.14(iii)] *For any partition λ,*

$$S^{\lambda'} \cong (S^\lambda)^{\sharp *}.$$

Next we turn to the simple modules D^λ, for λ e-regular. It follows from the cellularity of $\mathcal{H}_n$ that $(D^\lambda)^* \cong D^\lambda$. If we let $\lambda^\diamond$ denote the e-regular partition such that $(D^\lambda)^\sharp \cong D^{\lambda^\diamond}$, then $\diamond$ is an involutory bijection from the set of e-regular partitions of n to itself. This bijection is given combinatorially by Mullineux's algorithm [13]; we shall not describe this here, but we note that given an e-regular partition λ, the partition $\lambda^\diamond$ depends only on λ and e, and not on the underlying field.

The functor $M \mapsto M^\sharp$ is a self-equivalence of the category of $\mathcal{H}_n$-modules, and we have the following consequence for decomposition numbers.

Corollary 1.4. *For any partitions λ and μ with μ e-regular,*

$$[S^\lambda : D^\mu] = [S^{\lambda'} : D^{\mu^\diamond}].$$

Combining this with Lemma 1.1, we get the following.

Corollary 1.5. *Suppose λ and μ are partitions of n with μ e-regular. Then $[S^{\mu^{\diamond\prime}} : D^\mu] = 1$, while $[S^\lambda : D^\mu] = 0$ unless $\lambda \trianglerighteq \mu^{\diamond\prime}$.*

1.1.4 Induction and restriction

If $0 \leqslant \kappa \leqslant n$, then $\mathcal{H}_{n-\kappa}$ is naturally a subalgebra of $\mathcal{H}_n$, and we have induction and restriction functors between the module categories of these two algebras. Given a module M lying in a block B of $\mathcal{H}_n$ and given a block A of $\mathcal{H}_{n-\kappa}$, we write $M{\downarrow}_A^B$ for the projection onto A of the restriction of M from $\mathcal{H}_n$ to $\mathcal{H}_{n-\kappa}$. Similarly, we write $N{\uparrow}_A^B$ for a module induced from A to $\mathcal{H}_n$ and then projected onto B.

For Specht modules and simple modules, there are various 'branching rules' describing the effects of these functors. Suppose that we have an abacus display for B, and that r_1 and r_2 are runners on the abacus with r_2 immediately to the right of r_1. Let A be the block of $\mathcal{H}_{n-\kappa}$ whose abacus display is obtained from that for B by moving κ beads from r_2 to r_1. Given a partition λ in B, let $\lambda^{-1}, \dots, \lambda^{-s}$ be the distinct partitions which may be obtained by moving κ beads on runner r_2 one place to the left. Similarly, given a partition ν in A, let $\nu^{+1}, \dots, \nu^{+t}$ be those partitions which may be obtained by moving κ beads on runner r_1 one place to the right.

Theorem 1.6. The Branching Rule [12, Corollary 6.2]
The module $S^\lambda\downarrow^B_A$ has a filtration in which the factors are $S^{\lambda^{-1}}, \dots, S^{\lambda^{-s}}$, each occurring $\kappa!$ times. Similarly, $S^\nu\uparrow^B_A$ has a filtration in which the factors are $S^{\nu^{+1}}, \dots, S^{\nu^{+t}}$, each occurring $\kappa!$ times.

For the simple modules D^λ, the situation is rather more complicated. Given A and B as above, suppose that we have e-regular partitions λ in B and μ in A. We define the *signature* of λ to be the sequence of $+$ and $-$ signs obtained by examining runners r_1 and r_2 from bottom to top, writing a $+$ whenever there is a bead on runner r_1 with no bead immediately to the right, and writing a $-$ when there is a bead on runner r_2 with no bead immediately to the left. We now form the *reduced signature* by successively deleting all adjacent pairs $+-$. If there are any $-$ signs in the reduced signature, the corresponding beads on runner r_2 are called *normal*. If there are at least κ normal beads, then we let λ^- be the (e-regular) partition obtained by moving the κ highest normal beads one place to the left.

We form the reduced signature of ν in exactly the same way; if this contains any $+$ signs, then the corresponding beads on runner r_1 of the abacus are called *conormal*. If there are at least κ conormal beads, then we let ν^+ be the (e-regular) partition obtained by moving the κ lowest conormal beads one place to the right.

Theorem 1.7. [1, §2.5]
- *If there are fewer than κ normal beads in the abacus display for λ, then $D^\lambda\downarrow^B_A = 0$. If there are exactly κ normal beads, then $D^\lambda\downarrow^B_A \cong (D^{\lambda^-})^{\oplus\kappa!}$.*
- *If there are fewer than κ conormal beads in the abacus display for ν, then $D^\nu\uparrow^B_A = 0$. If there are exactly κ conormal beads, then $D^\nu\uparrow^B_A \cong (D^{\nu^+})^{\oplus\kappa!}$.*

We now consider the relationship between the modular branching rule and the Mullineux map. Given an abacus display for B, it is a simple matter to construct an abacus display for $B^\sharp$: we simply rotate the abacus through 180°, and then replace each bead with an empty space and each empty space with a bead. If we do this with the abacus display described above for B and do the same for A, then we find that the abacus display for $A^\sharp$ is obtained from that for $B^\sharp$ by moving κ beads from one runner to the runner immediately to its left. Given an e-regular partition μ in $B^\sharp$, we form the reduced signature using these two runners as described above, and if there are at least κ normal beads, we define the partition μ^-. Then we have the following, which is essentially the main result of [6].

Proposition 1.8. *Suppose λ is an e-regular partition in B. Then λ^- is defined in A if and only if $(\lambda^\diamond)^-$ is defined in $A^\sharp$, and if these partitions are defined then we have $(\lambda^-)^\diamond = (\lambda^\diamond)^-$.*

1.1.5 The Jantzen–Schaper formula

One of the most important tools for calculating and estimating the decomposition numbers of $\mathcal{H}_n$ is the (q-analogue of the) Jantzen–Schaper formula. We shall use this in several places in this paper. Details of the formula may be found in [9]. We note that the formula allows a strengthening of Lemma 1.1 and Corollary 1.5, by using a coarser form of the dominance order. Given partitions λ and μ of n and with e defined as above, we say that λ *dominates μ in the Jantzen–Schaper order* if $\lambda \trianglerighteq \mu$ and if the Young diagram for μ may be obtained from that for λ by removing a rim hook of length divisible by e and then adding a rim hook of the same length, or equivalently if an abacus display for μ may be obtained from an abacus display for λ by moving one bead up its runner and moving another bead down its runner. We extend this order transitively to give a partial order, of which the usual dominance order is a refinement. We use the symbol $\trianglerighteq$ for this new order, which we use exclusively from now on. Although this order depends on the integer e, no confusion should arise.

1.1.6 Scopes equivalences

Various Morita equivalences for blocks of the same weight were found by Scopes [15]; her results were generalised to Iwahori–Hecke algebras by Jost [10].

Suppose that A is a block of $\mathcal{H}_{n-\kappa}$ of weight w, and B a block of $\mathcal{H}_n$ of weight w. Suppose that there is an abacus display for B with runners r_1 and r_2 such that:

- r_2 lies immediately to the right of r_1;
- there are exactly κ more beads on runner r_2 than on runner r_1;
- by interchanging runners r_2 and r_1, we obtain an abacus display for A.

Then we say that A and B form a $[w:\kappa]$*-pair*.

Suppose that A and B form a $[w:\kappa]$-pair with $w \leqslant \kappa$, and let λ be a partition in B. Then there are exactly κ beads on runner r_2 in the abacus display for λ which do not have beads immediately to their left. If we move each of these beads one place to the left, we obtain a partition in A, which we denote as $\Phi(\lambda)$. We have the following.

Theorem 1.9. [12, p. 127] *Let A, B and Φ be as above. Then:*

- *Φ is a bijection between the set of partitions in B and the set of partitions in A;*
- *$\Phi(\lambda)$ is e-regular if and only if λ is e-regular;*
- *for any partition λ in B,*
$$S^{\lambda}{\downarrow}_A^B \sim (S^{\Phi(\lambda)})^{\kappa!}, \qquad S^{\Phi(\lambda)}{\uparrow}_A^B \sim (S^{\lambda})^{\kappa!};$$
- *for any e-regular partition λ in B,*
$$D^{\lambda}{\downarrow}_A^B \cong (D^{\Phi(\lambda)})^{\oplus\kappa!}, \qquad D^{\Phi(\lambda)}{\uparrow}_A^B \cong (D^{\lambda})^{\oplus\kappa!};$$
- *the correspondence $D^{\lambda} \leftrightarrow D^{\Phi(\lambda)}$ is induced by a Morita equivalence between B and A.*

In view of Theorem 1.9, we say two blocks are *Scopes equivalent* if they form a $[w:\kappa]$-pair for some $\kappa \geqslant w$. We extend this reflexively and transitively to define an equivalence relation on the set of blocks of weight w, and we refer to an equivalence class as a *Scopes class*.

1.1.7 Pyramids

In order to understand the combinatorics of Scopes classes, Richards [14] defined the notion of a *pyramid*. Let γ be an e-core, and choose an abacus display for γ in which there is at least one bead on each runner. Let $p_1 < \cdots < p_e$ be those integers such that there is a bead at position p_i but no bead at position $p_i + e$, for each i. Then exactly one p_i lies in each congruence class modulo e. We number the runners of the abacus so that the bead at position p_i lies on runner i for each i. For $i < j$ the integer $p_j - p_i$ is a positive integer not divisible by e, and it does not depend on the choice of abacus display for γ. Given $w \geqslant 0$, we define

$${}_ia_j = \begin{cases} \left\lfloor \frac{p_j - p_i}{e} \right\rfloor & (p_j - p_i < we) \\ w-1 & (p_j - p_i > we) \end{cases}$$

for $1 \leqslant i \leqslant j \leqslant e$. We extend this notation to include all pairs of integers $i \leqslant j$ by putting ${}_ia_j = w-1$ if $i \leqslant 0$ or $j > e$. If B is the block of $\mathcal{H}_n$ with core γ and weight w, then the set of integers ${}_ia_j$ is called the pyramid for B; we shall write ${}_ia_j(B)$ when it is not clear to which block we are referring. We shall also use shorthands such as ${}_i0_j$ to mean ${}_ia_j = 0$ and ${}_i1^+{}_j$ to mean ${}_ia_j \geqslant 1$.

Note that our definition of the pyramid is slightly different from that of Richards. He defines the pyramid using the integers ${}_iA_j = w - 1 - {}_ia_j$.

A crucial property of pyramids is the following.

Proposition 1.10. [14, Lemma 3.1 & Proposition 3.3] *Two blocks of weight w are Scopes equivalent if and only if they have the same pyramid.*

2 Blocks of small weight

In this section, we review some notation and basic results for blocks of small weight. In the interests of brevity, we state only essential results.

2.1 Blocks of weight 1

The following theorem is well known.

Proposition 2.1. *Suppose B is a block of $\mathcal{H}_n$ of weight 1. Then there are exactly e partitions in B, which are totally ordered by the Jantzen–Schaper ordering: $\lambda^1 \rhd \cdots \rhd \lambda^e$. Furthermore, λ^i is e-regular if and only if $1 \leqslant i \leqslant e-1$, and the decomposition number $[S^{\lambda^i} : D^{\lambda^j}]$ equals 1 if $j = i$ or $j = i-1$, and 0 otherwise, irrespective of* $\mathrm{char}(\mathbb{F})$. *In particular, the adjustment matrix for B is trivial.*

2.2 Adjustment matrices for blocks of weight 2

In this section, we give the results which describe the adjustment matrix for a block of $\mathcal{H}_n$ of weight 2; we shall need this in order to find the adjustment matrices for weight 3 blocks.

First we need to describe some notation for partitions in blocks of weight 2. Suppose B is a block of $\mathcal{H}_n$ of weight 2, and take an abacus display for B. We number the runners of the abacus as in Section 1.1.7. If λ is a partition in B, then the abacus display for λ is obtained from the abacus display for its core by moving two beads down one space each or by moving one bead down two spaces. We write:

- $\lambda = [i, j]$ if the abacus display for λ is obtained by moving two beads down one space each, on runners i and j (where i may equal j);
- $\lambda = [i]$ if the abacus display for λ is obtained by moving the lowest bead on runner i down two spaces.

If the numbers of beads on the runners of the abacus are $b_1, \ldots, b_e$ from left to right, we may refer to this as the $\langle b_1, \ldots, b_e \rangle$ notation. Note that our numbering of runners means that this notation for λ does not depend on the choice of abacus display.

Now we can describe the adjustment matrix for B.

Theorem 2.2. *Suppose B is a weight 2 block of $\mathcal{H}_n$, and that λ and μ are e-regular partitions in B.*

1. [14] *If* $\mathrm{char}(\mathbb{F}) \geqslant 3$, *then* $a_{\lambda\mu} = \delta_{\lambda\mu}$.
2. [4, Corollary 2.4] *If* $\mathrm{char}(\mathbb{F}) = 2$, *then*

$$a_{\lambda\mu} = \begin{cases} 1 & (\lambda = [i,i],\ \mu = [i],\ {}_{i-1}1_i1_{i+1},\ 2 \leqslant i \leqslant e) \\ 1 & (\lambda = [i,i],\ \mu = [i,i+1],\ {}_{i-1}1_i0_{i+1},\ 2 \leqslant i < e) \\ \delta_{\lambda\mu} & (\text{otherwise}). \end{cases}$$

2.3 Blocks of weight 3

2.3.1 Notation for weight 3 blocks

In this section, we describe notation for blocks of weight 3, and note some basic results concerning $[3:\kappa]$-pairs.

Suppose B is a block of $\mathcal{H}_n$ of weight 3, and fix an abacus display for B. If λ is a partition in B, then we write:

- $\lambda = [i]$ if the display for λ is obtained from the display for the core of B by moving the lowest bead on runner i down three spaces;
- $\lambda = [i,j]$ if the display for λ is obtained by moving the lowest bead on runner i down two spaces, and a bead on runner j down one space (where possibly $i = j$);
- $\lambda = [i,j,k]$ if the display for λ is obtained by moving three beads down one space each on runners i, j and k (which may coincide).

As with partitions of weight 2, we may refer to this as the $\langle b_1, \dots, b_e \rangle$ notation; we may even write the partition $[i]$ as $[i]_B$ or $[i \mid b_1, \dots, b_e]$ (and similarly for $[i,j]$ and $[i,j,k]$) to emphasise which block or abacus display we are using. Where there is a risk of confusion, we shall be explicit about the weight of a partition described using this notation.

An advantage of using our numbering of the runners of the abacus is that if A and B are blocks forming a $[3:\kappa]$-pair with $\kappa \geqslant 3$, then the map Φ described in §1.1.6 becomes

$$\begin{array}{ccc} [i,j,k] & \longmapsto & [i,j,k], \\ [i,j] & \longmapsto & [i,j], \\ [i] & \longmapsto & [i], \end{array}$$

for all i,j,k.

2.3.2 Rouquier blocks

There is a class of blocks which is particularly well understood. These blocks are defined for any weight, but we restrict our attention to blocks of weight 3.

Suppose B is a weight 3 block of $\mathcal{H}_n$ with pyramid $({}_ia_j)$. We say that B is *Rouquier* if ${}_ia_j = 2$ for all $i < j$.

It is easy to verify which partitions in a Rouquier block are e-regular:

- $[i]$ is e-regular if and only if $i \geqslant 2$;
- $[i,j]$ is e-regular if and only if $i,j \geqslant 2$;
- $[i,j,k]$ is e-regular if and only if $i,j,k \geqslant 2$.

One particular advantage of Rouquier blocks is that, using a theorem of James, Lyle and Mathas, we can derive information about the adjustment matrices in a very direct way from the decomposition numbers. For blocks of weight 3, this is actually sufficient to calculate the adjustment matrices completely. In order to state the result we need, we define an equivalence relation on the partitions in a Rouquier block. Given an abacus display for a partition λ, we reach the display for the core of λ by moving a bead up one space on the abacus three times. We define the *i-mass* of λ to be the number of these moves which take place on runner i. For example, the i-mass of the partition $[i,i,j]$ is 2 while the k-mass of this partition is 0, if $i \neq j \neq k \neq i$. Given partitions λ and μ, we write $\lambda \leftrightarrow \mu$ if and only if the i-mass of λ equals the i-mass of μ for each i.

Proposition 2.3. [8, Proposition 2] *Suppose B is a Rouquier block of $\mathcal{H}_n$. If λ and μ are e-regular partitions in B, then*

$$a_{\lambda\mu} = \begin{cases} [S^\lambda : D^\mu] & (\lambda \leftrightarrow \mu) \\ 0 & (\textit{otherwise}). \end{cases}$$

We shall use this result later in order to calculate the adjustment matrices for Rouquier blocks explicitly. These will be used to deal with difficult cases in the inductive proof of our main theorem.

2.3.3 $[3 : \kappa]$-pairs

In studying weight 3 blocks, $[3 : \kappa]$-pairs are a vital tool. Since blocks forming a $[3 : \kappa]$-pair with $\kappa \geqslant 3$ are Morita equivalent, the study of blocks of weight 3 centres around $[3 : 1]$- and $[3 : 2]$-pairs. Here we set up some notation and prove some basic results for such pairs, following Martin and Russell [11].

Suppose A and B form a $[3 : \kappa]$-pair, and that the abacus display for B is obtained from that for A by swapping the adjacent runners j and k, where $j < k$ (recall our numbering system for runners in Section 1.1.7). We say that a partition λ in B is *exceptional* for this $[3 : \kappa]$-pair if there are more than κ beads on runner k of the abacus display for B with no bead immediately to the left, and *non-exceptional* otherwise. Similarly, we say that a partition ν in A is exceptional if there are more than κ beads on runner j of the abacus display for ν with no bead immediately to the right. Note that if λ is a partition in B, then there are always at least κ normal beads on runner k of the abacus display for λ; we define $\Phi(\lambda)$ to be the partition in A obtained by moving the κ highest normal beads one place to the left.

If λ is a non-exceptional partition, then there are exactly κ beads on runner k with no bead immediately to the left, and so $\Phi(\lambda)$ is obtained by moving these κ beads to the right. In particular, if $\kappa \geqslant 3$, then the definition of Φ agrees with the definition in §1.1.6, since in that case every partition in B is non-exceptional.

If λ is e-regular, then we say that the simple module D^λ is exceptional if there are more than κ normal beads on runner k of the abacus display for λ. We make a similar definition for A: if ν is an e-regular partition in A we say that D^ν is exceptional if there are more than κ conormal beads on runner j.

The following is then a familiar result in the study of weight 3 blocks.

Proposition 2.4. *Suppose that A and B form a $[3 : \kappa]$-pair as above, and that λ is a partition in B.*

- *Φ is a bijection between the set of partitions in B and the set of partitions in A.*
- *$\Phi(\lambda)$ is e-regular if and only if λ is.*
- *If λ is non-exceptional, then*

$$S^\lambda{\downarrow}_A^B \sim (S^{\Phi(\lambda)})^{\kappa!}, \qquad S^{\Phi(\lambda)}{\uparrow}_A^B \sim (S^\lambda)^{\kappa!}.$$

- *If λ is e-regular and D^λ is non-exceptional, then*

$$D^\lambda{\downarrow}_A^B \cong (D^{\Phi(\lambda)})^{\oplus\kappa!}, \qquad D^{\Phi(\lambda)}{\uparrow}_A^B \cong (D^\lambda)^{\oplus\kappa!}.$$

- *If λ is e-regular and D^λ is exceptional, then $D^\lambda{\downarrow}_A^B$ and $D^{\Phi(\lambda)}{\uparrow}_A^B$ are not semi-simple.*

We also need the following.

Lemma 2.5. *Suppose A and B form a* $[3:\kappa]$*-pair, and define* Φ *as above. Let* Φ' *be the function from the set of partitions in* $B^\sharp$ *to the set of partitions in* $A^\sharp$ *defined in the same way. If* λ *is an e-regular partition in B, then*

$$\Phi'(\lambda^\diamond) = (\Phi(\lambda))^\diamond,$$

and D^λ *is exceptional if and only if* $D^{\lambda^\diamond}$ *is exceptional.*

Proof. This follows immediately from Propositions 1.8 and 2.4. □

Now we examine the cases $\kappa = 1$ and 2 in more detail.

2.3.4 $[3:1]$-pairs

Suppose that A and B form a $[3:1]$-pair, and that the abacus display for B is obtained from that for A by swapping runners j and k. Then there are $3e$ exceptional partitions in each of A and B, which we denote as follows (with $1 \leqslant l \leqslant e$):

A

$$\overline{\alpha}_l = \begin{cases} [j,l] & (l \neq k) \\ [j] & (l = k) \end{cases}$$
$$\overline{\beta}_l = \begin{cases} [j,k,l] & (l \neq k) \\ [k,j] & (l = k) \end{cases}$$
$$\overline{\gamma}_l = \begin{cases} [k,k,l] & (l \neq j,k) \\ [k,k,k] & (l = j) \\ [k,k] & (l = k) \end{cases}$$

B

$$\alpha_l = \begin{cases} [k,k,l] & (l \neq j,k) \\ [k,k,k] & (l = j) \\ [k,k] & (l = k) \end{cases}$$
$$\beta_l = \begin{cases} [j,k,l] & (l \neq k) \\ [k,j] & (l = k) \end{cases}$$
$$\gamma_l = \begin{cases} [j,l] & (l \neq k) \\ [j] & (l = k). \end{cases}$$

The exceptional simple modules in A and B are the modules $D^{\overline{\alpha}_l}$ and D^{α_l} for those l such that α_l is e-regular. The bijection Φ acts on the exceptional partitions as follows:

$$\begin{aligned} \Phi : \alpha_l &\longmapsto \overline{\alpha}_l \\ \beta_l &\longmapsto \overline{\gamma}_l \\ \gamma_l &\longmapsto \overline{\beta}_l. \end{aligned}$$

We now give some results on the decomposition numbers of blocks forming a $[3:1]$-pair. Let A and B be as above, and let C be the block of weight 1 whose abacus display is obtained from that for B by moving a bead from runner j to runner k. We let $\lambda^1 \rhd \cdots \rhd \lambda^e$ be the partitions in C. We get the following result on induction and restriction between B and C from Theorems 1.6 and 1.7.

Proposition 2.6. *Let B and C be as above. Then there is a permutation* $\pi \in \mathfrak{S}_e$ *such that:*

1. *if* λ *is a partition in B, then*

$$S^\lambda \uparrow_B^C \cong \begin{cases} S^{\lambda^l} & (\textit{if } \lambda \textit{ is of the form } \alpha_{\pi(l)}, \beta_{\pi(l)} \textit{ or } \gamma_{\pi(l)}) \\ 0 & (\textit{otherwise}); \end{cases}$$

2. *if* λ *is an e-regular partition in B, then*

$$D^\lambda \uparrow_B^C \cong \begin{cases} D^{\lambda^l} & (\textit{if } \lambda \textit{ is of the form } \alpha_{\pi(l)}) \\ 0 & (\textit{otherwise}). \end{cases}$$

Corollary 2.7. *The partition $\alpha_{\pi(l)}$ is e-regular if and only if $1 \leqslant l \leqslant e-1$. In this case, $D^{\alpha_{\pi(l)}}$ appears exactly once as a composition factor of each of*

$$S^{\alpha_{\pi(l)}},\quad S^{\beta_{\pi(l)}},\quad S^{\gamma_{\pi(l)}},\qquad S^{\alpha_{\pi(l+1)}},\quad S^{\beta_{\pi(l+1)}},\quad S^{\gamma_{\pi(l+1)}},$$

and does not appear as a composition factor of any other Specht module.

Proof. This follows at once from Proposition 2.6, the decomposition matrix of C described in Section 2.1, and the fact that induction is an exact functor. □

Corollary 2.8. *If $1 \leqslant l \leqslant e-1$, then*

$$\alpha_{\pi(l)}{}^{\diamond\prime} = \gamma_{\pi(l+1)}.$$

Proof. By Corollary 1.5, if λ is an e-regular partition then $\lambda^{\diamond\prime}$ is the least dominant partition such that $[S^{\lambda^{\diamond\prime}} : D^{\lambda}] > 0$. The result follows since $\alpha_{\pi(l)} \rhd \alpha_{\pi(l+1)}$ and $\alpha_l \rhd \beta_l \rhd \gamma_l$ for any l. □

2.3.5 [3 : 2]-pairs

In this section we review some background on [3 : 2]-pairs; the notation here is less complex than for [3 : 1]-pairs.

Suppose A and B form a [3 : 2]-pair, and that an abacus display for B is obtained by swapping runners j and k of an abacus display for A. We use the following notation for the exceptional partitions in A and B:

A	B
$\overline{\alpha} = [j]$	$\alpha = [k,k,k]$
$\overline{\beta} = [j,k]$	$\beta = [j,k,k]$
$\overline{\gamma} = [j,k,k]$	$\gamma = [j,k]$
$\overline{\delta} = [k,k,k]$	$\delta = [j]$

The partitions α and $\overline{\alpha}$ are always e-regular, and the only exceptional simple modules for this pair are $D^{\overline{\alpha}}$ and D^{α}. The bijection Φ has the following effect on the exceptional partitions:

$$\begin{aligned}\Phi : \alpha &\longmapsto \overline{\alpha}\\ \beta &\longmapsto \overline{\delta}\\ \gamma &\longmapsto \overline{\gamma}\\ \delta &\longmapsto \overline{\beta}.\end{aligned}$$

Let C be the block of weight zero whose abacus display is obtained from that for B by moving a bead from runner j to runner k. Let ν denote the unique partition in C.

Proposition 2.9.

1. *If λ is a partition in B, then*

$$S^{\lambda}\uparrow_B^C \cong \begin{cases} S^{\nu} & (\text{if } \lambda = \alpha,\ \beta,\ \gamma \text{ or } \delta)\\ 0 & (\text{otherwise}).\end{cases}$$

If in addition λ is e-regular, then

$$D^{\lambda}\uparrow_B^C \cong \begin{cases} D^{\nu} & (\lambda = \alpha)\\ 0 & (\lambda \neq \alpha).\end{cases}$$

2. *The simple module D^α appears exactly once as a composition factor of each of $S^\alpha, S^\beta, S^\gamma, S^\delta$, and does not appear as a composition factor of any other Specht module.*

Proof. Part (1) follows from Theorems 1.6 and 1.7. Part (2) then follows from the exactness of induction and the fact that $S^\nu = D^\nu$. □

3 The main theorem

The statement of the main theorem is rather complicated. To begin with, we consider Rouquier blocks. In the symmetric group case, the adjustment matrices for Rouquier blocks of all weights have been computed (in terms of decomposition matrices for Schur algebras) by Turner [17], and it is conjectured that an analogue of his result holds for Iwahori–Hecke algebras generally. Here, we effectively prove this conjecture for weight 3 blocks; it turns out to be straighforward to compute the adjustment matrices using the Jantzen–Schaper formula in this case.

With the aid of Proposition 2.3, it suffices to calculate the decomposition numbers $d_{\lambda\mu}$ for those pairs (λ, μ) of e-regular partitions with $\mu \rhd \lambda$ and $\lambda \leftrightarrow \mu$. For weight 3 blocks, such pairs are as follows:

- $\lambda = [i,i],\ \mu = [i]\ (2 \leqslant i \leqslant e)$;
- $\lambda = [i,i,i],\ \mu = [i]\ (2 \leqslant i \leqslant e)$;
- $\lambda = [i,i,i],\ \mu = [i,i]\ (2 \leqslant i \leqslant e)$;
- $\lambda = [i,i,k],\ \mu = [i,k]\ (2 \leqslant i,k \leqslant e,\ i \neq k)$.

Using the Jantzen-Schaper formula, we obtain the following.

Proposition 3.1.

1. *Suppose* $\mathrm{char}(\mathbb{F}) = 2$, *and B is a weight* 3 *Rouquier block of $\mathcal{H}_n$. Then*

$$[S^\lambda : D^\mu] = \begin{cases} 0 & (\lambda = [i,i], \mu = [i]) \\ 1 & (\lambda = [i,i,i], \mu = [i]) \\ 0 & (\lambda = [i,i,i], \mu = [i,i]) \\ 1 & (\lambda = [i,i,k], \mu = [i,k], i \neq k). \end{cases}$$

2. *Suppose* $\mathrm{char}(\mathbb{F}) = 3$, *and B is a weight* 3 *Rouquier block of $\mathcal{H}_n$. Then*

$$[S^\lambda : D^\mu] = \begin{cases} 1 & (\lambda = [i,i], \mu = [i]) \\ 0 & (\lambda = [i,i,i], \mu = [i]) \\ 1 & (\lambda = [i,i,i], \mu = [i,i]) \\ 0 & (\lambda = [i,i,k], \mu = [i,k], i \neq k). \end{cases}$$

The proof of Proposition 3.1 is completely straightforward; note that to estimate the decomposition number $[S^\lambda : D^\mu]$ using the Jantzen–Schaper formula, it suffices to calculate the decomposition numbers $[S^\nu : D^\mu]$ for those partitions ν with $\mu \rhd \nu \rhd \lambda$. If $\lambda = [i]$ and $\mu = [i,i,i]$, then the only such ν is $[i,i]$, while if (λ, μ) is any of the other pairs given above, then there is no such ν. We leave the calculations to the reader.

It turns out that the non-zero entries in the adjustment matrices for weight 3 blocks 'come from' the Rouquier blocks by induction. We shall make this precise shortly, but first we consider how induction and restriction of simples in a $[3:\kappa]$-pair are related to adjustment matrices.

Suppose A and B are blocks of $\mathcal{H}_{n-\kappa}$ and $\mathcal{H}_n$ forming a $[3:\kappa]$-pair, and recall the map Φ defined in Section 2.3.3.

Lemma 3.2. [5, Lemma 4.3] *Suppose that A and B are as above. Suppose λ, μ are e-regular partitions in B.*

1. *If D^μ is exceptional, then*
$$a_{\lambda\mu} = a_{\Phi(\lambda)\Phi(\mu)} = \delta_{\lambda\mu}.$$
2. *If D^λ is non-exceptional, then*
$$a_{\lambda\mu} = a_{\Phi(\lambda)\Phi(\mu)}.$$

Using Proposition 3.1 and Lemma 3.2, it is possible to calculate many entries in the adjustment matrices of weight 3 blocks. We define a *Scopes sequence* to be a sequence $B_0,\dots,B_r$ of weight 3 blocks such that for $1 \leqslant i \leqslant r$, B_{i-1} and B_i form a $[3:\kappa_i]$-pair for some κ_i. If λ is an e-regular partition in B_0 and $\check{\lambda}$ is an e-regular partition in B_r, then we say that λ *induces semi-simply to* $\check{\lambda}$ *via* $B_0,\dots,B_r$ if there are e-regular partitions $\lambda = \lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(r)} = \check{\lambda}$ lying in $B_0,\dots,B_r$ respectively such that $D^{\lambda^{(i)}}$ is non-exceptional for the $[3:\kappa_i]$-pair (B_{i-1}, B_i), and $\Phi(\lambda^{(i)}) = \lambda^{(i-1)}$ for each i, where Φ is the map defined in Section 2.3.3 for this pair. If there is a similar sequence $\lambda = \lambda^{(0)},\dots,\lambda^{(r)} = \check{\lambda}$ such that $\Phi(\lambda^{(i)}) = \lambda^{(i-1)}$ for each i, and exactly one $D^{\lambda^{(i)}}$ is exceptional, then we say that D^λ *induces almost semi-simply to* $D^{\check{\lambda}}$ *via* $B_0,\dots,B_r$. If λ and $\check{\lambda}$ are e-regular partitions lying in weight 3 blocks B and C respectively, then we say that λ induces (almost) semi-simply to $\check{\lambda}$ if there is some Scopes sequence $B = B_0,\dots,B_r = C$ such that λ induces (almost) semi-simply to $\check{\lambda}$ via $B_0,\dots,B_r$.

Now we can state our main theorem. Suppose λ and μ are e-regular partitions lying in a weight 3 block B of $\mathcal{H}_n$.

If $\mathrm{char}(\mathbb{F}) = 2$, we define $\hat{a}_{\lambda\mu}$ to equal 1 if the pair (λ,μ) satisfies one of the following conditions:

- there is a Rouquier block C and $2 \leqslant i \leqslant e$ such that λ induces semi-simply or almost semi-simply to $[i,i,i]_C$, while μ induces semi-simply to $[i]_C$;
- there is a Rouquier block C and $2 \leqslant i \neq k \leqslant e$ such that λ induces semi-simply to $[i,i,k]_C$, while μ induces semi-simply to $[i,k]_C$.

If λ and μ do not satisfy either of the above conditions, then we set $\hat{a}_{\lambda\mu} = \delta_{\lambda\mu}$.

If $\mathrm{char}(\mathbb{F}) = 3$, we define $\hat{a}_{\lambda\mu}$ to equal 1 if the pair (λ,μ) satisfies one of the following conditions:

- there is a Rouquier block C and $2 \leqslant i \leqslant e$ such that λ induces semi-simply to $[i,i,i]_C$, while μ induces semi-simply to $[i,i]_C$;
- there is a Rouquier block C and $2 \leqslant i \leqslant e$ such that λ induces semi-simply to $[i,i]_C$, while μ induces semi-simply to $[i]_C$.

If λ and μ do not satisfy either of the above conditions, then we set $\hat{a}_{\lambda\mu} = \delta_{\lambda\mu}$.

Theorem 3.3. *Suppose B is a weight 3 block of $\mathcal{H}_n$, and that λ and μ are e-regular partitions in B. Then $a_{\lambda\mu} = \hat{a}_{\lambda\mu}$.*

It will help with the proof of Theorem 3.3 (and it may also be more useful to the reader) to have a more explicit description of those pairs (λ,μ) satisfying $\hat{a}_{\lambda\mu} = 1$. In fact, we give a complete description of partitions inducing semi-simply to any given e-regular partition in a Rouquier block, as well as those inducing almost semi-simply to the partition $[i,i,i]$. An explicit (induction-free) description of the adjustment matrices for weight 3 blocks may be inferred from this.

Proposition 3.4. *Suppose λ is an e-regular partition lying in a block B, and that $[*]$ is a symbol of the form $[i]$, $[i,j]$ or $[i,j,k]$ $(2 \leqslant i < j < k)$. Then λ induces semi-simply to the partition $[*]_C$ in some Rouquier block C if and only if λ, $[*]$ and the pyramid for B satisfy one of the following sets of conditions.*

$[*]$	λ	Conditions on pyramid for B
$[i]$	$[i]_B$	${}_i2_{i+1}$
	$[i,i+1]_B$	${}_i1^-{}_{i+1},\ {}_i1^+{}_{i+2}$
	$[i,i+1,i+2]_B$	${}_i0_{i+2}$
$[i,i]$	$[i,i]_B$	${}_{i-1}1^+{}_i1^+{}_{i+1}$
	$[i,i,i+1]_B$	${}_{i-1}1^+{}_i0_{i+1}$
$[i,i,i]$	$[i,i,i]_B$	${}_{i-1}2_i$
$[i,j]$	$[i,j]_B$	${}_i1^+{}_{i+1},\ {}_i2_j$
	$[i,i+1,j]_B$	${}_i0_{i+1}1^+{}_j$
	$[i,j,j]_B$	${}_{i-1}1^+{}_j,\ {}_i1^-{}_j,\ {}_{i+1}0_j$
$[j,i]$	$[j,i]_B$	${}_i1^+{}_j1^+{}_{j+1}$
	$[j,j]_B$	${}_{i-1}1^+{}_j,\ {}_i0_j1^+{}_{j+1}$
	$[i,j,j+1]_B$	${}_i1^+{}_j0_{j+1}$
	$[j,j,j+1]_B$	${}_{i-1}1^+{}_j,\ {}_i0_j0_{j+1}$
$[i,i,j]$	$[i,i,j]_B$	${}_{i-1}1^+{}_i1^+{}_j$
	$[j,j,j]_B$	${}_{i-1}2_j,\ {}_i0_j$
$[i,j,j]$	$[i,j,j]_B$	${}_i2_j,\ {}_{j-1}1^+{}_j$
	$[j,j,j]_B$	${}_{i-1}2_j,\ {}_i1_j,\ {}_{j-1}1_j$
$[i,j,k]$	$[i,j,k]_B$	${}_i1^+{}_j1^+{}_k$
	$[i,k,k]_B$	${}_i2_k,\ {}_{j-1}1^+{}_k,\ {}_j0_k$
	$[j,j,k]_B$	${}_{i-1}1^+{}_j,\ {}_i0_j1^+{}_k$
	$[k,k,k]_B$	${}_{i-1}2_k,\ {}_i1_k,\ {}_{j-1}1_k,\ {}_j0_k$

Moreover, if one of these sets of conditions holds, then for any Scopes sequence $B = B_0, \ldots, B_r$ with B_r a Rouquier block, λ induces semi-simply to $[*]_{B_r}$ via $B_0, \ldots, B_r$.

Proof. Given $[*]$, let $\mathcal{S}[*]$ be the set of partitions λ given in the table. For any weight 3 block B there is a Scopes sequence $B = B_0, \ldots, B_r$ with B_r a Rouquier block [3, Lemma 3.1], and so it suffices to prove the following two statements.

1. If λ is an e-regular partition lying in a Rouquier block B, then $\lambda \in \mathcal{S}[*]$ if and only if $\lambda = [*]_B$.
2. If A and B are weight 3 blocks forming a $[3 : \kappa]$-pair and λ is an e-regular partition lying in B, then:
   (a) if D^λ is non-exceptional for this pair, then $\lambda \in \mathcal{S}[*]$ if and only if $\Phi(\lambda) \in \mathcal{S}[*]$;
   (b) if D^λ is exceptional for this pair, then $\Phi(\lambda) \notin \mathcal{S}[*]$.

Part (1) is easy to verify, given the pyramid for a Rouquier block. Part (2) is straightforward (albeit tedious) to check, given the descriptions of Φ in 2.3. □

We need the corresponding result for 'almost semi-simple' induction of λ to $[i,i,i]$.

Proposition 3.5. *Suppose λ is an e-regular partition lying in a weight 3 block B of $\mathcal{H}_n$, and $2 \leqslant i \leqslant e$. Then λ induces almost semi-simply to the partition $[i,i,i]_C$ for some Rouquier block C if and only if λ and the pyramid* *for B satisfy one of the following conditions.*

λ	Conditions on pyramid for B
$[i-1]$	${}_{i-1}1_i,\ {}_{i-1}2_{i+1}$
$[i-1,i+1]$	${}_{i-1}1_i,\ {}_{i-1}1_{i+1}$
$[i,i-1]$	${}_{i-1}0_i1^+{}_{i+1}$
$[i-1,i,i+1]$	${}_{i-1}0_i0_{i+1},\ {}_{i-1}1_{i+1}$
$[i,i+1,i+1]$	${}_{i-1}0_{i+1},\ {}_{i-2}1^+{}_{i+1}$

Moreover, if λ satisfies any of these conditions and $B = B_0, \dots, B_r$ is a Scopes sequence with B_r a Rouquier block, then λ induces almost semi-simply via this sequence to $[i,i,i]_{B_r}$.

Proof. This is proved similarly to Proposition 3.4. Let $\mathcal{S}$ be the set of partitions described. Then it suffices to prove the following two statements.

1. $\mathcal{S}$ does not contain any partition lying in a Rouquier block.
2. If A and B are weight 3 blocks forming a $[3:\kappa]$-pair and λ is an e-regular partition lying in B, then:
   (a) if D^λ is non-exceptional for this pair, then $\lambda \in \mathcal{S} \Leftrightarrow \Phi(\lambda) \in \mathcal{S}$;
   (b) if D^λ is exceptional for this pair, then $\Phi(\lambda) \in \mathcal{S}$ if and only if λ induces semi-simply to $[i,i,i]_C$ for some Rouquier block C.

Again, (1) is easy, while (2) can be checked using the description of Φ and that of the partitions inducing semi-simply to $[i,i,i]_C$ listed in Proposition 3.4. □

For the remainder of this section, we state some simple results about adjustment matrices which will help us to prove Theorem 3.3, and we show that our main theorem is compatible with these.

Lemma 3.6. [5, Lemma 4.2] *Suppose B is a block of $\mathcal{H}_n$, and that λ and μ are e-regular partitions in B.*

1. $a_{\lambda\mu} = a_{\lambda^\diamond\mu^\diamond}$.
2. *If $a_{\lambda\mu} \neq 0$, then $\mu \trianglerighteq \lambda$ and $\lambda^{\diamond\prime} \trianglerighteq \mu^{\diamond\prime}$.*

The next result shows how certain entries in the adjustment matrix may be derived from the adjustment matrices for blocks of weight less than 3. Suppose μ is an e-regular partition lying in a weight 3 block B of $\mathcal{H}_n$. We say that μ is *lowerable* if there is a block C of $\mathcal{H}_{n-1}$ of weight 0, 1 or 2 such that $D^\mu{\downarrow}_C^B \neq 0$.

Proposition 3.7. *Suppose λ and μ are e-regular partitions in a weight 3 block B of $\mathcal{H}_n$, and suppose C is a block of $\mathcal{H}_{n-1}$ of weight 0, 1 or 2 with $D^\mu{\downarrow}_C^B \neq 0$.*

1. *If $D^\lambda{\downarrow}_C^B = 0$, then $a_{\lambda\mu} = 0$.*
2. *If $D^\lambda{\downarrow}_C^B \neq 0$, then there are e-regular partitions λ^- and μ^- in C such that*

$$D^\lambda{\downarrow}_C^B \cong D^{\lambda^-}, \qquad D^\mu{\downarrow}_C^B \cong D^{\mu^-}$$

and

$$a_{\lambda\mu} = a_{\lambda^-\mu^-}.$$

Proof. Given an abacus display for B, the abacus display for C is obtained by moving a bead from runner j to runner i, where runner i lies immediately to the left of runner j. The fact that C has weight 0, 1 or 2 means that in the abacus display for B the number of beads on runner i is at least that on runner j. Using Theorem 1.7, we find that the restriction of any simple module from B to C is either 0 or simple. So certainly there is an e-regular partition μ^- with $D^\mu\downarrow^B_C\cong D^{\mu^-}$, and either $D^\lambda\downarrow^B_C=0$ or there is an e-regular λ^- with $D^\lambda\downarrow^B_C\cong D^{\lambda^-}$. Moreover, D^μ is the only simple module in B which restricts to give D^{μ^-}.

Let B^0 and C^0 be the blocks of $\mathcal{H}_{\mathbb{C},q'}(\mathfrak{S}_n)$ and $\mathcal{H}_{\mathbb{C},q'}(\mathfrak{S}_{n-1})$ corresponding to B and C. Let $\mathtt{D}$ and $\mathtt{E}$ be the decomposition matrices for B^0 and C^0; let $\mathtt{B}$ and $\mathtt{C}$ be the adjustment matrices for B and C, so that the decomposition matrices for B and C are $\mathtt{DB}$ and $\mathtt{EC}$ respectively.

Let $\mathtt{S}$ be the 'Specht branching matrix' for restriction from B to C: this has rows indexed by partitions in B and columns indexed by partitions in C, with the (λ,ν)-entry being the multiplicity of S^ν in the Specht filtration for $S^\lambda\downarrow^B_C$ given by Theorem 1.6. Since the Branching Rule is independent of characteristic, $\mathtt{S}$ is also the Specht branching matrix for restriction from B^0 to C^0. Let $\mathtt{T}$ be the 'simple branching matrix' for restriction from B to C: here the rows and columns are indexed by e-regular partitions in B and C respectively, and the (μ,ξ)-entry is the composition multiplicity $[D^\mu\downarrow^B_C : D^\xi]$. By Theorem 1.7 the restriction of a simple module from B to C (or from B^0 to C^0) is either simple or zero, and if it is non-zero it is described in a characteristic-free way, so $\mathtt{T}$ is also the simple branching matrix for restriction from B^0 to C^0.

By exactness of restriction from B to C, we get

$$\mathtt{DBT} = \mathtt{SEC},$$

and by exactness of restriction from B^0 to C^0 we get

$$\mathtt{DT} = \mathtt{SE},$$

so that

$$\mathtt{DBT} = \mathtt{DTC}.$$

Since $\mathtt{D}$ has full column rank, we may cancel it to get

$$\mathtt{BT} = \mathtt{TC}.$$

We compare the (λ,μ^-)-entries of both sides. We have

$$(\mathtt{BT})_{\lambda\mu^-} = \sum_\nu a_{\lambda\nu}[D^\nu\downarrow^B_C : D^{\mu^-}] = a_{\lambda\mu}.$$

On the other hand, we have

$$\begin{aligned}(\mathtt{TC})_{\lambda\mu^-} &= \sum_\xi [D^\lambda\downarrow^B_C : D^\xi] a_{\xi\mu^-}\\ &= \begin{cases} a_{\lambda^-\mu^-} & (D^\lambda\downarrow^B_C\cong D^{\lambda^-}\neq 0)\\ 0 & (D^\lambda\downarrow^B_C=0).\end{cases}\end{aligned}$$

□

In order to use Lemma 3.2 and Lemma 3.6(2), we must show that they are compatible with Theorem 3.3.

Lemma 3.8. *If λ and μ are e-regular partitions lying in a weight 3 block B of $\mathcal{H}_n$, then $\hat{a}_{\lambda^\diamond\mu^\diamond} = \hat{a}_{\lambda\mu}$.*

Proof. This is certainly true if B is a Rouquier block, where the effect of the Mullineux map may be read from [16, Proposition 3.3]. For an arbitrary weight 3 block B, take a Scopes sequence $B = B_0, \ldots, B_r$ such that B_r is Rouquier; then λ and μ induce semi-simply to partitions $\check{\lambda}$ and $\check{\mu}$ via $B_0, \ldots, B_r$ if and only if $\lambda^\diamond$ and $\mu^\diamond$ induce semi-simply to $\check{\lambda}^\diamond$ and $\check{\mu}^\diamond$ via $B_0^\sharp, \ldots, B_r^\sharp$, by Lemma 2.5; a similar statement applies for the case where λ induces almost semi-simply to $[i,i,i]_{B_r}$. □

Proposition 3.9. *Suppose A and B are blocks of $\mathcal{H}_{n-\kappa}$ and $\mathcal{H}_n$ forming a $[3:\kappa]$-pair, and that λ and μ are e-regular partitions in B.*

1. *If D^μ is exceptional, then*
$$\hat{a}_{\lambda\mu} = \hat{a}_{\Phi(\lambda)\Phi(\mu)} = \delta_{\lambda\mu}.$$
2. *If D^λ is non-exceptional, then*
$$\hat{a}_{\lambda\mu} = \hat{a}_{\Phi(\lambda)\Phi(\mu)}.$$

Proof.

1. If $\lambda \neq \mu$ and either $\hat{a}_{\lambda\mu} = 1$ or $\hat{a}_{\Phi(\lambda)\Phi(\mu)} = 1$, then it is easy to check, using the definition of $\hat{a}_{\lambda\mu}$ and Propositions 3.4 and 3.5, that D^μ cannot be an exceptional simple module for the pair (A,B).
2. By (1), we can assume that D^μ is non-exceptional. Then, by the last statement in Proposition 3.4, we find that λ and μ induce semi-simply to given partitions $[*]_C$ and $[\dagger]_C$ for some Rouquier block C if and only if $\Phi(\lambda)$ and $\Phi(\mu)$ do; a similar statement holds for the case where λ induces almost semi-simply to $[i,i,i]_C$, using the last statement of Proposition 3.5.

□

The following corollary follows immediately from Proposition 3.9 and Lemma 3.2, since Theorem 3.3 clearly holds for Rouquier blocks by Proposition 3.1.

Corollary 3.10. *Suppose λ and μ are e-regular partitions lying in a block B of weight 3. If λ induces semi-simply to some partition lying in a Rouquier block, then Theorem 3.3 holds for the pair (λ,μ), i.e. $a_{\lambda\mu} = \hat{a}_{\lambda\mu}$.*

Now we describe the strategy of our proof of Theorem 3.3, which is by induction on n. Suppose λ and μ are e-regular partitions in a weight 3 block B of $\mathcal{H}_n$, and there is a block A of $\mathcal{H}_{n-\kappa}$ forming a $[3:\kappa]$-pair with B. If D^λ is non-exceptional or D^μ is exceptional for this pair, then Theorem 3.3 holds for the pair (λ,μ) by induction, Lemma 3.2 and Proposition 3.9. So we may assume that for every such A, D^λ is exceptional and D^μ is non-exceptional for the pair (A,B). As D^λ cannot be exceptional for two different such pairs, we may assume that there is at most one such A, and that we have $\kappa = 1$ or 2. This means that B is one of three types of block, which we deal with in the remaining three sections.

4 The principal block of $\mathcal{H}_{3e}$

As an initial case for our inductive proof of Theorem 3.3, we consider the block B of $\mathcal{H}_{3e}$ with core $\varnothing$. This has a pyramid $({}_ja_k)$ with ${}_j0_k$ whenever $1 \leqslant j \leqslant k \leqslant e$, and may be represented on an abacus with the $\langle 3^e \rangle$ notation.

Note that every e-regular partition in B is lowerable. This enables us to calculate the adjustment matrix for B using Proposition 3.7. In characteristic 3, we immediately deduce $a_{\lambda\mu} = \delta_{\lambda\mu}$ for all e-regular λ and μ in B, by Theorem 2.2(1). If $\mathrm{char}(\mathbb{F}) = 2$, we need to consider some weight 2 blocks of

$\mathcal{H}_{3e-1}$. There are $e-1$ of these, which we label $B_1, \dots, B_{e-1}$. The block B_i has core $(i, 1^{e-1-i})$, and its abacus display is obtained from that for B by moving a bead from runner $i+1$ to runner i. Thus B_i may be represented with the $\langle 3^{i-1}, 4, 2, 3^{e-i-1}\rangle$ notation for partitions of weight 2. The pyramid for B_i has

$$_ja_k = \begin{cases} 0 & (1 \leqslant j \leqslant k \leqslant i \text{ or } 2 \leqslant j \leqslant k \leqslant e-1 \text{ or } i+1 \leqslant j \leqslant k \leqslant e) \\ 1 & (\text{otherwise}). \end{cases}$$

Looking at Theorem 2.2(2), we see that B_i has a non-trivial adjustment matrix if and only if there is some j with $_{j-1}1_j$. This happens only for $i = 1,\ j = 2$ and $i = e-1,\ j = e$.

In B_1, we find that we have $a_{\nu\xi} = 1$ with $\nu \neq \xi$ if and only if

$$\nu = [2,2], \qquad \xi = \begin{cases} [2,3] & (e \geqslant 3) \\ [2] & (e = 2). \end{cases}$$

To use Proposition 3.7 we need to find the e-regular partitions λ and μ in B such that $\nu = \lambda^-$ and $\xi = \mu^-$. By Theorem 1.7, these are

$$\lambda = \begin{cases} [3,3,2] & (e \geqslant 3) \\ [2,1] & (e = 2), \end{cases} \qquad \mu = \begin{cases} [2,3,4] & (e \geqslant 5) \\ [2,3] & (e = 3) \\ [2] & (e = 2). \end{cases}$$

(Note that e cannot equal 4 (or indeed any even integer greater than 2) when $\mathrm{char}(\mathbb{F}) = 2$.) So we get $a_{\lambda\mu} = 1$ for this pair.

In B_{e-1} we have $a_{\nu\xi} = 1$ with $\nu \neq \xi$ if and only if

$$\nu = [e,e], \qquad \xi = [e],$$

giving

$$\lambda = [e, e-1], \qquad \mu = [e].$$

Summarising, we have the following.

Proposition 4.1. *Suppose B is the block of $\mathcal{H}_{3e}$ with core $\varnothing$, and λ and μ are e-regular partitions in B.*

1. *If* $\mathrm{char}(\mathbb{F}) = 2$, *then we have*

$$a_{\lambda\mu} = \begin{cases} 1 & (\lambda = [e, e-1],\ \mu = [e]) \\ 1 & (\lambda = [3,3,2],\ \mu = [2,3,4],\ e \geqslant 5) \\ 1 & (\lambda = [3,3,2],\ \mu = [2,3],\ e = 3) \\ \delta_{\lambda\mu} & (\textit{otherwise}). \end{cases}$$

2. *If* $\mathrm{char}(\mathbb{F}) = 3$, *then we have* $a_{\lambda\mu} = \delta_{\lambda\mu}$.

By checking the definition of $\hat{a}_{\lambda\mu}$ together with Propositions 3.4 and 3.5, we find that Theorem 3.3 holds for B.

5 Blocks with rectangular cores

In this section, we suppose that B is a weight 3 block of $\mathcal{H}_n$ and that there is exactly one block A forming a $[3:\kappa]$-pair with B, with $\kappa = 1$. This means that B has a core of the form (x^z) for some $x, z > 0$ with $x + z \leqslant e$. We put $y = e - x - z$, and use the $\langle 3^x, 4^z, 3^y \rangle$ notation for partitions in B. We can easily calculate the pyramid for B: we have ${}_i0_j$ if $1 \leqslant i \leqslant j \leqslant x + y$ or $x + 1 \leqslant i \leqslant j \leqslant e$, while ${}_i1_j$ if $1 \leqslant i \leqslant x$ and $x + y + 1 \leqslant j \leqslant e$.

By induction, Lemma 3.2 and Proposition 3.9, we can show that $a_{\lambda\mu} = \hat{a}_{\lambda\mu}$ for partitions λ and μ in B unless D^λ is exceptional and D^μ is non-exceptional, for the $[3:1]$-pair (A, B). Thus we assume from now on that D^λ is an exceptional partition, and hence $\lambda = [x + y + 1, x + y + 1]$ or $[x + y + 1, x + y + 1, l]$ for some $l \neq x$, while D^μ is a non-exceptional partition, and we aim to show $a_{\lambda\mu} = \hat{a}_{\lambda\mu}$ for these partitions. Below we give the values of $\hat{a}_{\lambda\mu}$ for such λ and μ.

Lemma 5.1. *Suppose B is as above, and that λ and μ are e-regular partitions in B with D^λ an exceptional simple module and D^μ a non-exceptional simple module for the pair (A, B).*

- *If* $\mathrm{char}(\mathbb{F}) = 2$, *then*

$$\hat{a}_{\lambda\mu} = \begin{cases} 1 & (\lambda = [x+2, x+3, x+3],\ \mu = [x+2, x+3],\ y = 2,\ z = 1) \\ 1 & (\lambda = [x+2, x+3, x+3],\ \mu = [x+2, x+3, x+4],\ y = 2,\ z \geqslant 2) \\ 0 & (\textit{otherwise}). \end{cases}$$

- *If* $\mathrm{char}(\mathbb{F}) = 3$, *then*

$$\hat{a}_{\lambda\mu} = \begin{cases} 1 & (\lambda = [x+1, x+1],\ \mu = [x+1],\ y = 0,\ z = 1) \\ 1 & (\lambda = [x+1, x+1, x+2],\ \mu = [x+1, x+2],\ y = 0,\ z = 2) \\ 1 & (\lambda = [x+1, x+1, x+2],\ \mu = [x+1, x+2, x+3],\ y = 0,\ z \geqslant 3) \\ 0 & (\textit{otherwise}). \end{cases}$$

5.1 The case where μ is lowerable

First we suppose that μ is lowerable. This means that we may calculate $a_{\lambda\mu}$ using Proposition 3.7. We suppose C is a block of $\mathcal{H}_{n-1}$ of weight 0, 1 or 2 such that $D^\mu\!\downarrow^B_C \neq 0$. If C has weight 0 or 1 or if $\mathrm{char}(\mathbb{F}) = 3$ or if $D^\lambda\!\downarrow^B_C = 0$, then we get $a_{\lambda\mu} = 0$ from Proposition 3.7 and Theorem 2.2. So we suppose that C has weight 2, that $\mathrm{char}(\mathbb{F}) = 2$ and that $D^\lambda\!\downarrow^B_C \neq 0$. The fact that C has weight 2 means that the abacus display for C is obtained from that for B by moving a bead from runner j to runner $j - 1$, where $j \neq 1, x + 1, x + y + 1$. The fact that $D^\lambda\!\downarrow^B_C \neq 0$ then implies that $\lambda = [x + y + 1, x + y + 1, j]$ and that $j \neq x, x + y + 2$. We then find that $D^\lambda\!\downarrow^B_C \cong D^{\tilde{\lambda}}$, where

$$\tilde{\lambda} = \begin{cases} [x+y+1, x+y+1] & (2 \leqslant j \leqslant x-1) \\ [x+y, x+y] & (x+2 \leqslant j \leqslant x+y) \\ [x+y+2, x+y+2] & (x+y+3 \leqslant j \leqslant e). \end{cases}$$

In order for $a_{\lambda\mu}$ to be non-zero, $\tilde{\lambda}$ must be of the form $[i, i]$ with ${}_{i-1}1_i$ in C. Examining the pyramid for C, we see that this happens if and only if $y = 2$ and $j = x + y$. In this case we get $a_{\tilde{\lambda}\tilde{\mu}} = 1$ only when $\tilde{\mu} = [x + y, x + y + 1]$, in which case $\mu = [x + y, x + y + 1]$ if $z = 1$ and $\mu = [x + y, x + y + 1, x + y + 2]$ if $z \geqslant 2$. Comparing this with Lemma 5.1, we see that $a_{\lambda\mu} = \hat{a}_{\lambda\mu}$ when μ is lowerable.

5.2 The case where μ is not lowerable

In this section, we suppose μ is not lowerable. A list of such μ is given in Table 1. This is essentially the same as Table 1 from [5], but contains nine extra cases which arise when e is less than 5. There are fifty-nine cases in all, each of which is labelled with a pair of letters. With each μ we give the partition $\mu^{\diamond\prime}$, which depends on the values of x, y and z. The labelling reflects the Mullineux map, so that the form of the partition $\mu^{\diamond}$ may be found by interchanging the two letters and interchanging x and z.

Our approach for these partitions will be to induce the simple modules D^{λ} and D^{μ} up to simple modules $D^{\tilde{\lambda}}$ and $D^{\tilde{\mu}}$ in a block C where we can easily calculate $a_{\tilde{\lambda}\tilde{\mu}}$. To aid us, we introduce some notation for induction. Suppose ν is an e-regular partition lying in a weight 3 block D, and take an abacus display for D. Suppose the number of beads on runner i of the abacus exceeds the number of beads on the runner to the immediate right by $\kappa \geqslant 1$, and let D_i be the block whose abacus is obtained by interchanging runner i with the runner to its right. Then D and D_i form a $[3:\kappa]$-pair. If D^{ν} is non-exceptional for this $[3:\kappa]$-pair, then define $\mathfrak{f}_i(\nu)$ to be the e-regular partition such that $D^{\nu}\uparrow_D^{D_i}\cong (D^{\mathfrak{f}_i(\nu)})^{\oplus\kappa!}$, and leave $\mathfrak{f}_i(\nu)$ undefined otherwise (so if $\mathfrak{f}_i(\nu)$ is defined, then $\Phi(\mathfrak{f}_i(\nu)) = \nu$, where Φ is the map defined in Section 2.3.3).

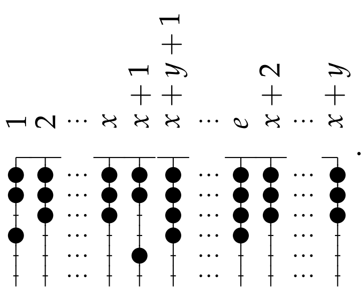
Recall that we are seeking to calculate $a_{\lambda\mu}$, where λ equals $[x+y+1, x+y+1]$ or $[x+y+1, x+y+1, l]$ for some l, and where μ is one of the partitions listed in Table 1. By Lemma 3.6(2), $\mu \rhd \lambda$ is a necessary condition for $a_{\lambda\mu} \neq 0$. This means that $a_{\lambda\mu} = 0$ whenever μ is in any of the cases J_*, K_*, L_*, M_*, N_*, since for these cases there is no exceptional λ with $\mu \rhd \lambda$. Checking with Lemma 5.1, we see that $a_{\lambda\mu} = \hat{a}_{\lambda\mu}$ for these cases. By applying Lemma 3.6(1) and Lemma 3.8, we also deal with cases $\mathrm{A_J}$, $\mathrm{A_K}$, $\mathrm{A_L}$, $\mathrm{A_M}$, $\mathrm{A_N}$, $\mathrm{C_L}$, $\mathrm{C_M}$, $\mathrm{C_N}$, $\mathrm{G_M}$.

5.2.1 Inducing D^{μ} to a lowerable simple module

Consider the (partial) function $\mathfrak{f} = \mathfrak{f}_{x+y+1}\mathfrak{f}_{x+y+2}\cdots\mathfrak{f}_e$. The effect of this is to move each of the runners $e, e-1, \dots, x+y+1$ in succession past runner $x+1$ (if $y>0$) or past runner 1 (if $y=0$). It is easy to see that $\mathfrak{f}(\lambda)$ is defined, and that if μ is in one of the following cases, then $\mathfrak{f}(\mu)$ is defined and is lowerable:

- case B_*: $[x+1, x+y+1]$ (with $y \geqslant 1,\ z = 1$),
- case E_*: $[1, x+y+1]$ (with $x \geqslant 2,\ y = 0$),
- case F_*: $[1, x+y+1, x+1]$ (with $y \geqslant 1$),
- case H_*: $[x+y+1, x+y+2, x+1]$ (with $y \geqslant 1,\ z \geqslant 2$),
- case I_*: $[x+y+1, x+1, x+2]$ (with $y \geqslant 2$).

Let C be the block in which $\mathfrak{f}(\lambda)$ and $\mathfrak{f}(\mu)$ lie. In characteristic 3, it is then immediate that $a_{\mathfrak{f}(\lambda)\mathfrak{f}(\mu)} = 0$; if $\mathrm{char}(\mathbb{F}) = 2$, it is easy to check using Lemma 2.2(2) that $a_{\mathfrak{f}(\lambda)\mathfrak{f}(\mu)} = 0$. Hence we have $a_{\lambda\mu} = 0$. As an example of the induction, we have $\mathfrak{f}([1, x+1, x+y+1]) = [x+1, 1 \mid 3^{x+1}, 4^{z}, 3^{y-1}]$; we easily see that this is lowerable from its abacus display:



	μ	Conditions on x, y, z	$\mu^{\diamond\prime}$
A_A	$[x+y+1]$	$x=1,\ y=0,\ z=1$	$[x,x,x]$
A_B	$[x+y+1]$	$x=1,\ y=1,\ z=1$	$[x+y,x+y,x]$
A_C	$[x+y+1]$	$x=2,\ y=0,\ z=1$	$[x,x,x-1]$
A_E	$[x+y+1]$	$y=0,\ z=2$	$[e,e,x]$
A_G	$[x+y+1]$	$x\geqslant 3,\ y=0,\ z=1$	$[x,x-1,x-2]$
A_H	$[x+y+1]$	$x\geqslant 2,\ y=1,\ z=1$	$[x+y,x,x-1]$
A_I	$[x+y+1]$	$y=2,\ z=1$	$[x+y,x+y-1,x]$
A_J	$[x+y+1]$	$y=0,\ z\geqslant 3$	$[e,e,e-1]$
A_K	$[x+y+1]$	$y\geqslant 1,\ z\geqslant 3$	$[e,e-1,x+y]$
A_L	$[x+y+1]$	$y\geqslant 2,\ z=2$	$[e,x+y,x+y-1]$
A_M	$[x+y+1]$	$y\geqslant 3,\ z=1$	$[x+y,x+y-1,x+y-2]$
A_N	$[x+y+1]$	$y=1,\ z=2$	$[e,e,x+y]$
B_A	$[x+1,x+y+1]$	$x=1,\ y=1,\ z=1$	$[x,x,x]$
B_B	$[x+1,x+y+1]$	$x=1,\ y=2,\ z=1$	$[x+y,x+y,x]$
B_C	$[x+1,x+y+1]$	$x=2,\ y=1,\ z=1$	$[x,x,x-1]$
B_G	$[x+1,x+y+1]$	$x\geqslant 3,\ y=1,\ z=1$	$[x,x-1,x-2]$
B_H	$[x+1,x+y+1]$	$x\geqslant 2,\ y=2,\ z=1$	$[x+y,x,x-1]$
B_I	$[x+1,x+y+1]$	$y\geqslant 3,\ z=1$	$[x+y,x+y-1,x]$
C_A	$[x+y+1,x+y+2]$	$x=1,\ y=0,\ z=2$	$[x,x,x]$
C_B	$[x+y+1,x+y+2]$	$x=1,\ y=1,\ z=2$	$[x+y,x+y,x]$
C_C	$[x+y+1,x+y+2]$	$x=2,\ y=0,\ z=2$	$[x,x,x-1]$
C_E	$[x+y+1,x+y+2]$	$y=0,\ z\geqslant 3$	$[e,e,x]$
C_G	$[x+y+1,x+y+2]$	$x\geqslant 3,\ y=0,\ z=2$	$[x,x-1,x-2]$
C_H	$[x+y+1,x+y+2]$	$x\geqslant 2,\ y=1,\ z=2$	$[x+y,x,x-1]$
C_I	$[x+y+1,x+y+2]$	$y=2,\ z=2$	$[x+y,x+y-1,x]$
C_L	$[x+y+1,x+y+2]$	$y\geqslant 2,\ z\geqslant 3$	$[e,x+y,x+y-1]$
C_M	$[x+y+1,x+y+2]$	$y\geqslant 3,\ z=2$	$[x+y,x+y-1,x+y-2]$
C_N	$[x+y+1,x+y+2]$	$y=1,\ z\geqslant 3$	$[e,e,x+y]$
D_F	$[x+y+1,x+1]$	$y\geqslant 1$	$[e,x+y,x]$
E_A	$[1,x+y+1]$	$x=2,\ y=0$	$[x,x,x]$
E_C	$[1,x+y+1]$	$x\geqslant 3,\ y=0$	$[x,x,x-1]$
F_D	$[1,x+y+1,x+1]$	$y\geqslant 1$	$[x+y,x,x]$
G_A	$[x+y+1,x+y+2,x+y+3]$	$x=1,\ y=0,\ z\geqslant 3$	$[x,x,x]$
G_B	$[x+y+1,x+y+2,x+y+3]$	$x=1,\ y=1,\ z\geqslant 3$	$[x+y,x+y,x]$
G_C	$[x+y+1,x+y+2,x+y+3]$	$x=2,\ y=0,\ z\geqslant 3$	$[x,x,x-1]$
G_G	$[x+y+1,x+y+2,x+y+3]$	$x\geqslant 3,\ y=0,\ z\geqslant 3$	$[x,x-1,x-2]$
G_H	$[x+y+1,x+y+2,x+y+3]$	$x\geqslant 2,\ y=1,\ z\geqslant 3$	$[x+y,x,x-1]$
G_I	$[x+y+1,x+y+2,x+y+3]$	$y=2,\ z\geqslant 3$	$[x+y,x+y-1,x]$
G_M	$[x+y+1,x+y+2,x+y+3]$	$y\geqslant 3,\ z\geqslant 3$	$[x+y,x+y-1,x+y-2]$
H_A	$[x+y+1,x+y+2,x+1]$	$x=1,\ y=1,\ z\geqslant 2$	$[x,x,x]$
H_B	$[x+y+1,x+y+2,x+1]$	$x=1,\ y=2,\ z\geqslant 2$	$[x+y,x+y,x]$
H_C	$[x+y+1,x+y+2,x+1]$	$x=2,\ y=1,\ z\geqslant 2$	$[x,x,x-1]$
H_G	$[x+y+1,x+y+2,x+1]$	$x\geqslant 3,\ y=1,\ z\geqslant 2$	$[x,x-1,x-2]$
H_H	$[x+y+1,x+y+2,x+1]$	$x\geqslant 2,\ y=2,\ z\geqslant 2$	$[x+y,x,x-1]$
H_I	$[x+y+1,x+y+2,x+1]$	$y\geqslant 3,\ z\geqslant 2$	$[x+y,x+y-1,x]$
I_A	$[x+y+1,x+1,x+2]$	$x=1,\ y=2$	$[x,x,x]$
I_B	$[x+y+1,x+1,x+2]$	$x=1,\ y\geqslant 3$	$[x+y,x+y,x]$
I_C	$[x+y+1,x+1,x+2]$	$x=2,\ y=2$	$[x,x,x-1]$
I_G	$[x+y+1,x+1,x+2]$	$x\geqslant 3,\ y=2$	$[x,x-1,x-2]$
I_H	$[x+y+1,x+1,x+2]$	$x\geqslant 2,\ y\geqslant 3$	$[x+y,x,x-1]$
J_A	$[1,2]$	$x\geqslant 3,\ y=0$	$[x,x,x]$
K_A	$[1,2,x+1]$	$x\geqslant 3,\ y\geqslant 1$	$[x,x,x]$
L_A	$[1,x+1,x+2]$	$x=2,\ y\geqslant 2$	$[x,x,x]$
L_C	$[1,x+1,x+2]$	$x\geqslant 3,\ y\geqslant 2$	$[x,x,x-1]$
M_A	$[x+1,x+2,x+3]$	$x=1,\ y\geqslant 3$	$[x,x,x]$
M_C	$[x+1,x+2,x+3]$	$x=2,\ y\geqslant 3$	$[x,x,x-1]$
M_G	$[x+1,x+2,x+3]$	$x\geqslant 3,\ y\geqslant 3$	$[x,x-1,x-2]$
N_A	$[1,x+1]$	$x=2,\ y=1$	$[x,x,x]$
N_C	$[1,x+1]$	$x\geqslant 3,\ y=1$	$[x,x,x-1]$

Table 1

So we find that if μ is in any of these cases, we have $a_{\lambda\mu} = 0$, which is equal to $\hat{a}_{\lambda\mu}$ by Lemma 5.1. Applying the Mullineux map and using Lemma 3.6(1), we may also deal with cases A_B, A_E, A_H, A_I, C_B, C_E, C_H, C_I, D_F, G_B, G_H, G_I.

5.2.2 Cases G_A, G_C and G_G

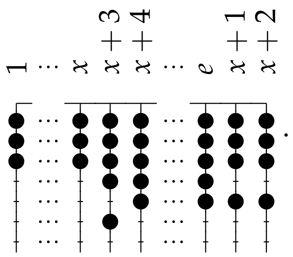
In these cases, we have $y = 0$, $z \geqslant 3$ and $\mu = [x+1, x+2, x+3]$. Suppose $a_{\lambda\mu} \neq 0$. Then the conditions $\mu \rhd \lambda$ and $\lambda^{\diamond\prime} \rhd \mu^{\diamond\prime}$ imply that λ is one of the exceptional partitions

$$[x+1, x+1, x+3], \qquad [x+1, x+1, x+2], \qquad [x-1, x+1, x+1] \text{ (if } x \geqslant 2).$$

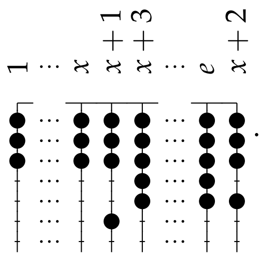
First we look at $\lambda = [x+1, x+1, x+3]$. We apply the partial function $\mathfrak{f} = (\mathfrak{f}_{x+3}\mathfrak{f}_{x+4} \cdots \mathfrak{f}_e)^{x+1}$ to both λ and μ. For μ, it is easy to see that

$$(\mathfrak{f}_{x+3} \cdots \mathfrak{f}_e)^x(\mu) = [x+1, x+2, x+3 \mid 3^x, 5^{z-2}, 4^2],$$

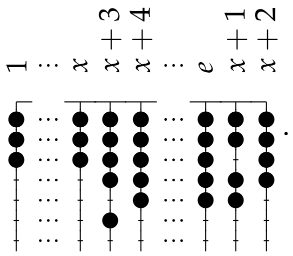
with abacus display



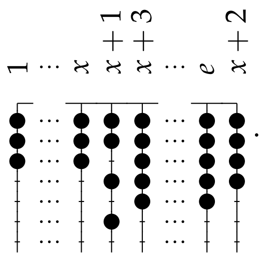
Applying $(\mathfrak{f}_{x+3} \cdots \mathfrak{f}_e)$ again, we find $\mathfrak{f}(\mu) = [x+1, x+2 \mid 3^x, 4, 5^{z-2}, 4]$:



For λ, applying $(\mathfrak{f}_{x+3} \cdots \mathfrak{f}_e)^x$ yields $[x+1, x+1, x+3 \mid 3^x, 5^{z-2}, 4^2]$:



Applying $(\mathfrak{f}_{x+3} \cdots \mathfrak{f}_e)$ again yields $\mathfrak{f}(\lambda) = [x+1, x+1 \mid 3^x, 4, 5^{z-2}, 4]$:



A very simple application of the Jantzen–Schaper formula yields $[S^{\mathfrak{f}(\lambda)} : D^{\mathfrak{f}(\mu)}] = 1$, irrespective of the underlying characteristic, which means that $a_{\mathfrak{f}(\lambda)\mathfrak{f}(\mu)} = 0$, and so $a_{\lambda\mu} = 0$ by Lemma 3.2.

Next we look at $\lambda = [x+1, x+1, x+2]$. Using Proposition 3.4, we see that this partition induces semi-simply to a partition in a Rouquier block (namely $[x+1, x+1]$), and so we may apply Corollary 3.10 to obtain $a_{\lambda\mu} = \hat{a}_{\lambda\mu}$.

Finally we assume $x \geqslant 2$ and look at $\lambda = [x-1, x+1, x+1]$. We apply the Mullineux map to λ and μ to get

$$\mu^\diamond = \begin{cases} [z+1, z+2, z+3 \mid 3^z, 4^x] & (x \geqslant 3) \\ [z+1, z+2 \mid 3^z, 4^x] & (x = 2), \end{cases}$$

$$\lambda^\diamond = \begin{cases} [z+1, z+1, z+3 \mid 3^z, 4^x] & (x \geqslant 3) \\ [z+1, z+1 \mid 3^z, 4^x] & (x = 2). \end{cases}$$

The case $x \geqslant 3$ corresponds to a case which we have dealt with in this subsection, and for this we have $a_{\lambda^\diamond\mu^\diamond} = 0$, which implies $a_{\lambda\mu} = 0$. In the case $x = 2$, a simple application of the Jantzen–Schaper formula yields $[S^{\lambda^\diamond} : D^{\mu^\diamond}] = 1$ regardless of the underlying characteristic. So we have $a_{\lambda^\diamond\mu^\diamond} = 0$, and so $a_{\lambda\mu} = 0$.

Checking with Lemma 5.1, we see that $a_{\lambda\mu} = \hat{a}_{\lambda\mu}$ for cases G_A, G_C, G_G. By applying the Mullineux map, we also deal with cases A_G, C_G.

5.2.3 Cases C_A and C_C

In these cases we have $x \leqslant 2$, $y = 0$, $z = 2$ and $\mu = [x+1, x+2]$. Suppose $a_{\lambda\mu} \neq 0$. Then the conditions $\mu \rhd \lambda$ and $\lambda^{\diamond\prime} \rhd \mu^{\diamond\prime}$ imply that λ is one of the partitions

$$[x+1, x+1], \qquad [x+1, x+1, x+2], \qquad [x-1, x+1, x+1] \text{ (if } x = 2).$$

If $\lambda = [x+1, x+1]$, then a very simple application of the Jantzen–Schaper formula gives $[S^\lambda : D^\mu] = 1$ independent of the characteristic, so that $a_{\lambda\mu} = 0$. If $\lambda = [x+1, x+1, x+2]$, then by Proposition 3.4 we find that λ induces semi-simply to a partition lying a Rouquier block, and we may apply Corollary 3.10 to obtain $a_{\lambda\mu} = \hat{a}_{\lambda\mu}$. If $x = 2$ and $\lambda = [x-1, x+1, x+1]$, then we apply the Mullineux map: we have $B^\sharp = B$, $\mu^\diamond = \mu$ and $\lambda^\diamond = [x+1, x+1]$, which is the first case dealt with here. This gives $a_{\lambda\mu} = a_{\lambda^\diamond\mu^\diamond} = 0$.

Checking with Lemma 5.1, we see that $a_{\lambda\mu} = \hat{a}_{\lambda\mu}$ for cases C_A and C_C. By applying the Mullineux map, we also deal with case A_C.

5.2.4 Case A_A

In this case we have $x = 1$, $y = 0$, $z = 1$, $\mu = [2]$ and $\lambda = [2, 2]$. We can easily apply the Jantzen–Schaper formula to get $[S^\lambda : D^\mu] = 1$ if $\operatorname{char}(\mathbb{F}) = 3$, and 0 otherwise. This shows that $a_{\lambda\mu} = 0$ if $\operatorname{char}(\mathbb{F}) \neq 3$, while $a_{\nu\mu} = 1$ for some $\mu \rhd \nu \unrhd \lambda$ if $\operatorname{char}(\mathbb{F}) = 3$. The condition $\mu \rhd \nu \unrhd \lambda$ forces $\lambda = \nu$. Checking with Lemma 5.1, we see that $a_{\lambda\mu} = \hat{a}_{\lambda\mu}$ for this final case too.

We have now dealt with all possible cases, and proved the following.

Proposition 5.2. *Suppose that A and B are weight 3 blocks as above, and that Theorem 3.3 holds for A. Then it holds for B.*

6 Blocks with birectangular cores

In this section, we suppose B is a weight 3 block of $\mathcal{H}_n$, and that there is exactly one block A forming a $[3:2]$-pair with B, and no block forming a $[3:1]$-pair with B. Then B has a core of the form $((2w+x)^z, w^{y+z})$ for some $w, x, y, z \geqslant 0$ with $w+x+y+z = e$ and $w, z > 0$. This may be represented on an abacus with the $\langle 3^w, 5^z, 4^y, 3^x \rangle$ notation. We have $a_{\lambda\mu} = \hat{a}_{\lambda\mu}$ for all e-regular λ and μ in B by induction using Lemma 3.2 and Proposition 3.9, except when D^λ is exceptional for the $[3:2]$-pair (A, B), i.e. when $\lambda = [w+x+y+1, w+x+y+1, w+x+y+1 \mid 3^w, 5^z, 4^y, 3^x]$. By Proposition 3.4, this λ induces semi-simply to a partition in a Rouquier block, so that we may apply Corollary 3.10. We deduce the following.

Proposition 6.1. *Suppose A and B are as above, and that Theorem 3.3 holds for A. Then it holds for B.*

We conclude this paper with the proof of Theorem 3.3.

Proof of Theorem 3.3. We proceed by induction. Given a weight 3 block B, we suppose first of all that there is no block A forming a $[3:\kappa]$-pair with B. Then B must be the principal block of $\mathcal{H}_{3e}$ discussed in Section 4, and the theorem holds for this block, from Section 4.

Now we suppose that there is at least one block A forming a $[3:\kappa]$-pair with B. If λ is an e-regular partition in B such that D^λ is non-exceptional for this $[3:\kappa]$-pair, then Theorem 3.3 holds for λ (and any μ) by induction, using Lemma 3.2 and Proposition 3.9. In particular, if $\kappa \geqslant 3$ (so that there are no exceptional simple modules), then Theorem 3.3 holds for B by induction. Also, if there are two different blocks A_1, A_2 forming $[3:\kappa]$-pairs with B, then the theorem holds by induction, since there cannot be a simple module in B which is exceptional for both of these pairs.

We are therefore left with the case where there is exactly one block forming a $[3:\kappa]$-pair with B, and $\kappa \leqslant 2$. If $\kappa = 1$, then Theorem 3.3 holds for B by induction using Proposition 5.2, while if $\kappa = 2$, then the theorem holds using Proposition 6.1. □

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