Regularising a partition on the abacus

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Abstract

We give an algorithm for computing the e-regularisation of a partition using an abacus display with e runners.

1 Introduction

This note concerns two combinatorial notions introduced by Gordon James in the context of representation theory of Iwahori–Hecke algebras of type A. Since this note is not intended for publication, we omit background and motivation, since these can easily be found elsewhere. The book by Mathas [M] is an excellent reference.

Recall that a *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \dots$ and the sum $|\lambda| = \lambda_1 + \lambda_2 + \dots$ is finite. We say that λ is a partition of $|\lambda|$. When writing partitions, we usually group together equal parts and omit trailiing zeroes. We write $\varnothing$ for the unique partition of 0.

λ is often identified with its *Young diagram*, which is the subset

$$[\lambda] = \{(i, j) \mid j \leqslant \lambda_i\}$$

of $\mathbb{N}^2$. We refer to elements of the latter set as *nodes*, and to elements of $[\lambda]$ as nodes of λ. We draw the Young diagram as an array of boxes using the English convention, so that i increases down the page and j increases from left to right.

Throughout this note, e denotes an integer greater than or equal to 2. A partition λ is said to be *e-regular* if there does not exist $i \geqslant 1$ such that $\lambda_i = \lambda_{i+e-1} > 0$. There is a function G, called *e-regularisation*, from the set of partitions to the set of e-regular partitions, which has representation-theoretic significance [J, BOX, FLM]. To define this map, we define the lth *ladder* in $\mathbb{N}^2$ to be the set

$$\mathcal{L}_l = \{(i, j) \in \mathbb{N}^2 \mid i + (j-1)(e-1) = l\},$$

for each $l \geqslant 1$. Now we define the lth ladder of a partition λ to be the intersection $\mathcal{L}_l \cap [\lambda]$. It is easy to see that λ is e-regular if and only if for each l the nodes in the lth ladder of λ are as high as possible; whether or not λ is e-regular, one can obtain an e-regular partition by moving the nodes in each ladder of λ to the highest positions in that ladder; this partition is the e-regularisation of λ.

Example. Suppose $e = 3$, and $\lambda = (5, 3^3, 1^5)$. Then the e-regularisation of λ is $(5, 4^2, 3, 2, 1)$, as we can see from the following Young diagrams, in which we label we label each node with the number of the ladder in which it lies.

1	3	5	7	9
2	4	6		
3	5	7		
4	6	8		
5				
6				
7				
8				
9				

1	3	5	7	9
2	4	6	8	
3	5	7	9	
4	6	8		
5	7			
6				

Another important concept is the notion of the abacus. With $e \geqslant 2$ still fixed, we take an abacus with e vertical runners, numbered $0, \dots, e-1$ from left to right, and we mark positions $0, 1, 2, \dots$ on these runners, reading from left to right along successive rows. Now given a partition λ, we choose a large integer r, and define

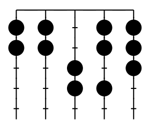
$$\beta_i = \lambda_i + r - i$$

for $i = 1, \dots, r$. We place a bead at position β_i for each i, and call the resulting configuration an *abacus display* for λ (with e runners). Abacus displays are also important in representation theory, chiefly in the classification of blocks of Iwahori–Hecke algebras.

Example. Suppose $e = 5$. Then the abacus is marked as follows.

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
⋮	⋮	⋮	⋮	⋮

If we take $\lambda = (7^2, 5, 4, 2^2, 1^4)$ and $r = 12$, then we get $(\beta_1, \dots, \beta_{12}) = (18, 17, 14, 12, 9, 8, 6, 5, 4, 3, 1, 0)$, so that the abacus display is as follows.



In an abacus display, we say that position t is *after* position s, or that position s is *before* position t, if $s < t$. We say that position s is *occupied* if there is a bead at position s, and *empty* otherwise. If we have an abacus display for a partition λ, then it is easy to see that λ is e-regular if and only if there is no $s \geqslant 0$ such that position s is empty while positions $s + 1, \dots, s + e$ are all occupied.

The purpose of this note is to give an algorithm for computing the e-regularisation of a partition using an e-runner abacus display. Although it is straightforward to translate between abacus displays and Young diagrams, this should provide a quicker method of regularising a partition if one is working with abacus displays. If you find this result useful, please cite this note.

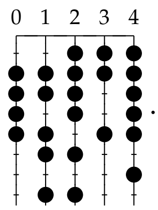
2 Regularisation on the abacus

Now we give our algorithm for regularising a partition using its abacus display. We begin by defining two functions on partitions.

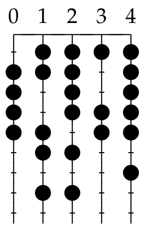
Suppose λ is a partition, and take an abacus display for λ. If λ is e-regular, set $A\lambda = \lambda$. Otherwise, there must be some empty position s on the abacus such that positions $s+1, \dots, s+e$ are occupied; let s be maximal with this property. (In terms of the Young diagram of λ, this corresponds to finding the rightmost column where λ is e-singular.) Suppose s lies on runner x, let $b_1 < \dots < b_l$ be the positions of the beads below s on this runner, and let $s_1 < s_2 < \dots$ be the positions of the empty spaces after position s not on runner x. Let $c \in \{1, \dots, l\}$ be minimal such that $s_c < b_{c+1}$ (this condition is to be regarded as automatically true in the case $c = l$).

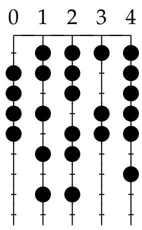
Now construct a new abacus display by moving a bead from b_k to $b_k - e$ and a bead from $s_k - e$ to s_k for $k = 1, \dots, c$ in turn; the definition of $s_1, \dots, s_c$ guarantees that you can do this, i.e. that positions b_k and $s_k - e$ are occupied while positions $b_k - e$ and s_k are empty, as long as k takes the values $1, \dots, c$ in order. Let $A\lambda$ be the partition defined by the resulting abacus display. It is quite easy to see that $A\lambda$ is independent of the choice of abacus display. Define $B\lambda$ in the same way, but replacing the condition '$s_c < b_{c+1}$' with '$s_{c+1} < b_{c+1}$'.

Example. Suppose $e = 5$, and $\lambda = (14^2, 13, 7^2, 6^2, 5^3, 4, 3^2, 2^{11})$:



We have $s = x = 1$, $(b_1, \dots, b_l) = (6, 11, 21, 26, 36)$ and $(s_1, s_2, \dots) = (13, 18, 22, 25, 28, 29, \dots)$. So in the definition of $A\lambda$ we take $c = 2$, while in the definition of $B\lambda$ we take $c = 3$. We find that

$A\lambda =$  $= (14^2, 13, 7^2, 6^2, 5^5, 4^2, 3, 2^2, 1^7)$;

$B\lambda =$  $= (14^2, 13, 7^2, 6^3, 5^3, 4^3, 3, 2^2, 1^7)$.

It is easy to check that $|A\lambda| = |\lambda|$. Also, if λ is e-singular then we have $s_c > b_c$, which implies that $A\lambda$ exceeds λ in the lexicographic ordering of partitions. Hence applying the function A repeatedly will terminate in an e-regular partition $\overline{A}\lambda$. A similar statement holds for B, yielding an e-regular partition $\overline{B}\lambda$. We shall prove the following statement, which yields an algorithm (in fact, a choice of algorithms) for regularising on the abacus.

Proposition 1. *For any λ, both $\overline{A}\lambda$ and $\overline{B}\lambda$ coincide with $G\lambda$.*

First we prove the following.

Lemma 2. *Suppose λ is a partition. Then either $B\lambda = A\lambda$ or $B\lambda = BA\lambda$.*

Proof. Choose an abacus display $\mathcal{A}$ for λ, and let $s, b_1, \dots, b_l$ and $s_1, s_2, \dots$ be as in the definitions. Let c be minimal such that $s_c < b_{c+1}$, and let d be minimal such that $s_{d+1} < b_{d+1}$. Then $d \geqslant c$; if $d = c$ then obviously we have $A\lambda = B\lambda$, so assume that $d > c$, i.e. $s_{c+1} > b_{c+1}$. Let C be the abacus display for $A\lambda$ obtained as in the definition.

Now in C position $b_{c+1} - e$ is empty; we claim that there must be beads in all positions $b_{c+1} - e + 1, \dots, b_{c+1}$. This is obviously true for position b_{c+1}, so consider a position $y \in \{b_{c+1} - e + 1, \dots, b_{c+1} - 1\}$. If y is occupied in $\mathcal{A}$, then it remains occupied when we pass to C, because on runners other than runner x we only move beads down to positions earlier than b_{c+1}. If position y is empty in $\mathcal{A}$, then we have $y = s_k$ for some $k \leqslant c$, so when we form C from $\mathcal{A}$ we move a bead down into position y, and we don't move it any further.

$\mathcal{A}$ and C are exactly the same after position b_{c+1}, and this implies that in C position $s' = b_{c+1} - e$ is the last empty position with e occupied positions immediately after it. The occupied positions below position s' on the same runner (i.e. runner x) are $b_{c+1}, \dots, b_l$, and the empty spaces after position s' not on runner x are $s_{c+1}, s_{c+2}, \dots$. So to construct $BA\lambda$, we must find the first d' such that $s_{c+d'+1} < b_{c+d'+1}$; from above, we have $d' = d - c$. Therefore an abacus display for $BA\lambda$ is constructed from C by moving the beads in positions $b_{c+1}, \dots, b_d$ each up one space, and moving a bead down into position s_k for $k = c + 1, \dots, d$ in turn. We deduce that $BA\lambda = B\lambda$. □

Corollary 3. *For any partition λ, we have $\overline{A}\lambda = \overline{B}\lambda$.*

Proof. This is immediate from Lemma 2, using induction on lexicographic order. □

Before proving Proposition 1, we give a lemma which relates moving a bead to ladder numbers.

Lemma 4. *Suppose λ is a partition, and take an abacus display for λ with r beads. Suppose that ξ is a partition whose abacus display is obtained by moving a bead from position $t + e$ to position t, for some t. Let m be the number of empty spaces before position t in either display.*

1. *Suppose positions $t+1, \dots, t+e-1$ are occupied in both abacus displays. Then $[\xi] \subset [\lambda]$, and $[\lambda] \setminus [\xi]$ consists of one node from each of the ladders numbered*

$$r - t + em - e + 1, \quad r - t + em - e + 2, \quad \dots, \quad r - t + em.$$

2. *Suppose $t < y < t + e$, and positions $t + 1, \dots, y - 1, y + 1, \dots, t + e - 1$ are occupied in both abacus displays, while position y is unoccupied. Then $[\xi] \subset [\lambda]$, and $[\lambda] \setminus [\xi]$ consists of one node from each of the ladders numbered*

$$r - y + em + 1, \quad r - y + em + 2, \quad \dots, \quad r - y + em + e.$$

Proof. It is straightforward enough to write down the actual nodes in each case. □

Proof of Proposition 1. We proceed by induction on $|\lambda|$, and for fixed $|\lambda|$ by induction on lexicographic order. Assuming that λ is e-singular, let $\mu = A\lambda$. By induction $\overline{A}\lambda = G\mu$, and so it suffices to show that λ and μ have the same e-regularisation, i.e. that the numbers of nodes in corresponding ladders are the same.

Assume the notation from the definition of $A\lambda$, and suppose first that $c > 1$. Let ξ be the partition obtained from λ by moving the bead from position $b_1 = s + e$ to position s. The fact that $c > 1$ means that every position between s and b_2 not on runner x is occupied. By the maximality of s, this means that $b_2 = b_1 + e$, and in the abacus display for ξ, position $s' = b_1$ is empty and immediately followed by e beads; since the abacus displays for λ and ξ agree after position b_1, s' must be maximal with this property. s' lies on runner x, and the beads below it on the same runner lie in positions $b_2, \dots, b_l$, while the empty spaces occurring after position s' not on runner x lie in positions $s_1, s_2, \dots$. So if we want to construct the partition $B\xi$, then we must find the smallest c' such that $s_{c'+1} < b_{c'+2}$. From what we already know, we see that $c' = c - 1$. So an abacus display for $B\xi$ is obtained from the given display for ξ by moving the beads at positions $b_2, \dots, b_c$ each up one space, and then moving a bead from position $s_k - e$ to position s_k for $k = 1, \dots, c - 1$ in turn. Hence μ may be obtained from $B\xi$ simply by moving a bead from position $s_c - e$ to position s_c.

Let m be the number of empty spaces before position s (in any of these abacus displays). By Lemma 4(1) (putting $t = s$) we see that the Young diagram of ξ may be obtained from the Young diagram of λ by removing one node from each of ladders

$$r - s + em - e + 1, \quad r - s + em - e + 2, \quad \dots, \quad r - s + em.$$

Now we apply Lemma 4(2) to μ and $B\xi$. We put $t = s_c - e$, and write t as $s + eu + v$, with $0 < v < e$. Each of the positions $t + 1, \dots, t + e - 1$ apart from position $y = s + e(u + 1)$ is occupied, by the construction of μ. And we claim that position y is unoccupied; indeed, the definition of c means that we have $b_c < s_{c-1} < s_c < b_{c+1}$, so that $b_c \leqslant y < b_{c+1}$, and by construction, all the positions $b_c, b_c + e, \dots, b_{c+1} - e$ are empty in the abacus display for μ.

In order to apply Lemma 4, we need to compute the number of empty spaces before position t in the abacus display for μ. Along with the m empty spaces before position s, there are an additional $u + 1 - c$ spaces on runner x (positions $s, s + e, \dots, s + ue$, minus positions $b_1 - e, \dots, b_c - e$), and an additional $c - 1$ spaces not on runner x (namely, positions $s_1 - e, s_2 - e, \dots, s_{c-1} - e$). Hence there are $m + u$ empty spaces before position t. Now Lemma 4(2) tells us that the Young diagram for $B\xi$ is obtained from the Young diagram for μ by removing a node from each of the ladders

$$r - s + em - e + 1, \quad r - s + em - e + 2, \quad \dots, \quad r - s + em.$$

By induction, ξ and $B\xi$ have the same regularisation (namely $\overline{A}\xi = \overline{A}B\xi$), and therefore λ and μ have the same regularisation, as required.

The case where $c = 1$ is very similar, except that we do not need to compute $B\xi$; we just replace $B\xi$ with ξ in the above argument. □

References

[BOX] C. Bessenrodt, J. Olsson & M. Xu, 'On properties of the Mullineux map with an application to Schur modules', *Math. Proc. Cambridge Philos. Soc.* **126** (1999), 443–59.

[FLM] M. Fayers, S. Lyle & S. Martin, '*p*-restriction of partitions and homomorphisms between Specht modules', *J. Algebra* **306** (2006), 175–90.

[J] G. James, 'On the decomposition matrices of the symmetric groups II', *J. Algebra* **43** (1976), 45–54.

[M] A. Mathas, *Iwahori–Hecke algebras and Schur algebras of the symmetric group*, University lecture series **15**, American Mathematical Society, Providence, RI, 1999.